ICNAAM 2012
Kos, Greece

# How individual response to habitat edges affects population persistence and spatial spread


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# Population Dynamics

- Number of individuals as a function of time

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- Explicit consideration of space is particularly important when we consider
  - Invasion processes
  - Heterogeneous environments

# Reaction Diffusion Equations

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## One single species in homogeneous space

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- Spatial heterogeneity can be incorporated by replacing the reaction and diffusion terms with the spatially dependent terms $f(u(x,t),x)$ and $\partial^2/\partial x^2(D(x)u(x,t))$

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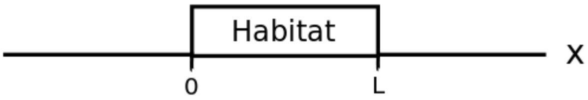
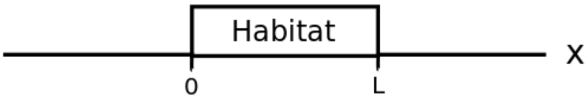
One then needs to impose interface conditions that relate the population densities and fluxes between two adjacent habitats.

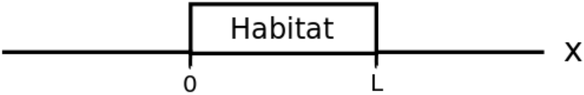
# Boundary Conditions


Habitat
x
0
L




### Dirichlet

$$u(0) = 0, \quad u(L) = 0$$ lethal boundaries

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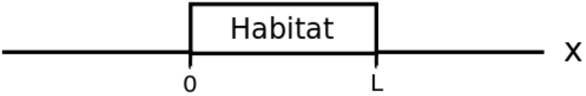
## Neumann

$$\frac{\partial u}{\partial x}(0) = 0, \quad \frac{\partial u}{\partial x}(L) = 0$$ no flux at boundaries

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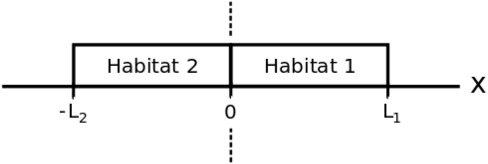
### Robin

$$\frac{\partial u}{\partial x}(0) = \xi u(0), \quad \frac{\partial u}{\partial x}(L) = \eta u(L)$$

- What conditions should be considered at interfaces?



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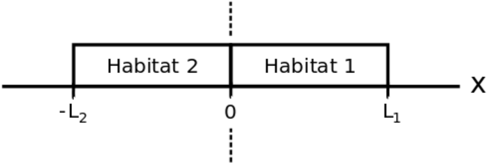


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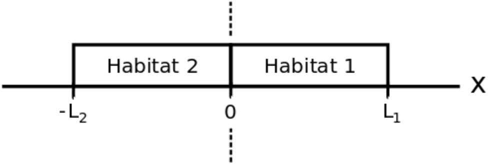


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$$u_1(0^+) = u_2(0^-), \quad D_1 \frac{\partial u_1}{\partial x}(0^+) = D_2 \frac{\partial u_2}{\partial x}(0^-)$$

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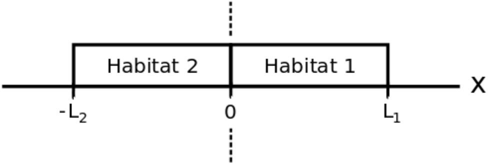


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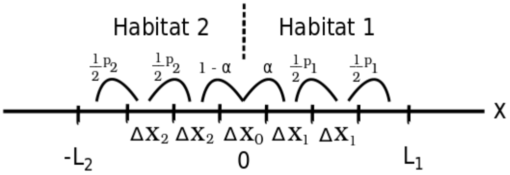
Are these natural assumptions?

