Modern Cosmology I

Raul Abramo
Physics Institute, Univ. of São Paulo
abramo@fma.if.usp.br

Outline:

- Review of General Relativity
- Einstein's equations
- Kinematics & dynamics of FLRW spacetimes
- Cosmological times & distances, momenta & energy
- Thermodynamics
- Big Bang, horizons, inflation & all that

Part 1: General Relativity

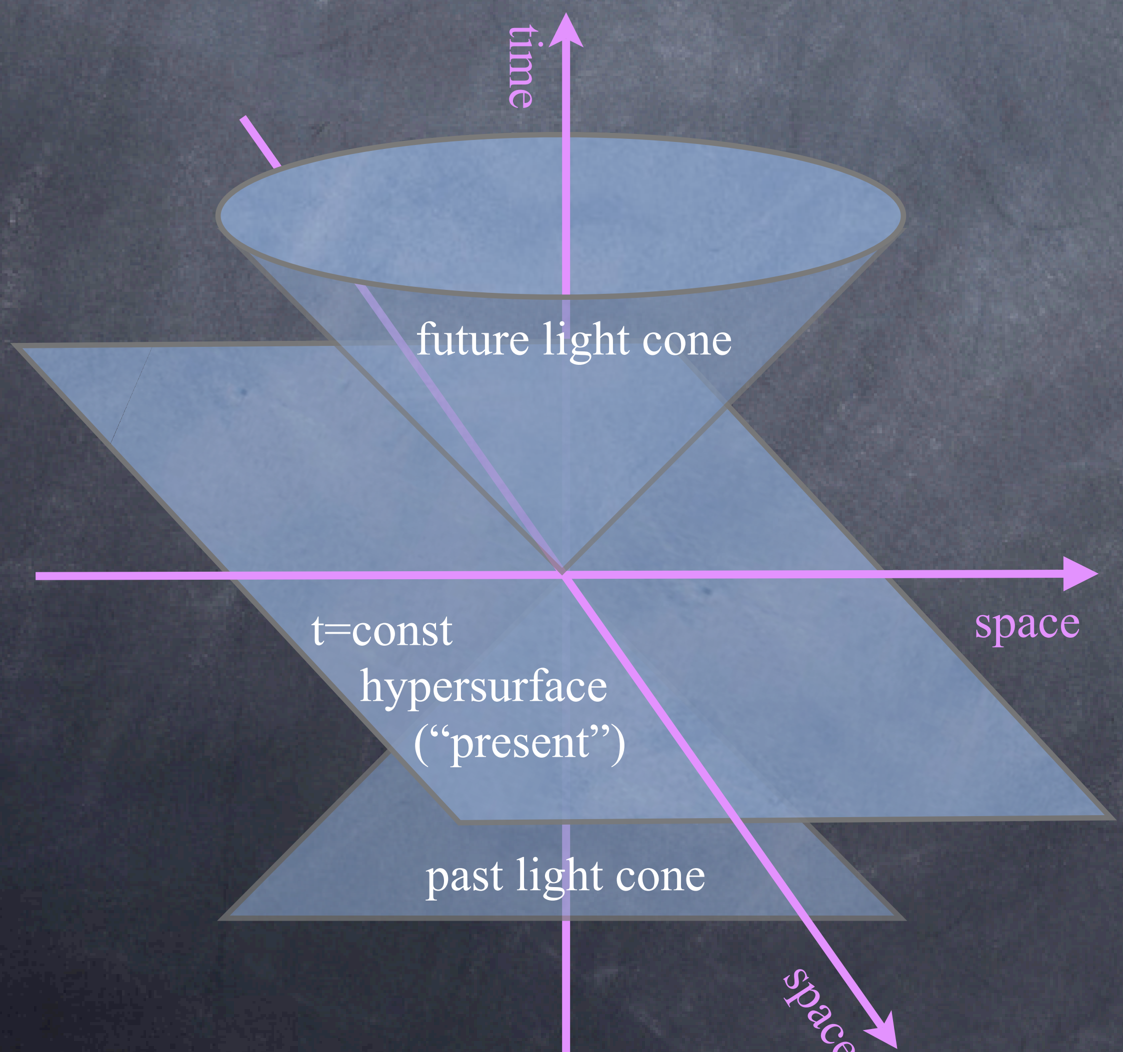
Bibliography:

C. Boyer, "A History of Mathematics"

H. Lorentz, A. Einstein, H. Minkowski & H. Weyl, "The Principle of Relativity"

S. Weinberg, "Gravitation and Cosmology" (notations/conventions!)

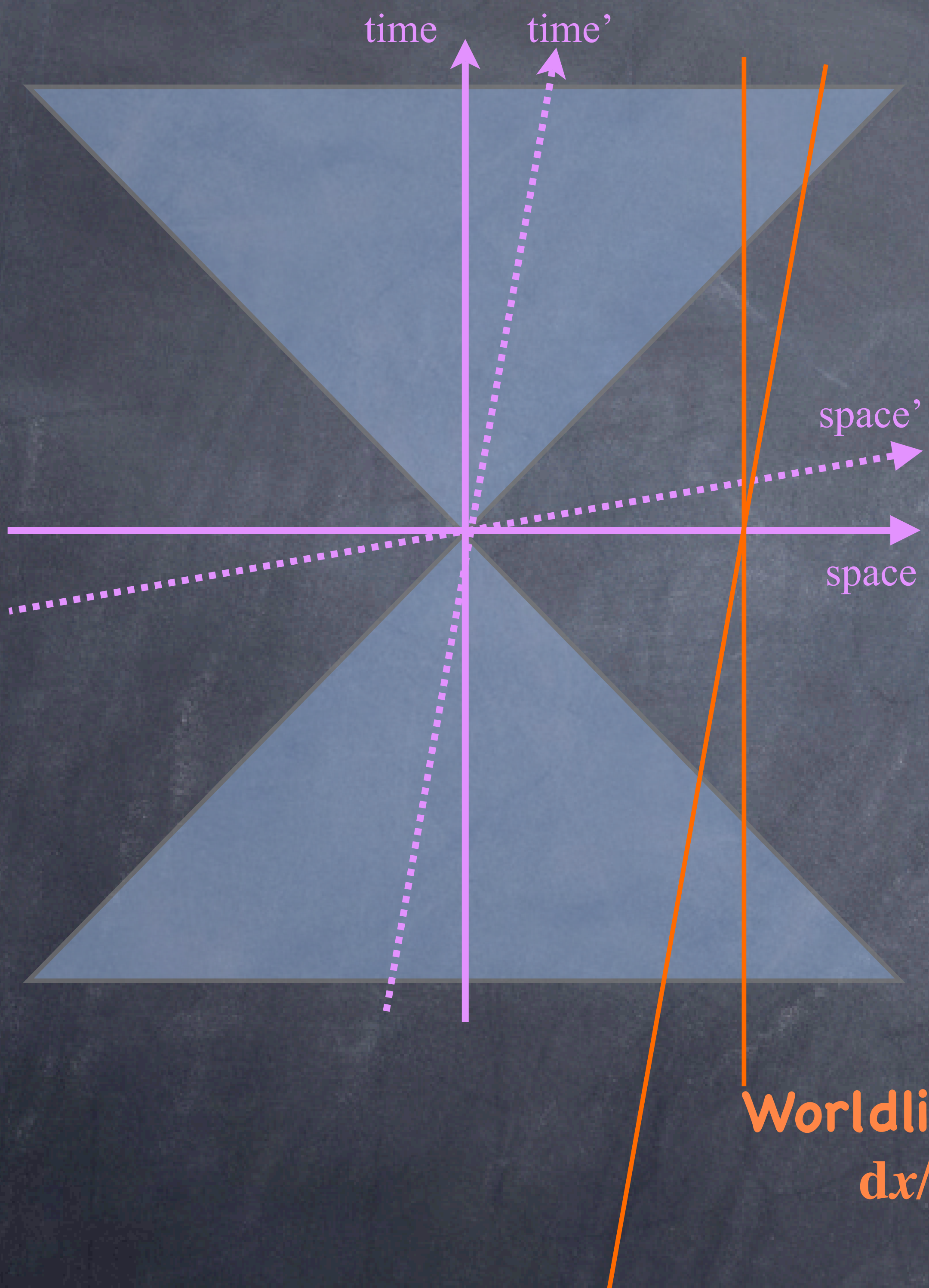
Special Relativity: the fundamental object is the light cone



Light cone:

$$ds^2 = -c^2\, dt^2 + d\vec{x}^2 \Rightarrow 0$$

Special Relativity: invariance of the light cone under **boosts**



$$dx^\mu \to dx'^\mu = \Lambda^\mu_{\ \nu}\, dx^\nu$$

$$\Lambda^\mu_{\ \nu} = \begin{pmatrix} \gamma & -\gamma \frac{v^i}{c} \\ -\gamma \frac{v_j}{c} & \delta_{ij} + (\gamma - 1)\frac{v^i\, v_j}{v^2} \end{pmatrix}$$

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

$$\Lambda^\mu_{\ \alpha}(v)\Lambda^\alpha_{\ \nu}(-v) = \delta^\mu_\nu$$

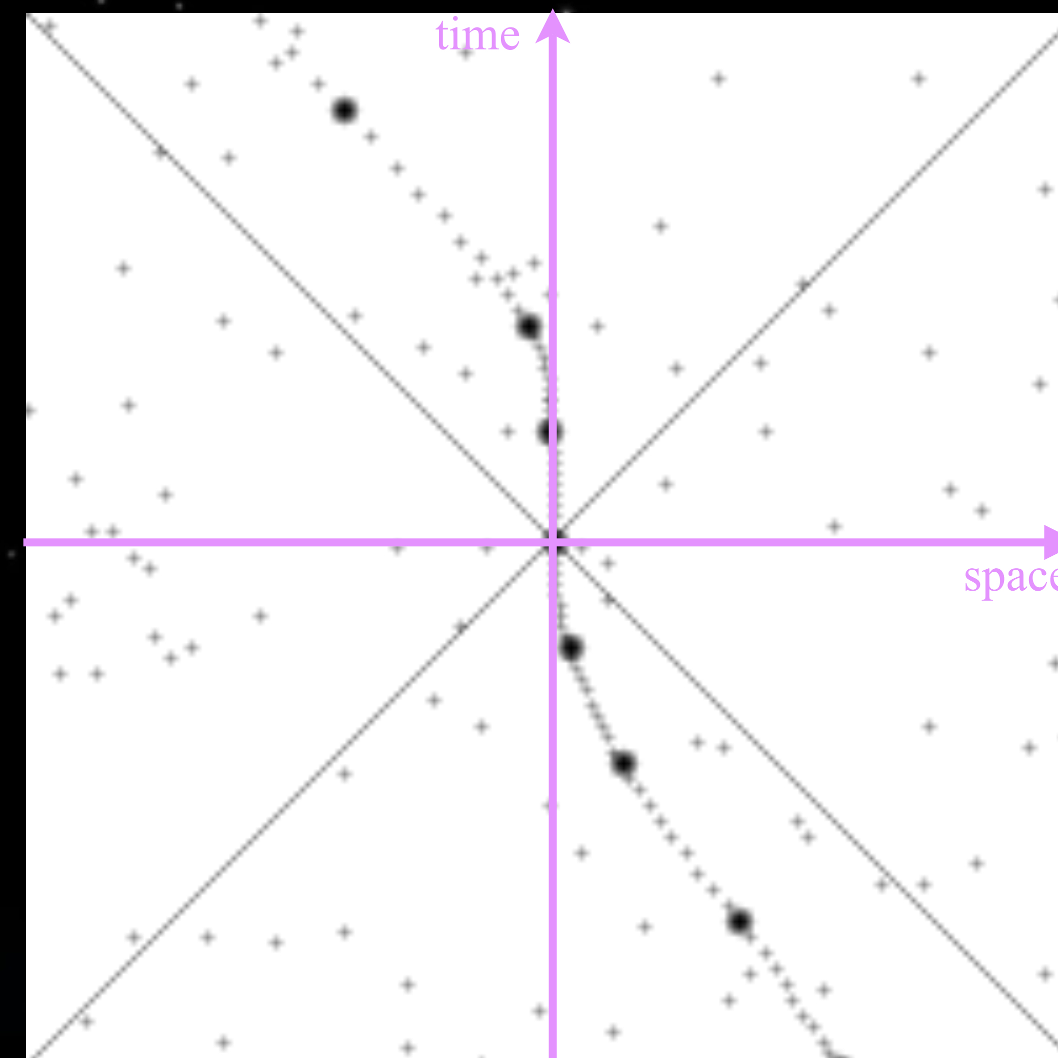
Worldline of observer with
$dx/dt=0$, $dx'/dt'=-v$

Worldline of observer with
$dx'/dt'=0$, $dx/dt=v$

Einstein's equivalence principle rehabilitates accelerated observers
➠ covariance under <u>general</u> coordinate transformations