- Ovaskainen and Cornell (2003) derived a discontinuous interface condition between two habitats



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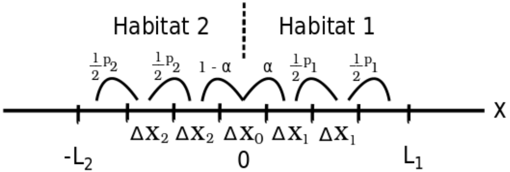


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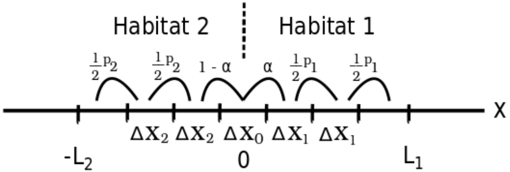


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## Discontinuous interface condition

$$\alpha \; p_2 \Delta x_2 u_2(0^-, t) = (1-\alpha) \; p_1 \Delta x_1 u_1(0^+, t)$$

$$D_2 \frac{\partial u_2}{\partial x}(0^-, t) = D_1 \frac{\partial u_1}{\partial x}(0^+, t)$$

$$\alpha\ p_2 \Delta x_2 u_2(0^-, t) = (1-\alpha)\ p_1 \Delta x_1 u_1(0^+, t)$$

- By making some assumptions this equation can be written in terms of diffusivities

$$D_2 = \lim_{\Delta x_2, \Delta t \to 0} p_2 \frac{\Delta x_2^2}{\Delta t},\ \ D_1 = \lim_{\Delta x_1, \Delta t \to 0} p_1 \frac{\Delta x_1^2}{\Delta t}$$

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$\Delta x_2 = \Delta x_1$

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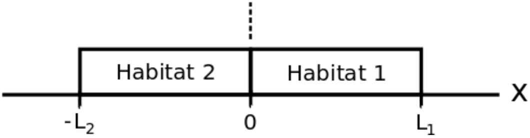
- 3 different interface conditions



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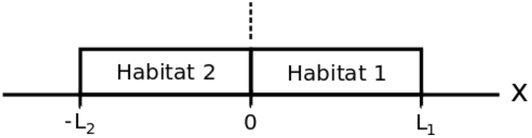


### Continuous

$$u_2(0^-, t) = u_1(0^+, t)$$

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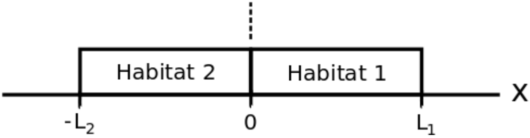


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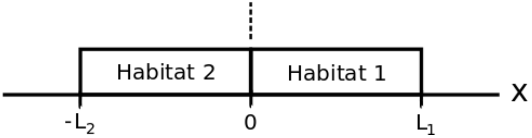


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### Discontinuous II

$$\alpha \; D_2 u_2(0^-, t) = (1-\alpha) \; D_1 u_1(0^+, t)$$

Plus flux continuity in all cases!

# Invasion in a periodically varying environment

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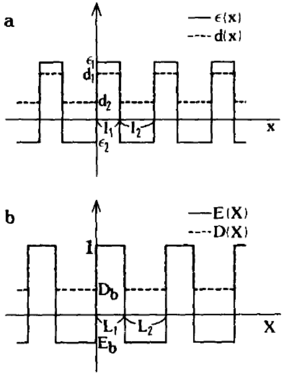


Figure: Shigesada *et al* (1986).

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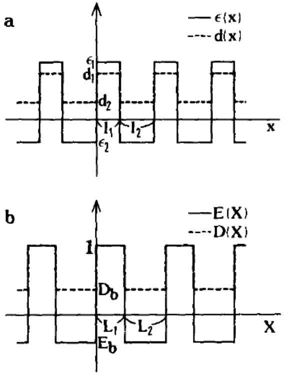


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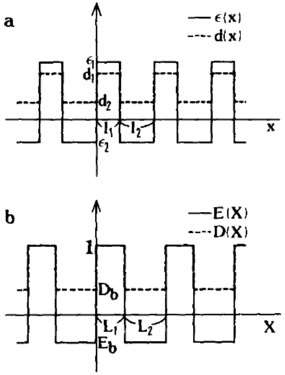


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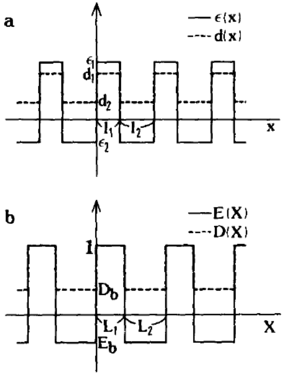


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3. Diffusivities in habitats 1 and 2 are $d_1$ and $d_2$

- Population dynamics is described by the reaction diffusion equations

$$
\begin{aligned}
\frac{\partial U_1}{\partial T} &= \frac{\partial^2 U_1}{\partial X^2} + (1 - U_1)U_1 \quad \text{in favorable habitats} \\
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where: $D_b = \frac{d_2}{d_1}$ and $E_b = \frac{\epsilon_2}{\epsilon_1}$.

- Persistence of the population is guaranteed if the zero solution of this equation is unstable
- We study then the solution of the linearized system

$$\begin{aligned}\frac{\partial U_1}{\partial T} &= \frac{\partial^2 U_1}{\partial X^2} + U_1 \quad \text{(favorable)}\\ \frac{\partial U_2}{\partial T} &= D_b \frac{\partial^2 U_2}{\partial X^2} + E_b U_2 \quad \text{(unfavorable)}\end{aligned}$$

**Given $U(X,T) = V(X)\exp(\lambda T)$, characteristic equations are**

Continuous boundary conditions

$$\sqrt{1-\lambda}\tan\left(\sqrt{1-\lambda}\frac{L_1}{2}\right) = \sqrt{(-E_b+\lambda)\,D_b}\tanh\left(\sqrt{\frac{-E_b+\lambda}{D_b}}\frac{L_2}{2}\right)$$

Discontinuous boundaries I

$$\sqrt{1-\lambda}\tan\left(\sqrt{1-\lambda}\frac{L_1}{2}\right) = \frac{1-\alpha}{\alpha}\sqrt{(-E_b+\lambda)}\tanh\left(\sqrt{\frac{-E_b+\lambda}{D_b}}\frac{L_2}{2}\right)$$

Discontinuous boundaries II

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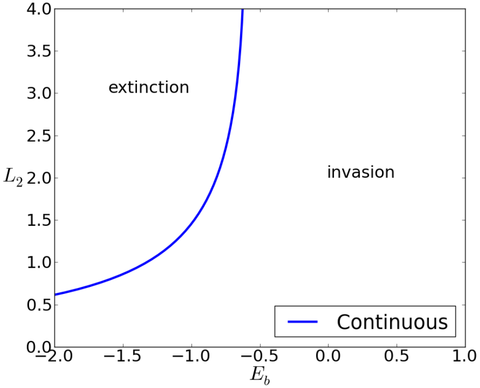
## Persistence conditions

- Unfavorable habitat size ($L_2$) x intrinsic growth rate in the unfavorable habitat ($E_b$)



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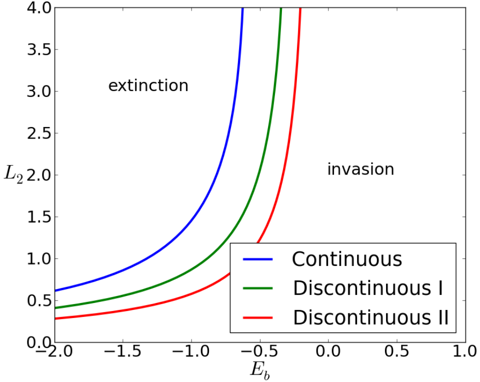

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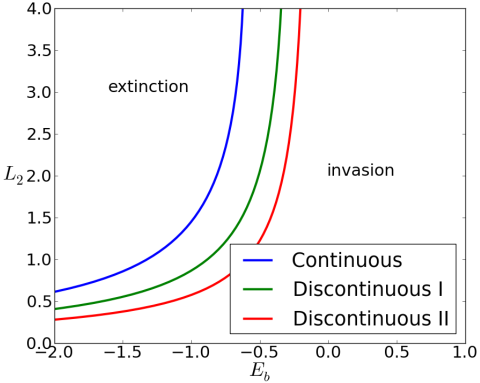


- No great qualitative changes are found

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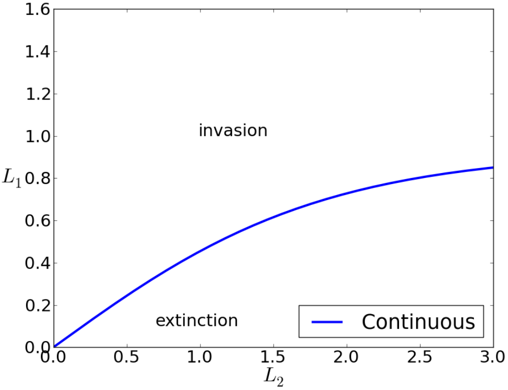