Stationary observer in gravitational field

Accelerated observer (free fall) = inertial observer



$$ds^2 = -c^2\, dt^2 + d\vec{x}^2$$

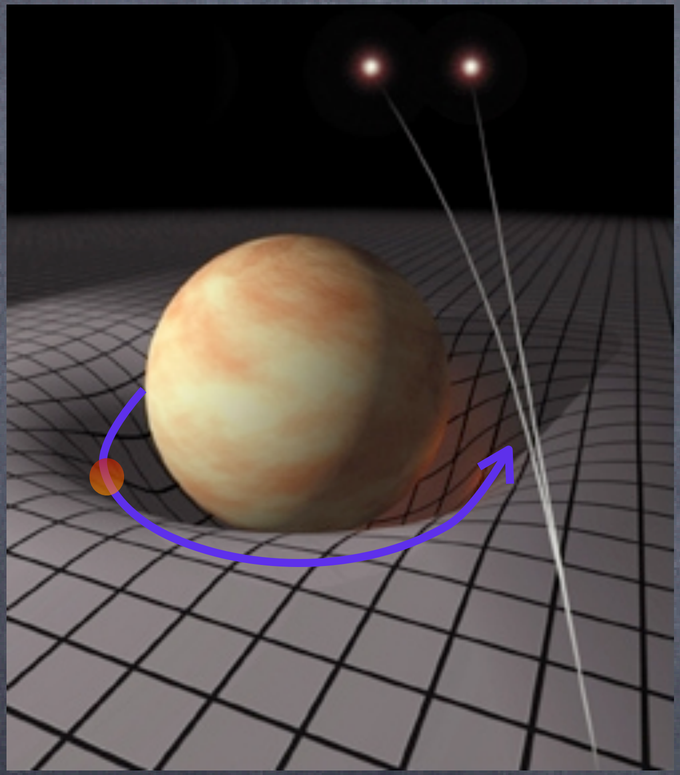
$$\Rightarrow ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

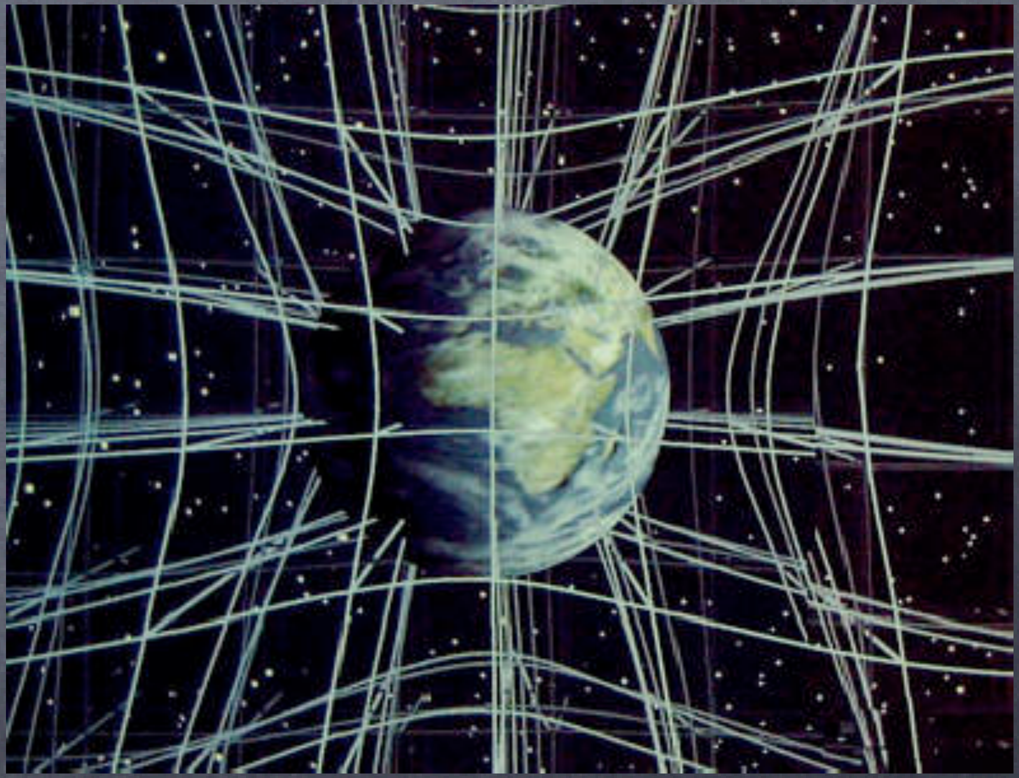
metric: geometry of spacetime

Generally covariant theories of gravity (including General Relativity) are based on the equivalence principle.

In these theories, the metric of spacetime (i.e., its geometry) has a dual role:

it both **causes** the motions of bodies... and it is **affected** by them.





- What is the geometry of spacetime?
- What causes it?
- How can we make measurements to test our theories?

Preamble: The pre-history of Differential Geometry

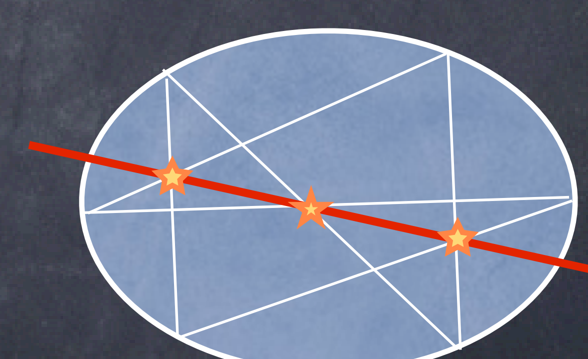
After Newton, physicists focused on Mechanics, Optics, Thermodynamics; Mathematics became obsessed with Analysis; Geometry was considered a second-rate subject.

Geometry's comeback started c. 1806, when **Charles J. Brianchon** (21 at the time) and **Gaspard Monge** ("Comte de Péluse") proved the following theorem:

> **The six sides of a hexagon circumscribes a conic section IFF the three lines common to the three pairs of opposite vertices have a point in common**

This was immediately recognized to be the **dual** (the "projective dual") to **Pascal**'s theorem of 1639 (Pascal was 16 at the time!), which states that:

> **If an arbitrary hexagon is inscribed in a conic section, then the three pairs of the continuations of opposite sides meet in points that lie on a line.**

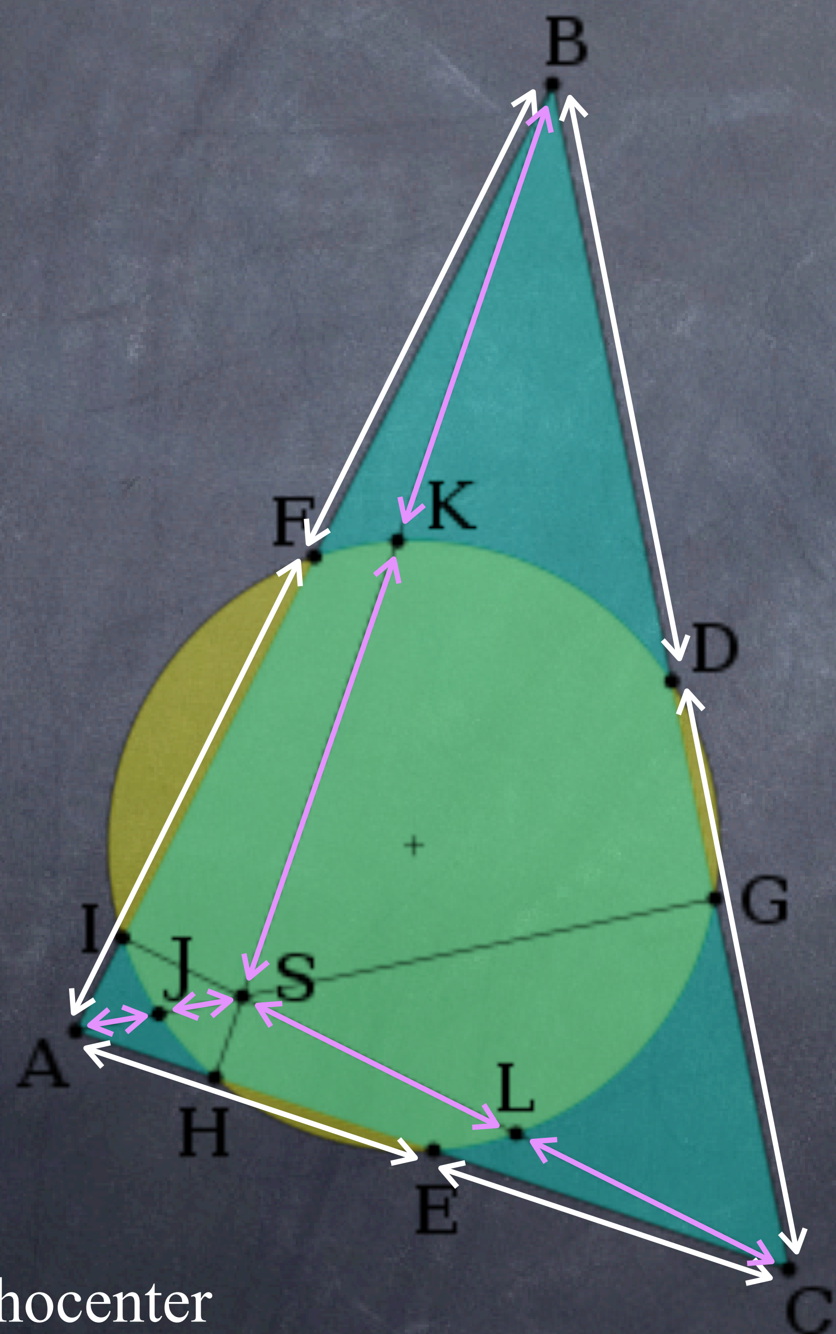


These results helped **Karl Feuerbach**, in 1822, to re-discover the properties of the **9-point circle** (Brianchon did this first)...

... and then to prove the **Feuerbach Theorem**...

AE=EC
CD=DB
BF=FA

SL=LC
SK=KB
SJ=JA

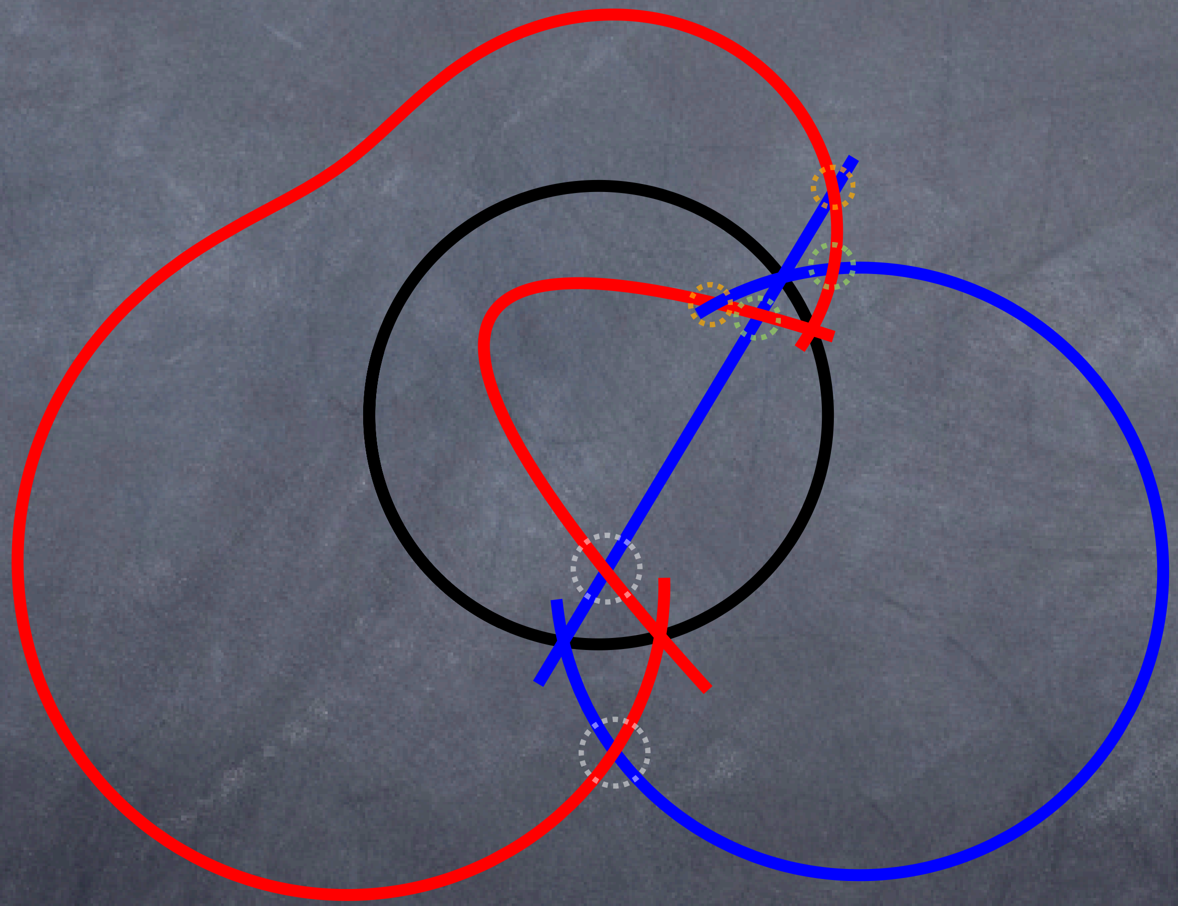


S: Orthocenter
AG, BH, CI: Altitudes

"the most beautiful theorem in Geometry since Euclid"