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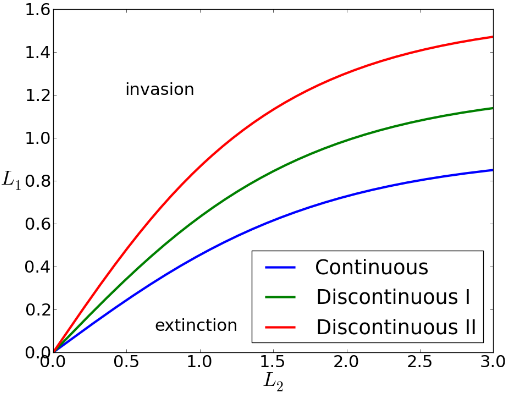

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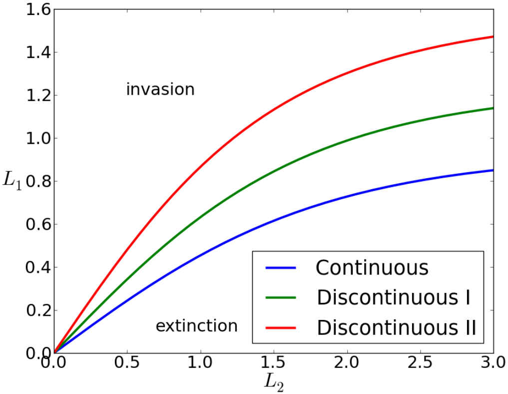
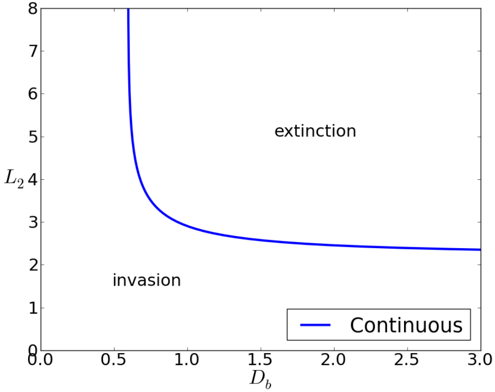



- Again curves have similar shape

# Invasion in a periodically varying environment

- Unfavorable habitat size ($L_2$) x diffusion coefficient in the unfavorable habitat ($D_b$)



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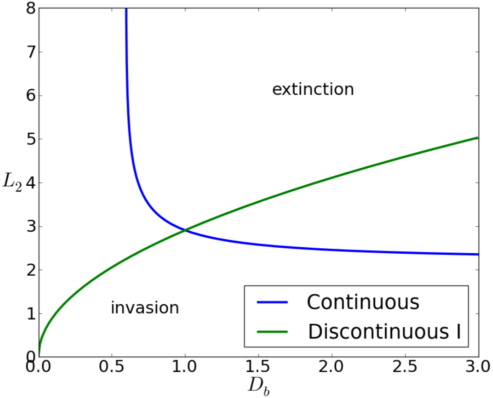

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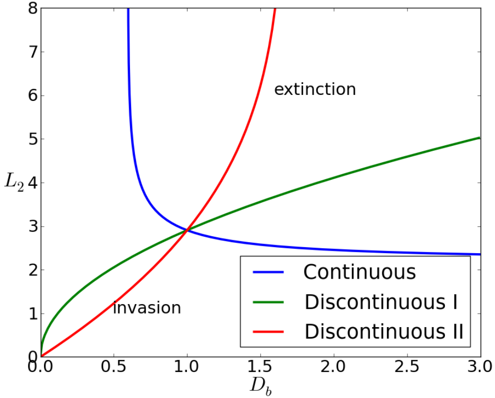

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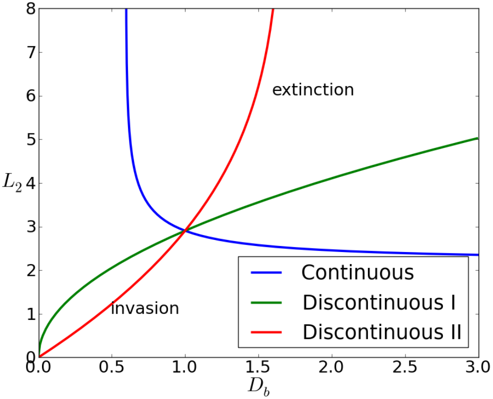