... which, in turn, inspired **Jakob Steiner** (Steiner/Geometry :: Gauss/Analysis) to discover, c. 1824, the laws of "inversive geometry": to every point inside (outside) a circle, corresponds another, outside (inside) that circle, found by the transformation (for unit radius):

$$x' = \frac{x}{x^2 + y^2} \quad , \quad y' = \frac{y}{x^2 + y^2}$$



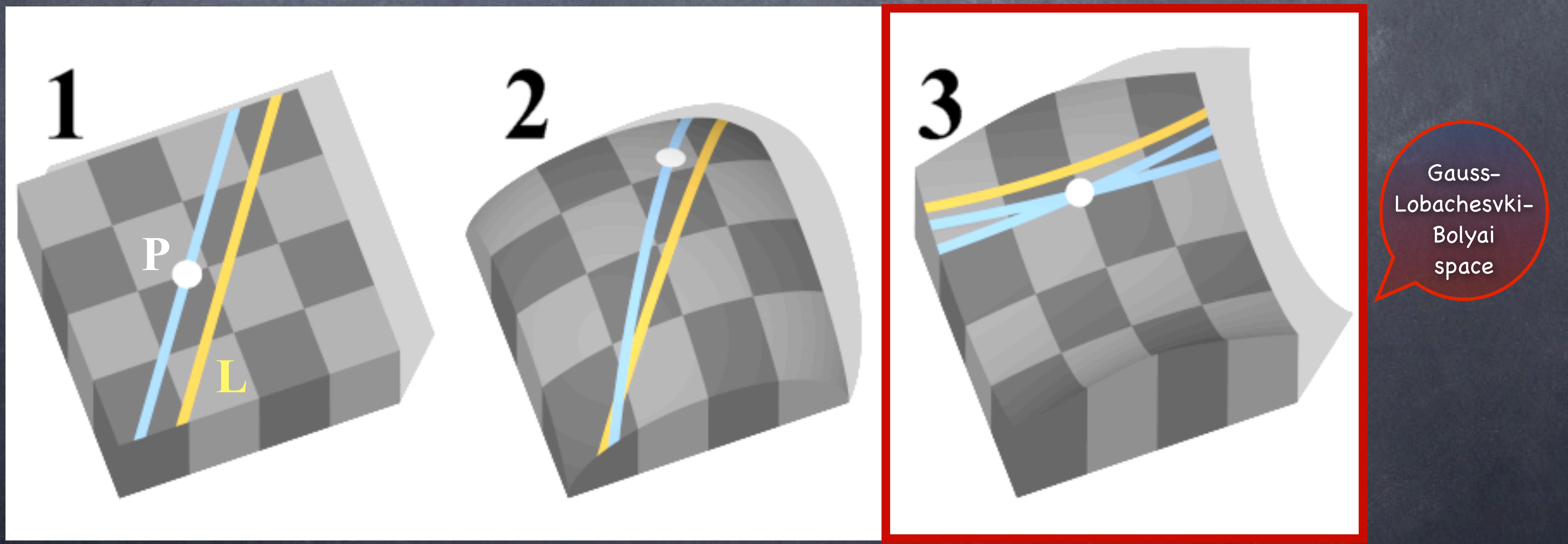
This is a **conformal transformation** - it leaves the **angles** of crossing lines **invariant**. (These types of transformations were later re-discovered by others, including **Lord Kelvin**, in the context of Electrostatics - as in the **method of images**.)

The study of coordinate transformations and dualities (e.g., points/lines) boosted interest in Geometry, turning it into a more respectable field.

Then, c. 1826 **Nicolai Lobachevski** (and, independently, **C. F. Gauss** and **János Bolyai**) addressed one of the pillars of Euclid's geometry: the "**parallel postulate**":

> **given a line L and a point P, there can be only one line through P that do not cross L.**

Lobachevski showed this to be **false**, by constructing 2D, **infinite** "curved" spaces (he called them "imaginary geometries") where **more than one** such lines exist.



1. Flat, infinite (Euclidean) | 2. Curved, finite (closed/elliptical) | 3. Curved, infinite (open/hyperbolic)

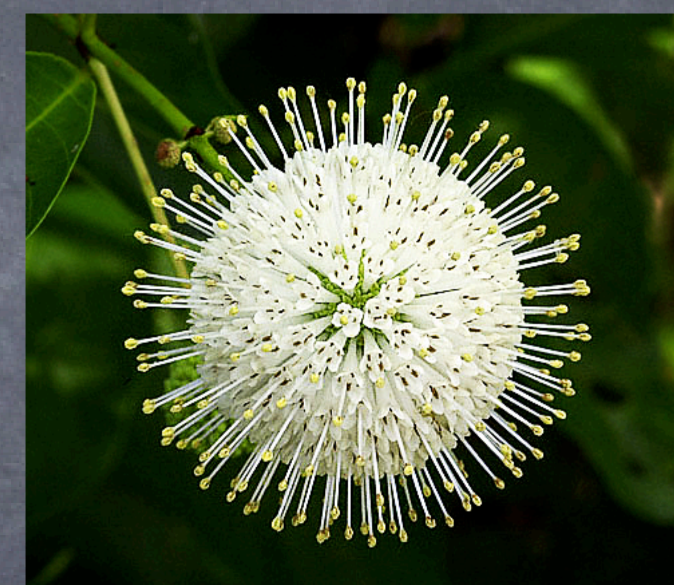
The flat (Euclidean), closed (i.e., spherical) and open (GLB) spaces are the **only manifolds** which obey a very simple principle: they are **homogeneous** and **isotropic**.

Homogeneity: space has the same properties at all points

Isotropy: space looks the same in all directions

Homogeneity without isotropy

Isotropy without homogeneity

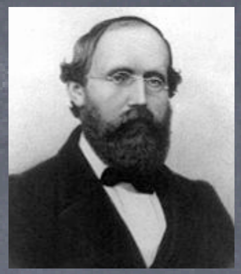


Cosmological principle: space is the same everywhere, looks the same in all directions

* Ehlers, Gehren & Sachs (1968): if all freely falling observers measure the same properties of matter (e.g., the cosmic microwave background), then the Universe is homogeneous and isotropic.

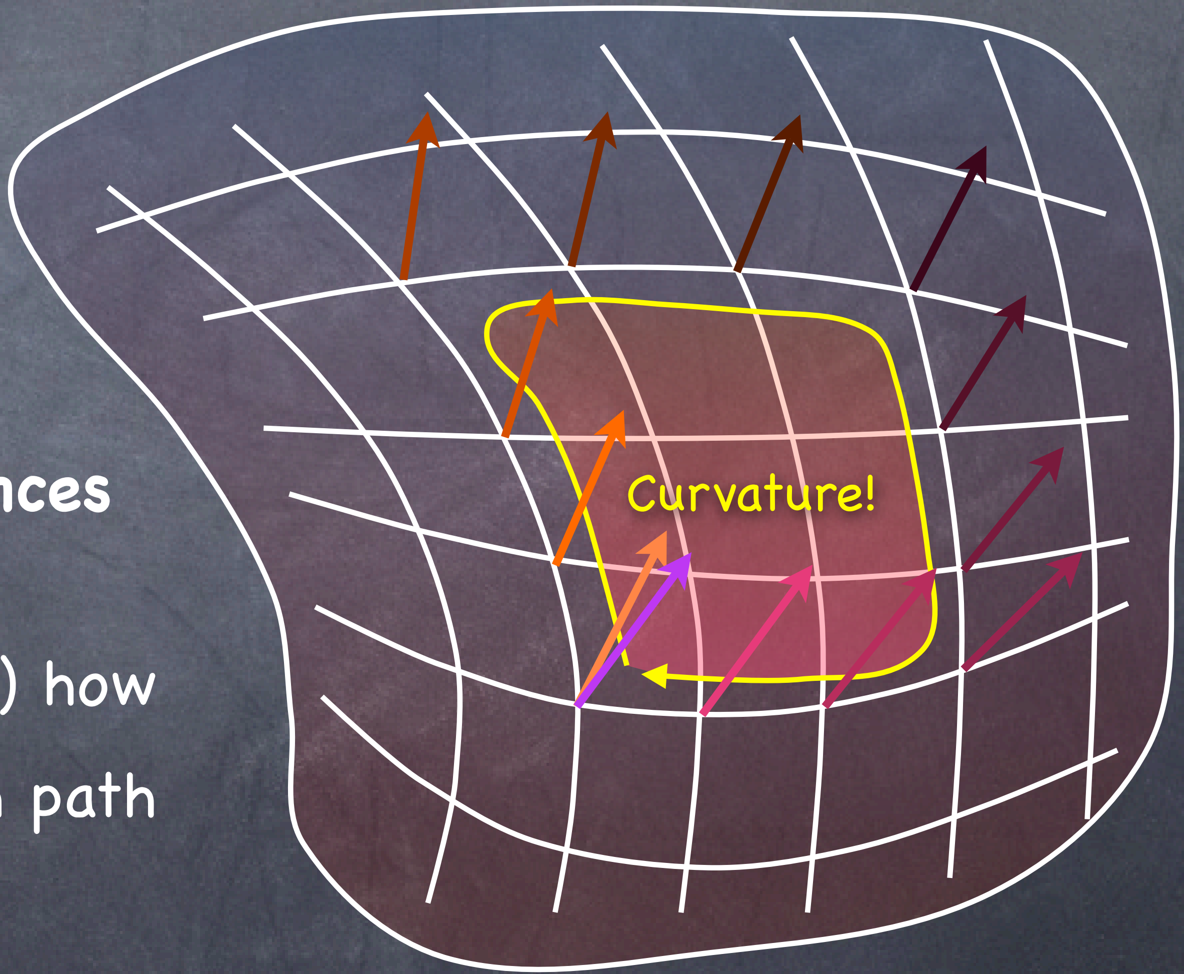
(Stoeger, Maartens & Ellis 1995 extended this result to approximately homog. and isotropic spaces)

Lobachevski's work was one of the motivations for **Georg Bernard Riemann**, in 1854, to propose in his thesis a global view of **Geometry as a study of manifolds of any number of dimensions, in any kind of space**.



These geometries are **essentially non-Euclidean**: the distance between two points is given in terms of a **metric**, which can itself be an **arbitrary, differentiable function.**

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$



The metric has a dual role:

i) it can be used to **measure** the **invariant distances** between any two points; and

ii) it determines (through the **affine connections**) how to **transport geometrical data** along any smooth path on the manifold - e.g.:

$$\frac{dV_\mu}{d\lambda} = \Gamma^\alpha_{\mu\nu} \frac{dx^\nu}{d\lambda} V_\alpha$$

$$\Delta V_\mu = \frac{1}{2} R^\alpha_{\ \beta\mu\nu} V_\alpha \oint dx^\mu x^\nu$$

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\frac{\partial g_{\beta\mu}}{\partial x^{\nu}} + \frac{\partial g_{\beta\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}} \right)$$

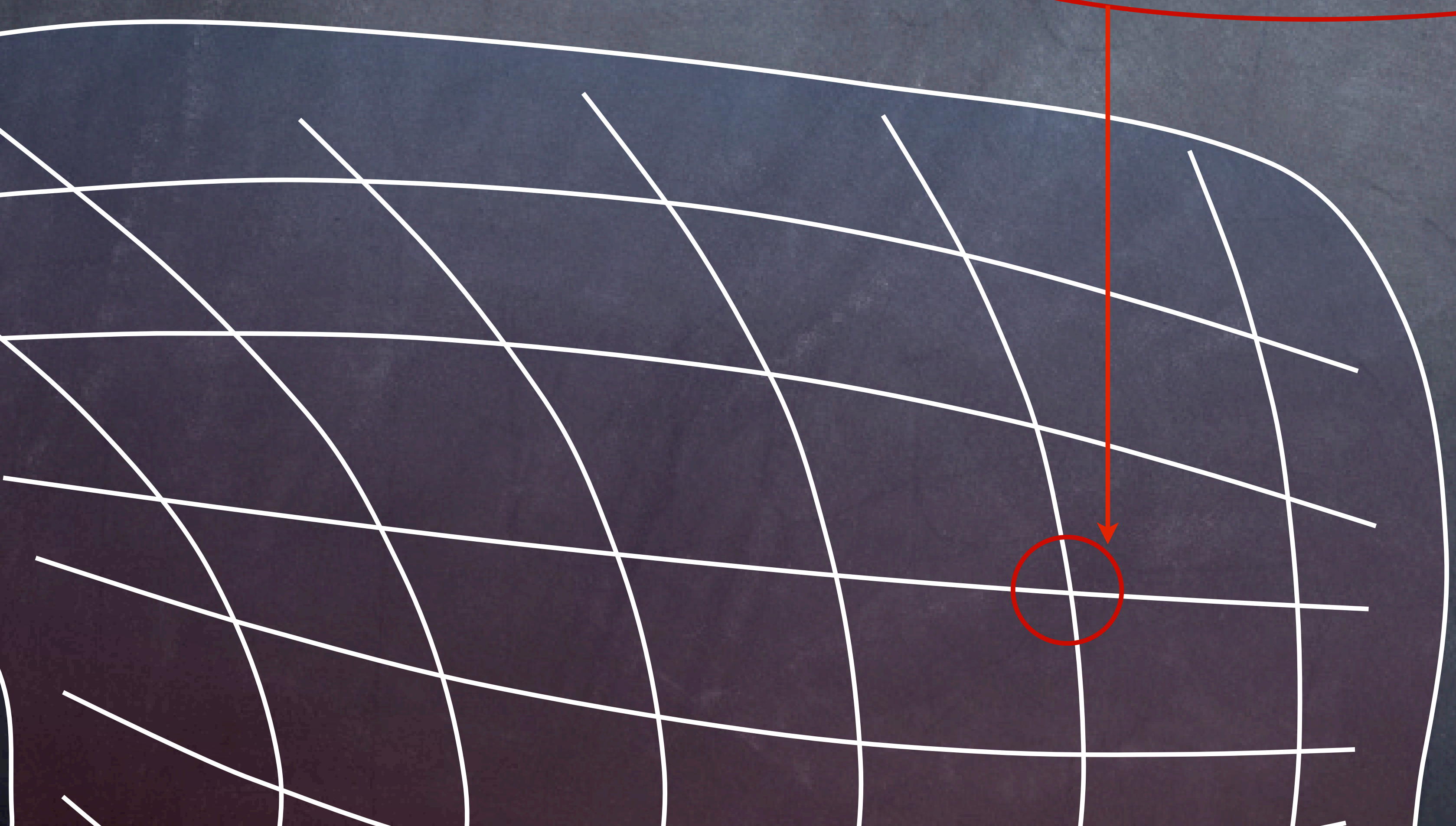
$$\frac{D V^{\alpha}}{D x^{\beta}} = V^{\alpha}_{;\beta} = \frac{\partial V^{\alpha}}{\partial x^{\beta}} + \Gamma^{\alpha}_{\mu\beta} V^{\mu}$$

$$R^{\alpha}_{\beta\mu\nu} = \frac{\partial \Gamma^{\alpha}_{\beta\mu}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\alpha}_{\beta\nu}}{\partial x^{\mu}} + \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu} - \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu}$$

$$\frac{D V_{\alpha}}{D x^{\beta}} = V_{\alpha\,;\,\beta} = \frac{\partial V_{\alpha}}{\partial x^{\beta}} - \Gamma^{\mu}_{\alpha\beta} V_{\mu}$$

The freedom to choose coordinates means that, on any given point, we can always use the "Einstein elevator" and go to a system where the metric is **locally Minkowski**, and the **connections vanish**:

$$g_{\mu\nu} \to \eta_{\mu\nu} \;,\; \Gamma^{\alpha}_{\mu\nu} \to 0$$



However, if space is **curved**, the **derivatives** of the connections cannot be made to vanish...

$$\partial_{\beta}\, \Gamma^{\alpha}_{\mu\nu} \neq 0 \quad !$$

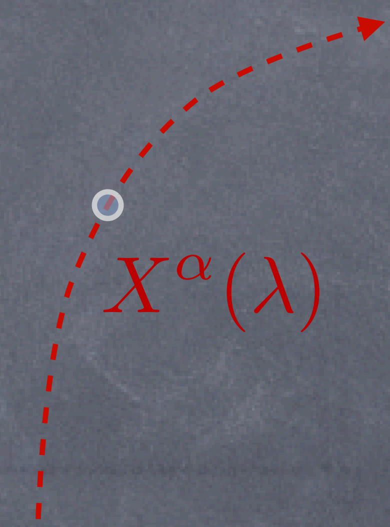
Hence, curvature cannot be "gauged away"

However, suppose we are given a space with some metric. What defines a freely falling ("inertial") observer at any point in that space?

➟ **Acceleration** over paths that go through that point should **vanish**:

$$\frac{D^2 X^\alpha}{D\lambda^2} = 0 \quad \Rightarrow \frac{d^2\, X^\alpha}{d\,\lambda^2} + \Gamma^\alpha_{\mu\nu} \frac{d\, X^\mu}{d\,\lambda} \frac{d\, X^\nu}{d\,\lambda} = 0$$

Geodesic equation



Notice that the geodesic equation determines **both** the **spatial** coordinates **and** the **time** coordinate of the inertial observer.

➟ "Proper time" is the $X^0 = \tau$ along a geodesic!

<u>**Newtonian limit**</u>

Small velocities...

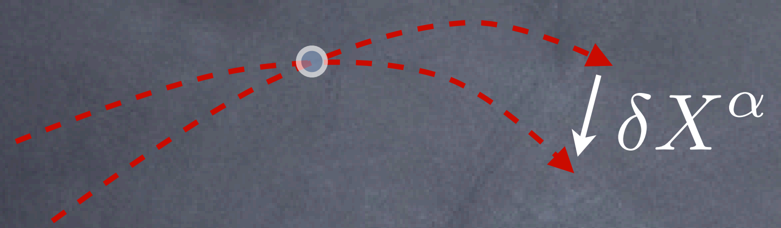
$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \approx \frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{00} \frac{dt}{d\tau} \frac{dt}{d\tau} = 0$$

$g_{00} = 1 - 2\phi$

Static, nearly Minkowski metric...

$$\Gamma^\alpha_{\mu\nu} \to \Gamma^i_{00} = -\frac{1}{2} \nabla^i g_{00} \quad \Rightarrow \begin{cases} \frac{d^2 x^i}{d\tau^2} = \frac{1}{2} \nabla^i g_{00} \left(1 \right)^2 \\ \frac{d^2 x^0}{d\tau^2} = 0 \end{cases}$$

We can also ask how any two geodesics that pass through the same point deviate from each other:



$$\frac{D^2\,\delta X^\alpha}{D\tau^2} = R^\alpha{}_{\beta\mu\nu}\,\delta X^\mu \frac{dX^\beta}{d\tau}\frac{dX^\nu}{d\tau}$$

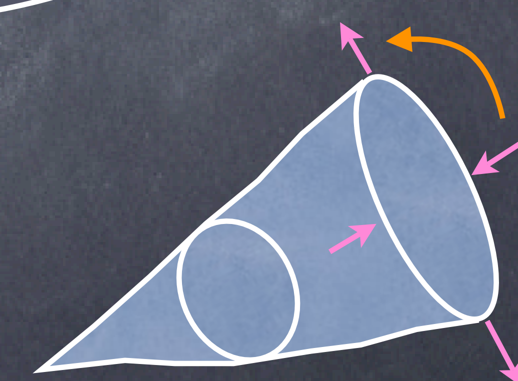
This is the **geodesic deviation** equation. A closely related result is the **Raychaudhuri Equation**. Let $u(\tau)$ be a timelike geodesic, $u^\mu u_\mu = -1$, and its 4-divergence:

$$\theta = D_\mu u^\mu = u^\mu{}_{;\mu}$$

This divergence then obeys the Raychaudhuri equation:

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$$

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 + \omega_{\mu\nu}\omega^{\mu\nu} - \sigma_{\mu\nu}\sigma^{\mu\nu} - R_{\mu\nu}u^\mu u^\nu$$



where the rotation tensor is: $\omega_{\mu\nu} = \frac{1}{2}\left(u_{\mu;\nu} - u_{\nu;\mu}\right)$

and the shear tensor is: $\sigma_{\mu\nu} = \frac{1}{2}\left(u_{\mu;\nu} + u_{\nu;\mu}\right) - \frac{1}{3}\theta(g_{\mu\nu} + u_\mu u_\nu)$

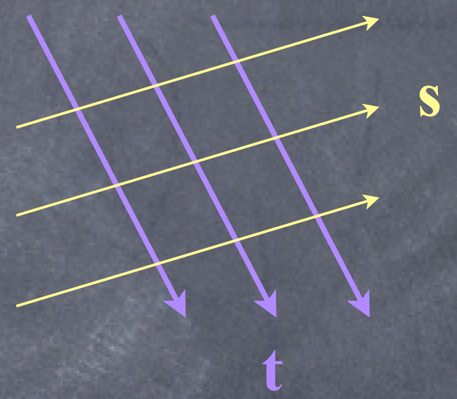
Problem #1

Let u be a geodesic curve. Show that: $u^\mu \, u_{\nu \, ; \, \mu} = 0$

Problem #2

Let $u(t,s)$ be a family of geodesic curves, and $v(t,s)$ the deviation vector for this family,

$$u^\mu = \frac{dX^\mu(t,s)}{dt} \quad , \quad v^\mu = \frac{dX^\mu(t,s)}{ds}$$



which implies that: $[u,v]^\mu = u^\nu v^\mu_{;\nu} - v^\nu u^\mu_{;\nu} = 0$.

Use this to compute the acceleration of v over t:

$$\frac{D^2 v^\mu}{D\, t^2} = u^\alpha D_\alpha (u^\beta D_\beta \, v^\mu)$$

and derive the geodesic deviation equation.

(N.B.: here t is just a parameter, it is **not** x^0 !)

Problem #3

Take a timelike geodesic, $U^\mu U_\mu = -1$, in a spacetime described by the metric $g_{\mu\nu}$.

Show that $h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu$ is the genuine **metric of the spacelike hypersurfaces**, as defined by this geodesic - and that $h_{\mu\nu}$ is a **projection operator** into that subspace.

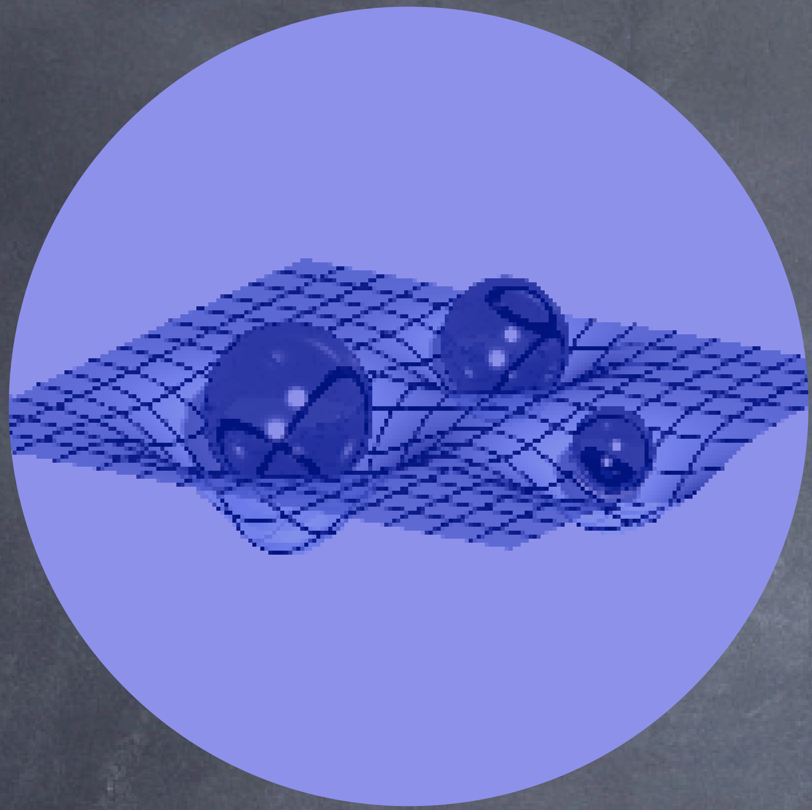
Problem #4

(a) Solve Raychaudhuri's equation for θ, assuming $\omega=\sigma=R=0$. - i.e., flat spacetime.

(b) Show that the family of timelike geodesics $U^\mu=\gamma(v)[1, v]$, with $v = r/t$, yields the solution for θ found in (a)

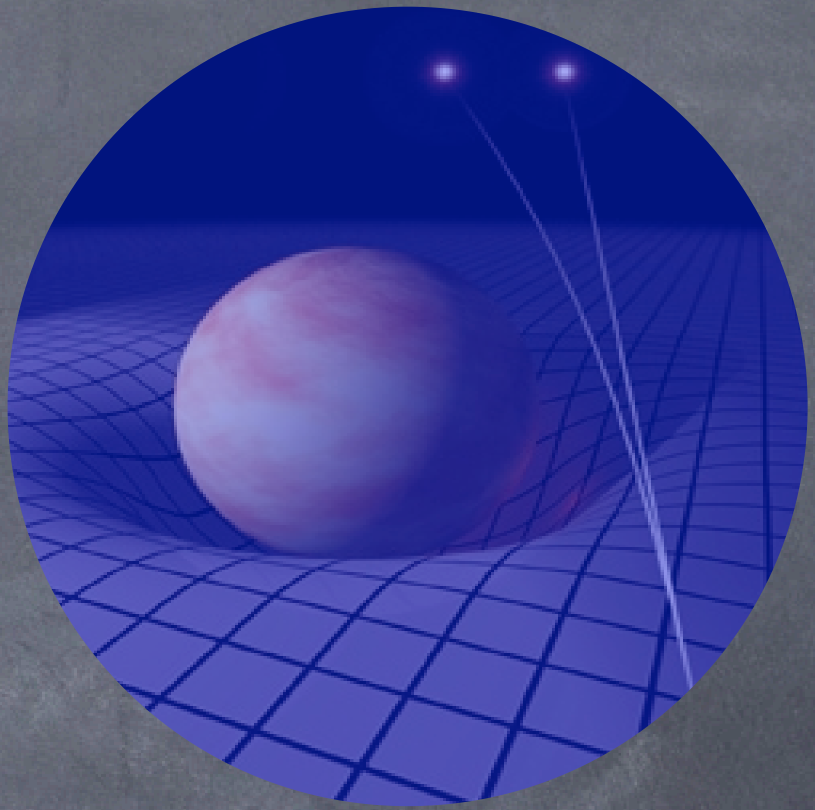
(c) What is the interpretation of θ? Is this well-defined for any r and any t?