- **Results are strongly affected by the different interface conditions!**

# Traveling periodic waves

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- We look then for traveling wave solutions

$$U(X,T) = f(Z)g(X), \quad Z = X - CT$$

where

$$g(X) = g(X+L), \quad L = L_1 + L_2$$

- Wave speeds x intrinsic growth rate in the unfavorable habitats ($E_b$) for different values of $L_2$

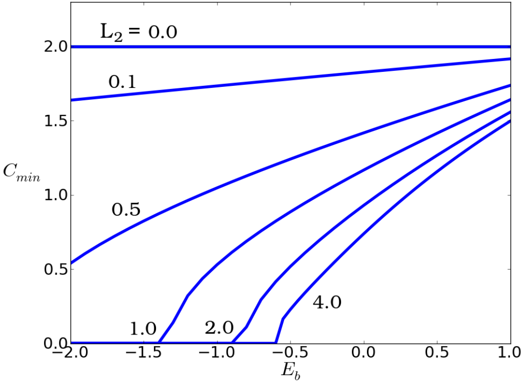


Figure: Continuous interfaces.

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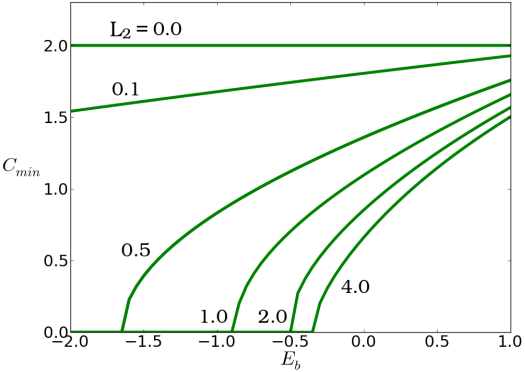


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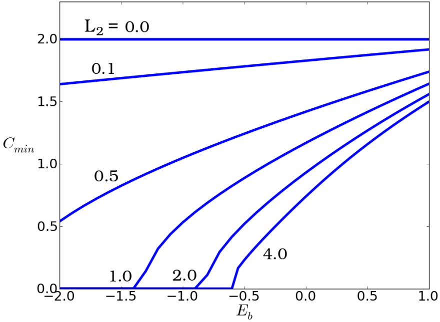


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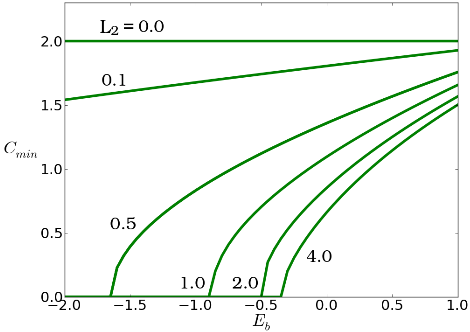


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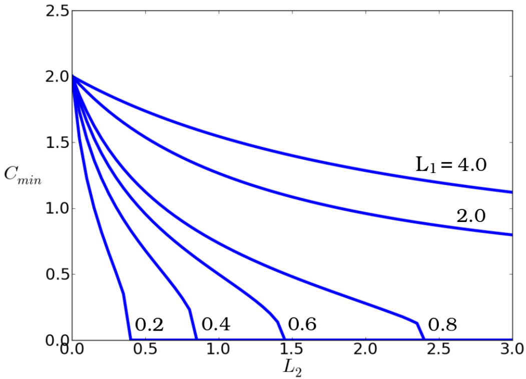


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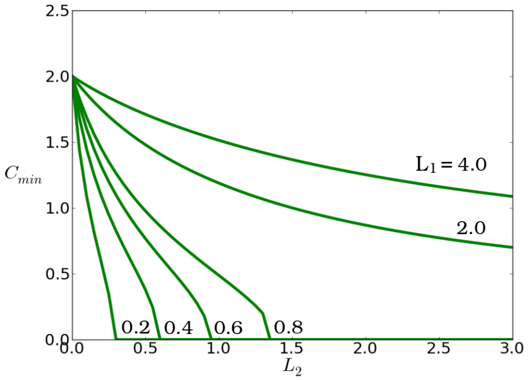


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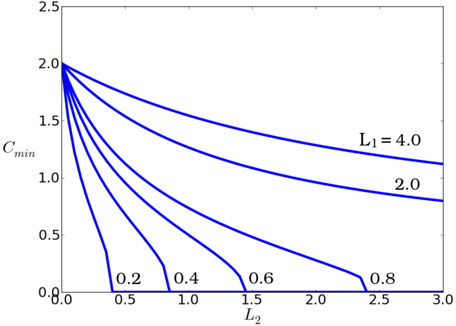


Figure: Continuous interfaces.