Part 2: Einstein's Equations



Matter curves space, determines metric...

... metric determines the kinematics of matter...



Matter and metric jointly determine the **dynamics**

Matter and gravity must be locked into a **self-consistent dynamics**

➟ **Fundamental symmetries imply conservation laws (Noether's theorem)**

Symmetries and conservation laws

Invariance under **time translations/reparametrizations** ➟ **Energy** conservation

Invariance under **spatial translations/reparametrizations** ➟ **Momentum** conservation

Invariance under **spatial rotations** ➟ Angular momentum conservation

But what about boosts (t-x rotations)? They are a symmetry as well...

Moreover, they mix energy and momentum! $P^\mu = mU^\mu \quad , \quad P'^\mu = \frac{\partial x'^\mu}{\partial x^\alpha} P^\alpha$

Energy conservation for classical point particle (non-relativ.):

$$t \to t + \delta t$$
$$q \to q + \dot q\, \delta t$$
$$\dot q \to \dot q + \ddot q \delta t + \dot q \delta \dot t$$

$$S = \int dt\, L(q, \dot q) \quad , \quad L(q, \dot q) = \frac{1}{2}\dot q^2 - V(q)$$

$$\delta S = \int dt \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\delta \dot q\right] = \int dt \left[\dot q(\ddot q\, \delta t + \dot q\, \delta \dot t) - \frac{dV}{dq}\dot q\, \delta t\right]$$

$$\delta S_i^f = [\cancel{\dot q^2 \delta t}]_i^f - \int_i^f dt\, \frac{d}{dt}\left[\frac{1}{2}\dot q^2 + V(q)\right] \delta t = 0$$

Energy = conserved "charge"

$t \to t + \delta t$: "global" symmetry

Gauge (local) symmetry of covariant theories: coordinate transformations

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu$$

Under a coordinate transformation, the **metric changes by:**

$$g^{\mu\nu} \to g'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta} g^{\alpha\beta} = g^{\mu\nu} + \underbrace{\xi^{\mu;\nu} + \xi^{\nu;\mu}}_{\delta g^{\mu\nu}} + \mathcal{O}(\xi^2)$$

In particular, if the **metric is invariant** under such a transformation, then ξ is a **Killing vector field.**

Problem #5

Take the 3-dimensional space with constant curvature (GLB) in spherical coordinates:

$$d\Sigma_3^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2$$

How many Killing vector fields does this space allow? What's their meaning?

The **dynamics** of matter should be independent of the coordinate system, and therefore the **matter action** should remain **invariant** under a coordinate transformation:

$$V^{\mu}_{\ ;\mu} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,V^\mu\right)$$

$$\Rightarrow \int_{\mathcal{V}} d^4x\,\sqrt{-g}\,V^{\mu}_{\ ;\mu} = \left(\sqrt{-g}\,V^\mu N_\mu\right)_{N_\mu:\mathcal{S}(\mathcal{V})}$$

$$S_m = \int d^4x\,\sqrt{-g}\,\mathcal{L}_m$$

Why is it safe to assume that this vanishes??...

$$\delta_\xi S_m = \int d^4x\left[\frac{\partial\sqrt{-g}\mathcal{L}_m}{\partial g^{\mu\nu}}\,\delta g^{\mu\nu} + \frac{\partial\sqrt{-g}\,\mathcal{L}_m}{\partial(\partial_\alpha g^{\mu\nu})}\,\partial_\alpha\delta g^{\mu\nu}\right]$$

$$\delta_\xi S_m = \int d^4x\,\frac{\sqrt{-g}}{2}\,T_{\mu\nu}(\xi^{\mu;\nu}+\xi^{\nu;\mu}) = \int d^4x\,\sqrt{-g}\,T^{\mu\nu}\xi_{\mu;\nu}$$

But since: $T^{\mu\nu}\xi_{\mu;\nu} = (T^{\mu\nu}\xi_\mu)_{;\nu} - T^{\mu\nu}_{\ \ ;\nu}\xi_\mu = \frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}\,T^{\mu\nu}\xi_\mu\right) - T^{\mu\nu}_{\ \ ;\nu}\xi_\mu$

$$\delta_\xi S_m = \int d^4x\left[\partial_\nu\left(\sqrt{-g}\,T^{\mu\nu}\xi_\mu\right) - T^{\mu\nu}_{\ \ ;\nu}\xi_\mu\right] \to 0$$

Therefore, we get the **conservation law**: $T^{\mu\nu}_{\ \ ;\nu} = 0$

$T_{\mu\nu}$:

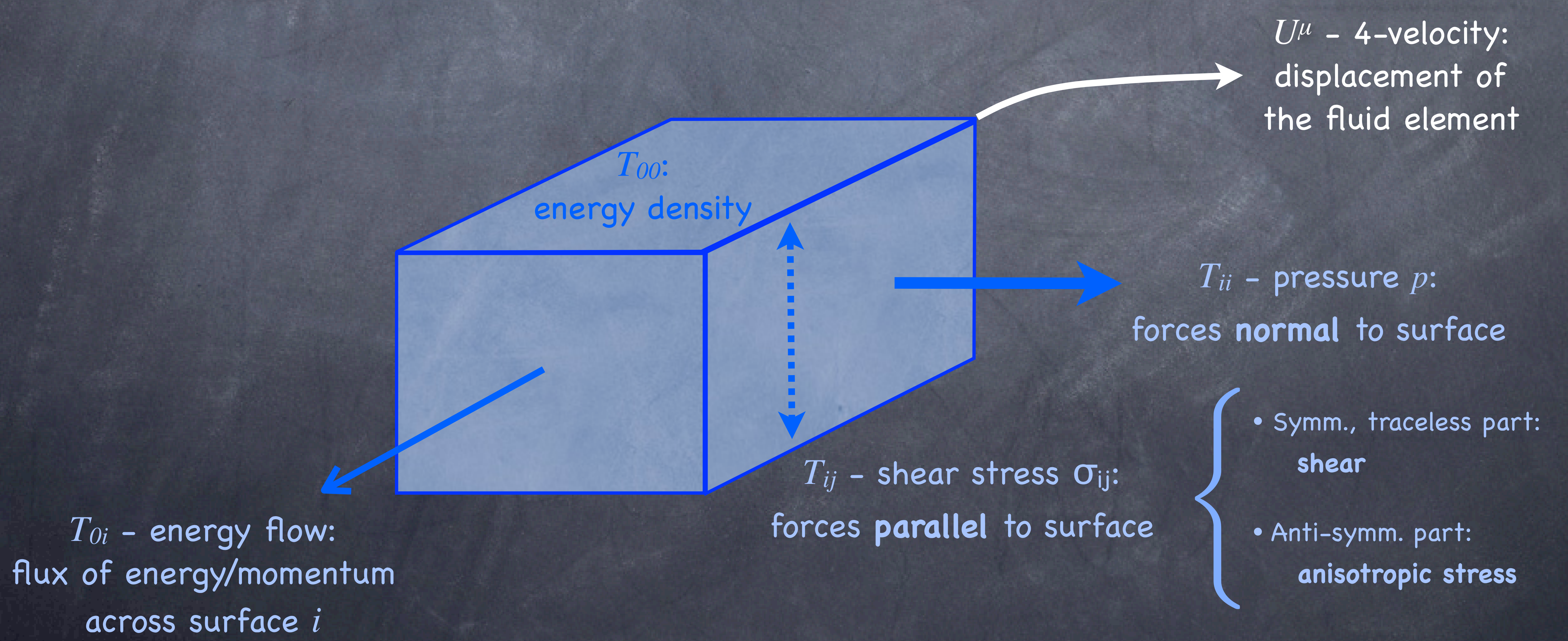
- Energy
- Momentum
- Stresses/energy flows

The energy-momentum tensor (or stress-energy tensor)

In general, it turns out to be more instructive to construct the EMT from first principles.

For a continuous media, the relevant quantities are: the **4-velocity**, the **energy density**, the **isotropic pressure**, and the **shear stress**.

Consider a fluid element:



In **Minkowsky spacetime** a fluid at rest, without any stresses, is given completely in terms of its energy density and pressure:

$$T^{00} = \rho \qquad T^{0i} = 0$$

$$T^{ij} = p\,\delta^{ij} \qquad U^\mu = (1,0,0,0)$$

Or, in terms of the 4-velocity: $T^{\mu\nu} = (\rho + p)U^\mu U^\nu + p\,\eta^{\mu\nu}$

A fluid in motion is still given by the same expression, if we replace the 4-velocity by:

$$U^\mu \to \gamma(v)(1,\vec{v})$$

We then get that:

$$T^{00} = \gamma^2(\rho + p) - p = \frac{\rho + p\,v^2}{1 - v^2}$$

$$T^{0i} = \frac{\rho + p}{1 - v^2}\,v^i \qquad \Rightarrow \partial_\mu T^{\mu 0} = \frac{\partial}{\partial t}\frac{\rho + p\,v^2}{1 - v^2} + \frac{\partial}{\partial x^i}\frac{(\rho + p)\,v^i}{1 - v^2}$$

$$T^{ij} = \frac{\rho + p}{1 - v^2}\,v^i\,v^j + p\delta^{ij}$$

The non-relativistic limit [i.e., neglect O(v²) terms], the conservation of the stress-energy tensor is the so-called **continuity equation**:

$$\partial_\mu T^{\mu 0} = \frac{\partial}{\partial t}\frac{\rho + p v^2}{1 - v^2} + \frac{\partial}{\partial x^i}\frac{(\rho+p)\, v^i}{1-v^2}$$

$$\simeq \dot\rho + \vec\nabla[(\rho+p)\,\vec v] \simeq \dot\rho + (\rho+p)\,\vec\nabla\,\vec v$$

Why is it OK to neglect $\vec\nabla(\rho+p)\cdot\vec v$?

But this is simply the well-known thermodynamic equation for **energy conservation:**

$$dE + p\,dV = 0$$

$$\Rightarrow \frac{1}{V}\frac{d(\rho V)}{dt} + p\,\frac{1}{V}\frac{dV}{dt} = \frac{d\rho}{dt} + (\rho+p)\,\frac{1}{V}\frac{dV}{dt} = 0$$

where the volume changes according to the divergence of the velocity, $\frac{1}{V}\frac{dV}{dt} = \vec\nabla\cdot\vec v$

Conservation of the stress-energy tensor:

- **Energy conservation**
- **Euler equation**

Later!

Problem #6

(a) A scalar field has the Lagrangean:

$$L = \sqrt{-g}\left[\frac{1}{2}\, g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi \,-\, V(\phi)\right]$$

Find its stress-energy tensor. Hint: $\delta(\sqrt{-g}) = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu}$

(b) Find the Klein-Gordon equation as a result of stress-energy conservation.

(c) Find the Klein-Gordon equation from the variational principle in terms of the scalar field itself.

(d) Can you write this stress-energy tensor in terms of that for a "fluid",

$$T_{\mu\nu} = (\rho + p)\, U_\mu\, U_\nu + p\, g_{\mu\nu} \quad ?$$

Problem #7

(a) Starting from the Lagrangean for EM:

$$L_{EM} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \qquad , \qquad F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$$

find the stress-energy tensor for the EM field.

(b) Show that this stress-energy tensor is conserved because of Maxwell's vacuum equations in curved spacetime.

(c) Show that, in 4D, vacuum EM is "conformally invariant", i.e., the solutions of Maxwell's equations in vacuum are invariant under the transformation:

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$

Einstein's equations, finally!

The metric counterpart of the stress-energy tensor must be conserved as well.

In fact, the Einstein-Hilbert action satisfies this constraint, and we have that:

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} + L_m \right] \quad \longrightarrow \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}$$

Because of the **Cosmological Principle**, to first approximation the left- and the right-hand sides of this equation must be **functions of time only**.

The only free parameters are **spacetime constants**:

Metric:

Spatial curvature
Cosmological constant

Matter:

Masses
Coupling constants
(and, e.g., ratios of densities)

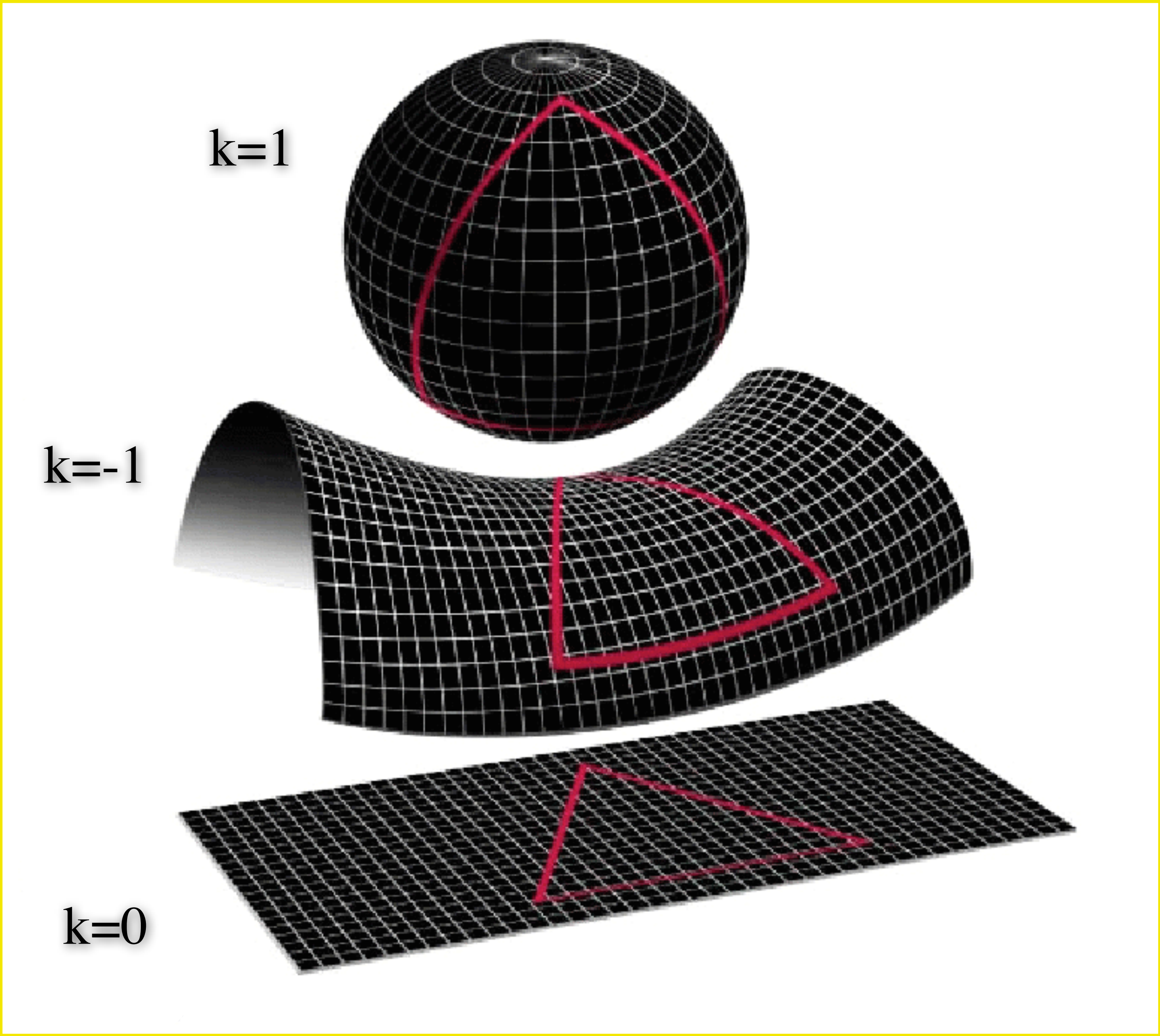
Part 3: Kinematics of FLRW spacetimes

A. Friedmann (1922-24), G. Lemaitre (1927), H. P. Robertson (1935-36), A. G. Walker(1937)

Spatial sections of constant curvature:

$$ds^2 = -dt^2 + a^2(t)\, d\Sigma^2$$

Def.: $a(t_0)=1$



$$d\Sigma^2 = \frac{dr^2}{1-r^2} + r^2 d\Omega^2$$

$(r/R_0)^2$

$$d\Sigma^2 = \frac{dr^2}{1+r^2} + r^2 d\Omega^2$$

$(r/R_0)^2$

$$d\Sigma^2 = dr^2 + r^2 d\Omega^2$$

Some other popular coordinates used to express FLRW spatial sections:

Polar coordinates:

$$d\Sigma^2 = \frac{dr^2}{1-kr^2} + r^2\, d\Omega^2$$

$k = \pm (R_0)^{-2}$

Hyperspherical coordinates:

$$r = \frac{1}{\sqrt{k}} \sin\left(\sqrt{k}\,\chi\right) \quad \Rightarrow \quad d\Sigma^2 = d\chi^2 + \frac{1}{k} \sin^2\left(\sqrt{k}\,\chi\right) d\Omega^2$$

Conformal-Cartesian coordinates:

$$r = \frac{R}{1+\frac{k}{4}R^2} \quad \Rightarrow \quad d\Sigma^2 = \frac{dR^2 + R^2 d\Omega^2}{(1+\frac{k}{4}R^2)^2}$$

Spatial sections are homogeneous, isotropic

The most common choice is the second one, since in hyperspherical coordinates the radial geodesics are trivial.

Problem #8

Show that the 3D metric of FLRW spatial sections in conformal-Cartesian coordinates,

$$d\Sigma^2 = \gamma_{ij}\, dx^i\, dx^j \quad , \quad \gamma_{ij} = \delta_{ij} \left(1 + \frac{1}{4}\, k\, \vec{x}^{\,2} \right)^{-2}$$

has a Riemann tensor given by:

$$R_{ijkl} = k\, (\gamma_{il}\gamma_{jk} - \gamma_{ik}\gamma_{jl})$$

Problem #9

The 3D volume can be defined as: $V = \int d^3x\, \sqrt{\det \gamma}$

What is the volume of a 3-sphere? Show that it reduces to the usual result when the spatial curvature is small.

Geodesics in FLRW spacetime

Let's take the FLRW metric in conformal-Cartesian coordinates:

$$ds^2 = -dt^2 + a^2(t)\, d\Sigma^2$$

$$d\Sigma^2 = \gamma_{ij}\, dx^i\, dx^j \quad , \quad \gamma_{ij} = \delta_{ij} \left(1 + \frac{1}{4} k\, \vec{x}^{\,2}\right)^{-2}$$

The connections are:

$$\Gamma^0_{00} = \Gamma^i_{00} = \Gamma^0_{i0} = 0$$

$$\Gamma^0_{ij} = \dot{a}\, a\, \gamma_{ij} \quad , \quad \Gamma^i_{0j} = \frac{\dot{a}}{a}\, \delta^i_j$$

$$\Gamma^k_{ij} = \frac{k}{2} \left(\frac{1}{1 + \frac{k}{4}\vec{x}^{\,2}} \right) \times \left(\delta_{ij} x^k - \delta_{ik} x^j - \delta_{jk} x^i \right)$$

Problem #10

Compute the Riemann tensor of FLRW in this coordinate system. How many independent components does it have?

A particle initally at rest in these coordinates has the 4-velocity:

$$U_0^\mu = \left.\frac{dx^\mu}{d\tau}\right|_0 = (1,0,0,0)$$

The geodesic equation is: $$\frac{dU^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} U^\alpha U^\beta = 0$$

which means the following set of equations:

$$\mu = 0 \quad \rightarrow \quad \frac{dU^0}{d\tau} + \Gamma^0_{ij} U^i U^j = 0$$

$$\mu = k \quad \rightarrow \quad \frac{dU^k}{d\tau} + 2\,\Gamma^k_{0i} U^0 U^i + \Gamma^k_{ij} U^i U^j = 0$$

The solution to this set of equations is: $U^\mu = U_0^\mu = (1,0,0,0)$

Hence, a particle at rest in any point in this spacetime will remain in the same position! This is the kinematic meaning of "homogeneity and isotropy"!

Problem #11

Consider a particle moving with some initial "peculiar" velocity v_0 :

$$U_0^\mu = \gamma(v_0)(1, \vec{v}_0)$$

Find the solution to the geodesic equation in this case. Make approximations if absolutely necessary. What happens with the peculiar velocity as a function of time?

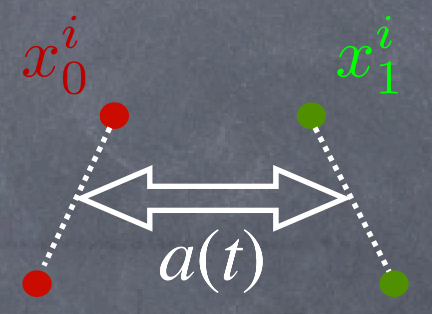
FLRW and expansion

Consider **two particles** at **rest** on **different spatial locations**.

The (spacelike) **distance** between them is given, at a **t=constant hypersurface**, by:

$$\Delta s^2 = \Delta l^2(t) = a^2(t)(\vec{x}_0 - \vec{x}_1)^2 = a^2(t)\,\Delta\vec{x}^{\,2}$$

constant!



The **speed** with which these particles "at rest" are moving is given by:

$$v = \frac{d}{dt}\Delta l = \frac{\dot{a}}{a}\Delta l \qquad \frac{\dot{a}}{a} \equiv H$$

Hubble parameter

Consider now a light ray propagating in the radial direction - and let's use the **hyperspherical coordinates**. We have:

$$ds_0^2 = -dt^2 + a^2(t)\,d\chi^2 = 0$$

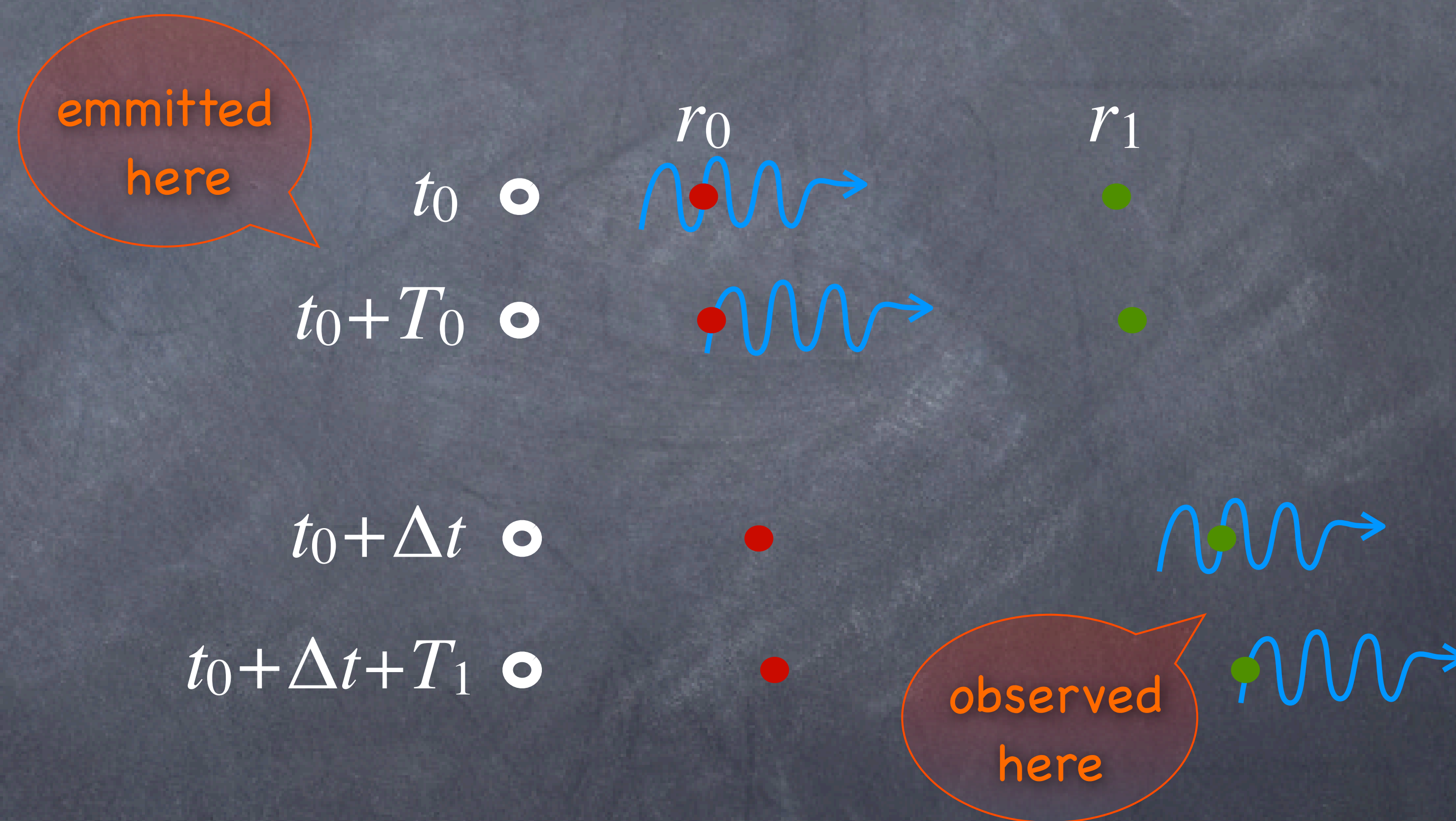
$$\Rightarrow \chi = \int_{t_0}^{t_0+\Delta t} \frac{dt}{a(t)}$$

Cosmological redshift

Suppose we have a light source emmitting radiation with a frequency ν_0 at the radial position r_0, and at the instant t_0. A time T_0 later, t_0+T_0 , the light source will be emmitting radiation at the same phase (+2π) as in t_0 .