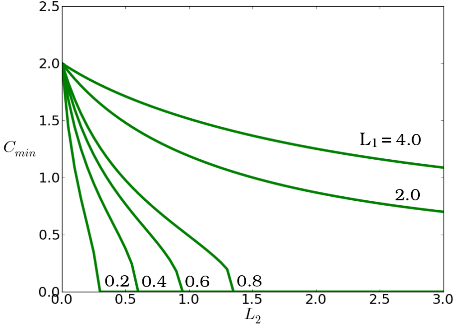


Figure: Discontinuous interfaces I.

- Wave speeds x diffusivity in the unfavorable ($D_b$) for different values of $L_2$

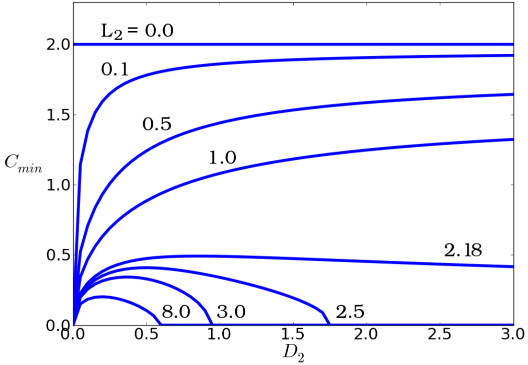


Figure: Continuous interfaces.

- Wave speeds x diffusivity in the unfavorable ($D_b$) for different values of $L_2$

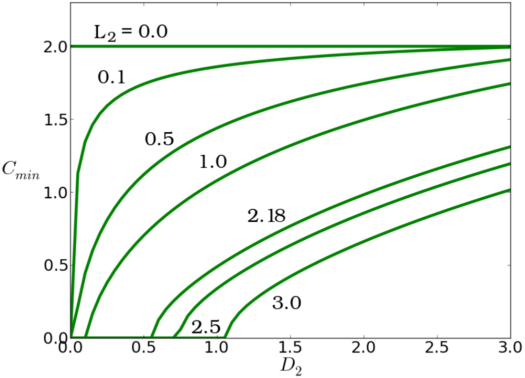


Figure: Discontinuous interfaces I.

- Wave speeds x diffusivity in the unfavorable ($D_b$) for different values of $L_2$

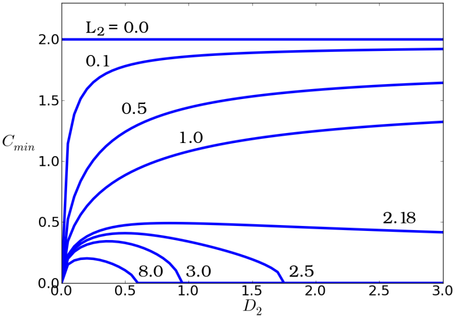


Figure: Continuous interfaces.

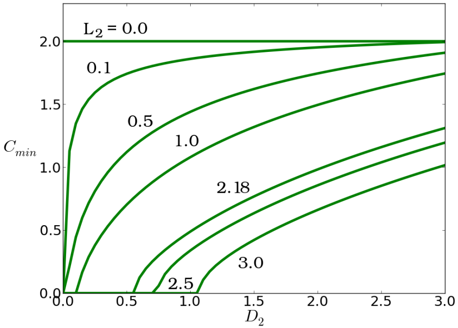


Figure: Discontinuous interfaces I.

# Summary

# Summary

- Interface conditions must reflect individual movement behaviour across an interface
- Edge behaviour and differential movement between habitats can lead to discontinuous interface conditions
- Results regarding species persistence and spread in heterogeneous environments are strongly affected by interface conditions

# Acknowledgments

FAPESP

CAPES

Thanks for your attention!