The light ray which was emmitted at t_0 is then observed at a position r_1 .

$$\chi = \int_{t_0}^{t_1} \frac{dt}{a(t)} = \int_{t_0+T_0}^{t_1+T_1} \frac{dt}{a(t)}$$



$$\Rightarrow \int_{t_0}^{t_0+T_0} \frac{dt}{a(t)} = \int_{t_1}^{t_1+T_1} \frac{dt}{a(t)} \qquad \Rightarrow \frac{T_0}{a(t_0)} \simeq \frac{T_1}{a(t_1)} \qquad \Rightarrow \frac{\nu_0}{\nu_1} = \frac{a(t_1)}{a(t_0)}$$

In terms of the **wavelength** of the light, we have:

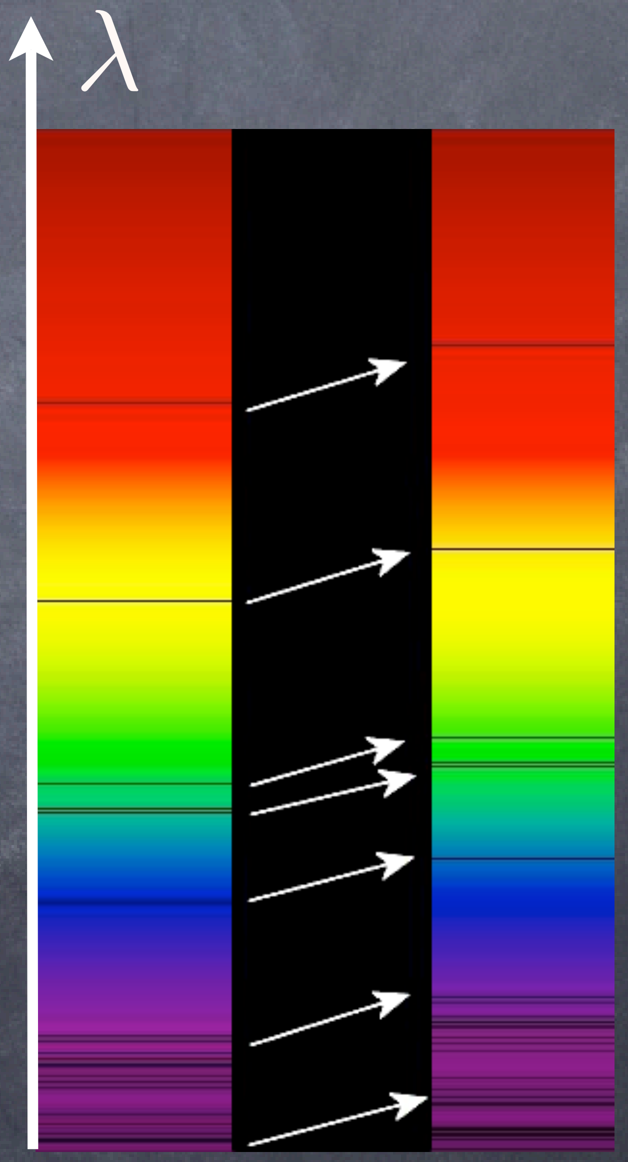
$$\frac{\lambda_{\rm obs}}{\lambda_{\rm emm}} = \frac{a(t_{\rm obs})}{a(t_{\rm emm})}$$

The "**redshift**" (or "blueshift") is defined as:

$$1+z = \frac{\lambda_{\rm obs}}{\lambda_{\rm emm}} = \frac{\nu_{\rm emm}}{\nu_{\rm obs}}$$

$$\Rightarrow \quad z = \frac{\lambda_{\rm obs} - \lambda_{\rm emm}}{\lambda_{\rm emm}} = \frac{\nu_{\rm emm} - \nu_{\rm obs}}{\nu_{\rm obs}}$$

$z = \frac{\Delta\lambda}{\lambda}$ ambiguous!



Absorption lines at the Sun

Same lines at a distant galaxy

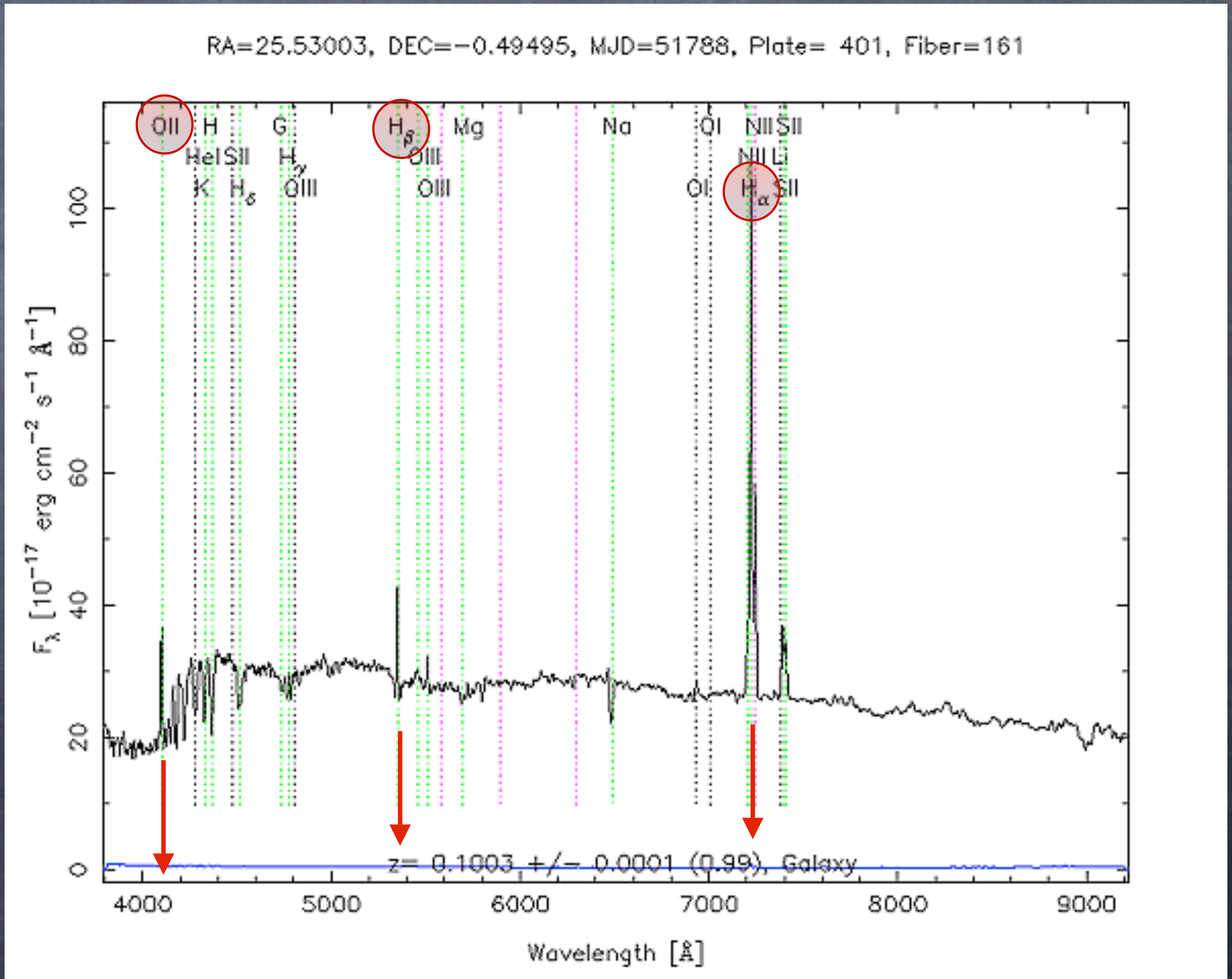
Any emmission or absorption line can be used to compute the redshift!

Typically, we observe **here** on Earth ($r=\chi=0$, $t=0$) the light emmitted by **distant galaxies** at time t.

Since by convention the scale factor today is $a_0=a(t_0)=1$, we have that the **redshift** of those distant galaxies is given by:

$$z = \frac{\lambda_{\rm obs} - \lambda}{\lambda} = \frac{a_0}{a(t)} - 1 = \frac{1}{a(t)} - 1$$

Example: SDSS galaxy at z=0.1003



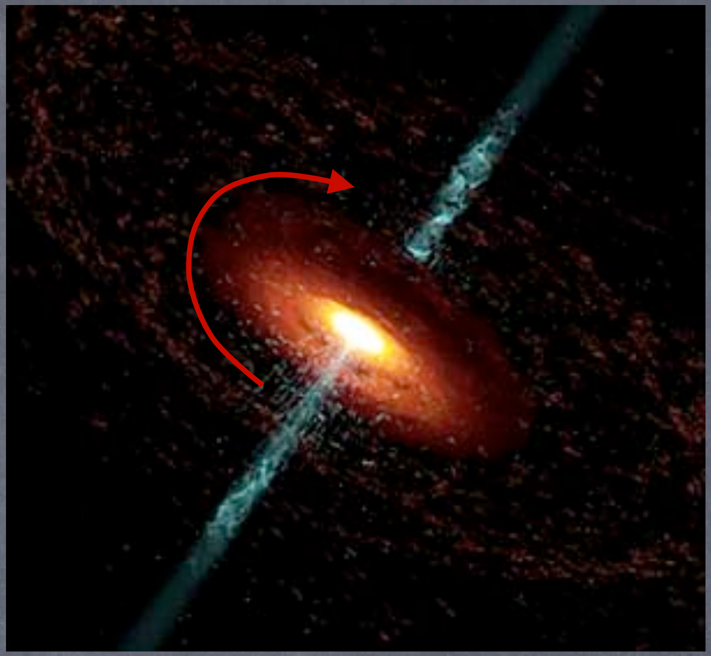
At rest, some of these lines are:

$$\mathrm{H}_\alpha : 6563\ \text{Å}$$

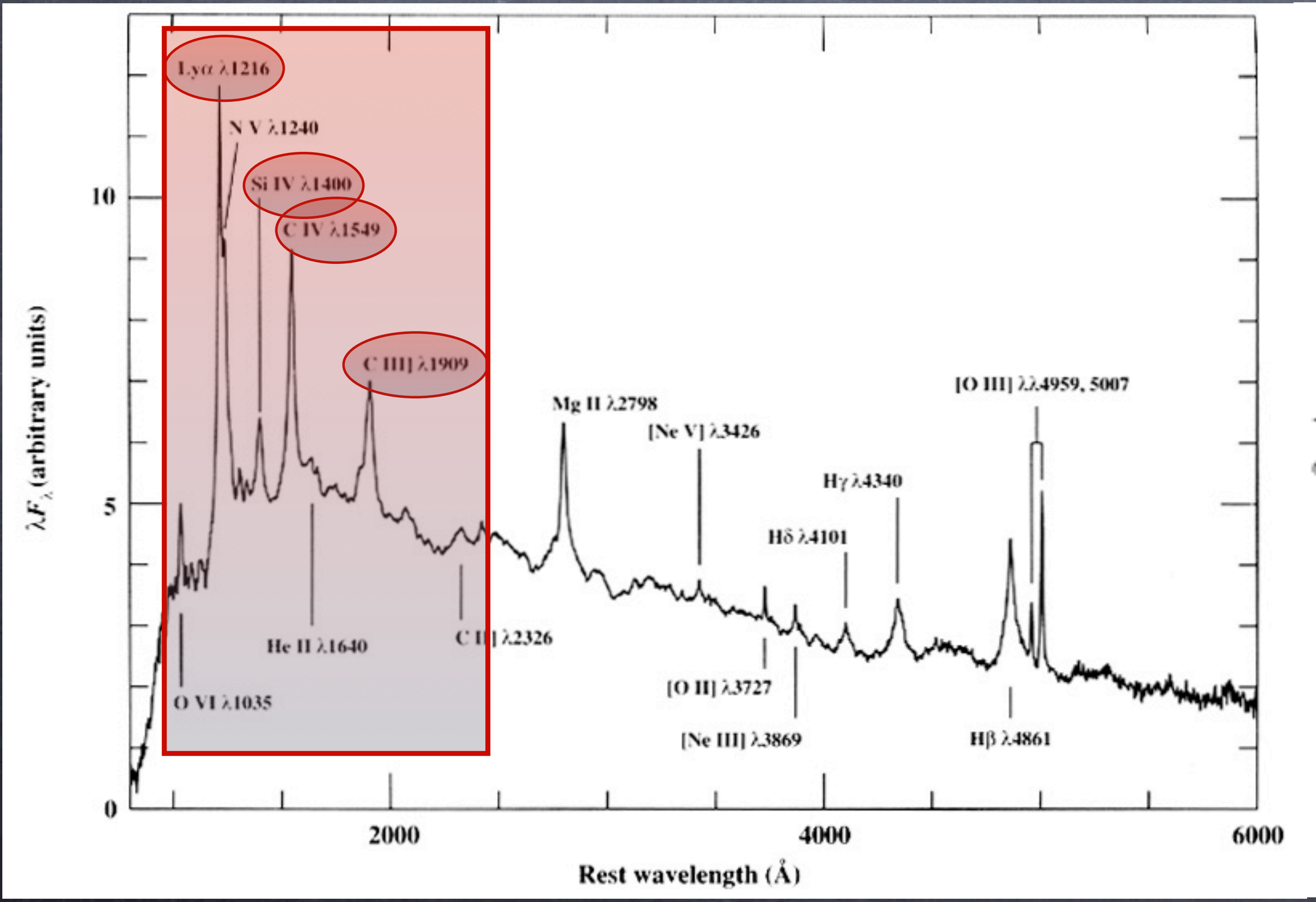
$$\mathrm{H}_\beta : 4861\ \text{Å}$$

$$\mathrm{OII} : 3727\ \text{Å}$$

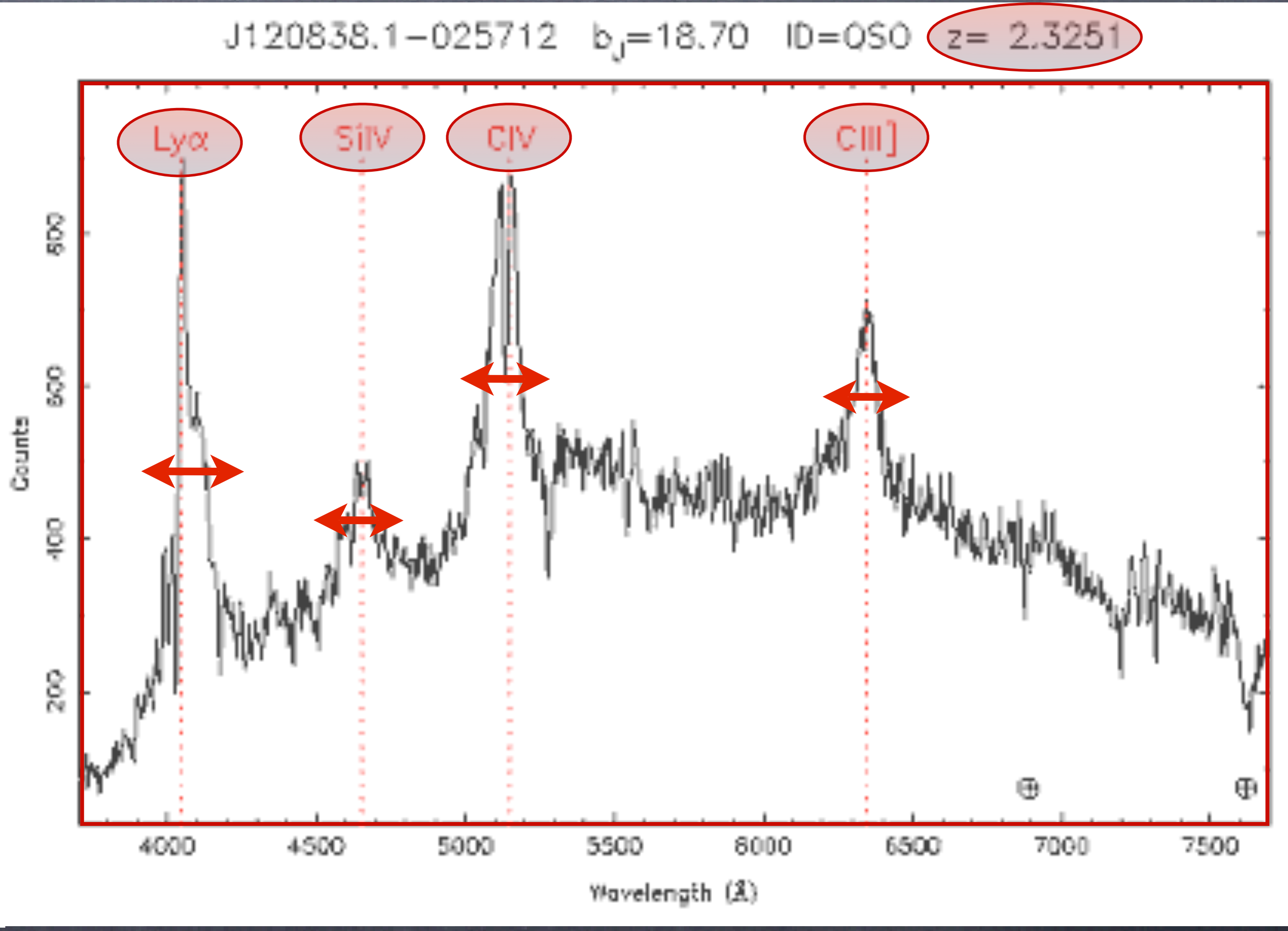
Another example: quasar ("quasi-stellar object")



Model (average quasar, restframe)



Actual quasar (2dF survey)



Cosmological redshift and Doppler effect are manifested in the same way!

Alternative derivation of the redshift

Since light is a null geodesics, we have: $l^\mu = (E = h\nu \, , \vec{p})$ $\qquad E = \frac{dT}{d\tau} \quad , \quad \vec{p} = \frac{d\vec{X}}{d\tau}$

with: $l^\mu l_\mu = g_{\mu\nu} \, l^\mu \, l^\nu = 0 \qquad \Rightarrow E = \frac{a\,p}{1 + \frac{k}{4} X^2}$

Let's use the conformal-Cartesian coordinates, and assume that the light ray is propagating in the direction X:

$$\frac{dl^0}{d\tau} + \Gamma^0_{ij} \, l^i l^j = 0 \qquad \Rightarrow \quad \frac{dE}{d\tau} + \frac{da}{dT} a \frac{1}{1 + \frac{k}{4} X^2} p^2 = 0$$

$$\Rightarrow \quad \frac{dE}{d\tau} + \frac{da}{d\tau}\frac{d\tau}{dT} a \frac{1}{1 + \frac{k}{4} X^2} p^2 = 0$$

$$\Rightarrow \quad \frac{dE}{d\tau} + \frac{da}{d\tau}\frac{1}{E}\frac{1}{a} E^2 = 0$$

$$\Rightarrow \quad \frac{dE}{da} + \frac{E}{a} = 0 \qquad \Rightarrow \quad E \sim \frac{1}{a}$$

$$\nu \sim \frac{1}{a}$$

$$\lambda \sim a$$

Problem #12

A quasar has a broad MgII emmission line at 5310 A. The observed width (FWHM - full width at half maximum) of that line is 70 A.

Given that, at rest, the MgII line lies at 2798 A, and its width is negligible, find:

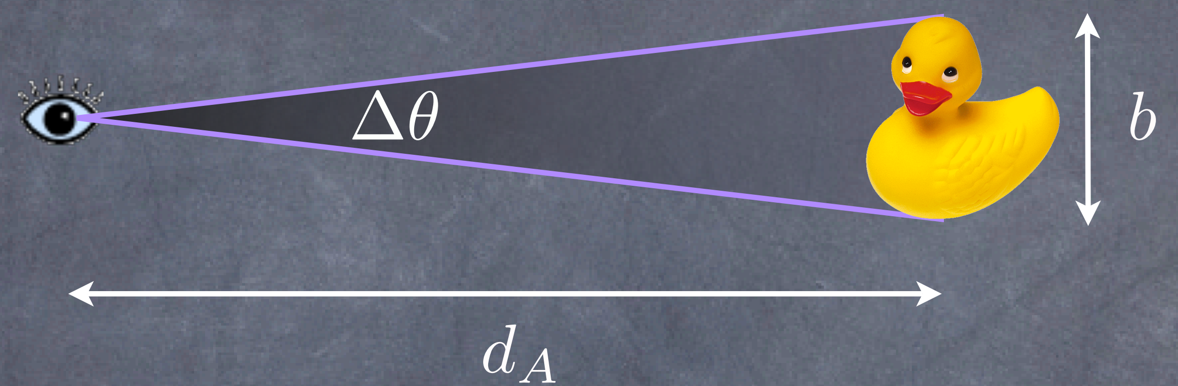
(a) The quasar's cosmological redshift.

(b) The averaged rotation velocity of the accretion disk near the region that emmits the MgII line.

Cosmological distance measurements

Angular distance (aka: angular diameter distance)

We can employ an object of known **transversal length** to estimate the distance to it from its angular size:



$$d_A = \frac{b}{\Delta\theta}$$

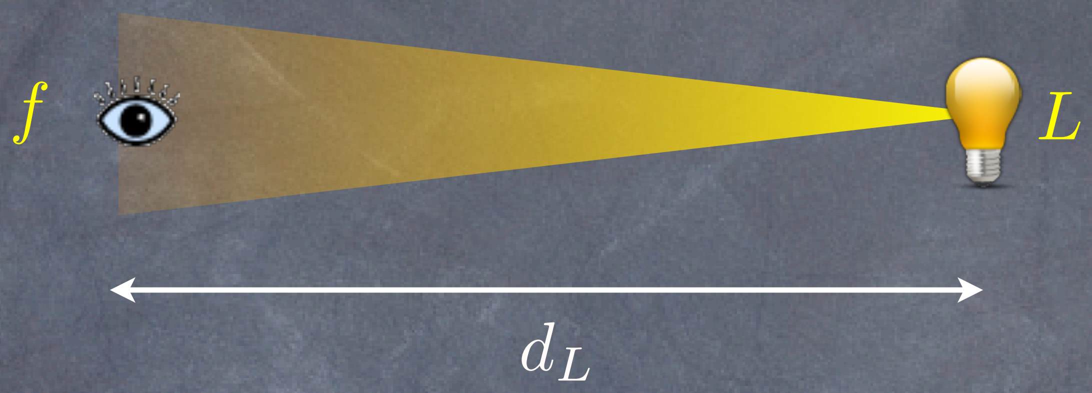
In FLRW these two distances are related. In polar and hyperspherical coordinates:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \qquad a^2(t)\, r^2 d\Omega^2 = b^2 \qquad d_A = a\, r = \frac{r}{1+z}$$

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + \frac{1}{k}\sin^2\left(\sqrt{k}\,\chi\right) d\Omega^2\right] \qquad a^2(t)\,\frac{1}{k}\sin^2\left(\sqrt{k}\,\chi\right) d\Omega^2 = b^2 \qquad d_A = \frac{1}{1+z}\frac{1}{\sqrt{k}}\sin\left(\sqrt{k}\,\chi\right)$$

Luminosity distance

We can employ an object of known **luminosity** to estimate the distance.



$$f = \frac{L}{4\pi\, d_L^2}$$

In FLRW the photons from the light source suffer two nontrivial effects:

1. Their energy falls (redshifts) as $E \sim 1/a$, and
2. The frequency with which they arrive is also "redshifted" by a factor of $1/a$.

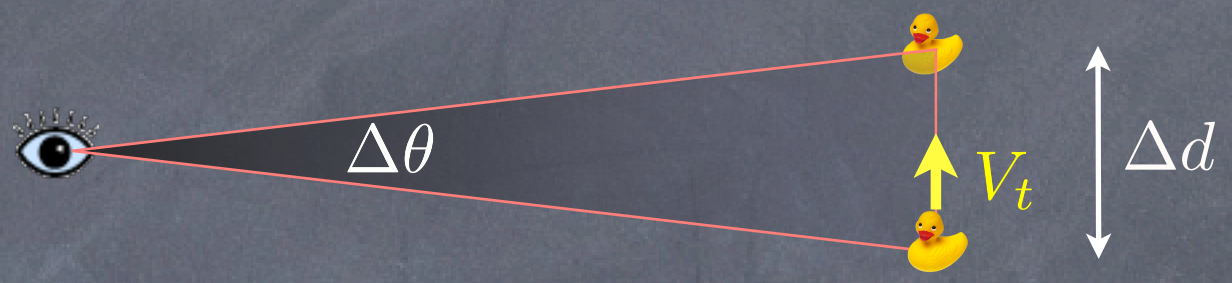
Hence, the observed flux (power) is given by:

$$f_0 = \left(\frac{a_e}{a_0}\right)^2 \frac{L}{4\pi\, r^2} \quad \Rightarrow \quad d_L = \frac{r}{a_e} = (1+z)\, r$$

Proper motion distance

If a source is moving with speed V_t, transverse to the line of sight, then over some time interval Δt it will move a distance Δd:

$$\Delta d = V_t\,\Delta t = V_t\,\frac{a(t)}{a_0}\,\Delta t_0$$



The angular distance that this source will travel in that time is given by the same reasoning used in the angular-diameter distance:

$$\Delta\theta = \frac{\Delta d}{r/(1+z)} = \frac{V_t\,\Delta t_0/(1+z)}{r/(1+z)} = \frac{V_t\,\Delta t_0}{r}$$

This relation allow us to define the **proper motion distance** as:

$$d_M = \frac{V_t}{\Delta\theta/\Delta t_0} \quad (=r)$$

To use this, we have to **know** the velocity!

Distance "duality" relations

From their definitions, we have that, for instance (Etherington, 1933):

$$\frac{d_L}{d_A} = (1+z)^2$$

What are the conditions under which this is true? (Bassett & Kunz 2004)

(a) photon number is conserved ("transparency")

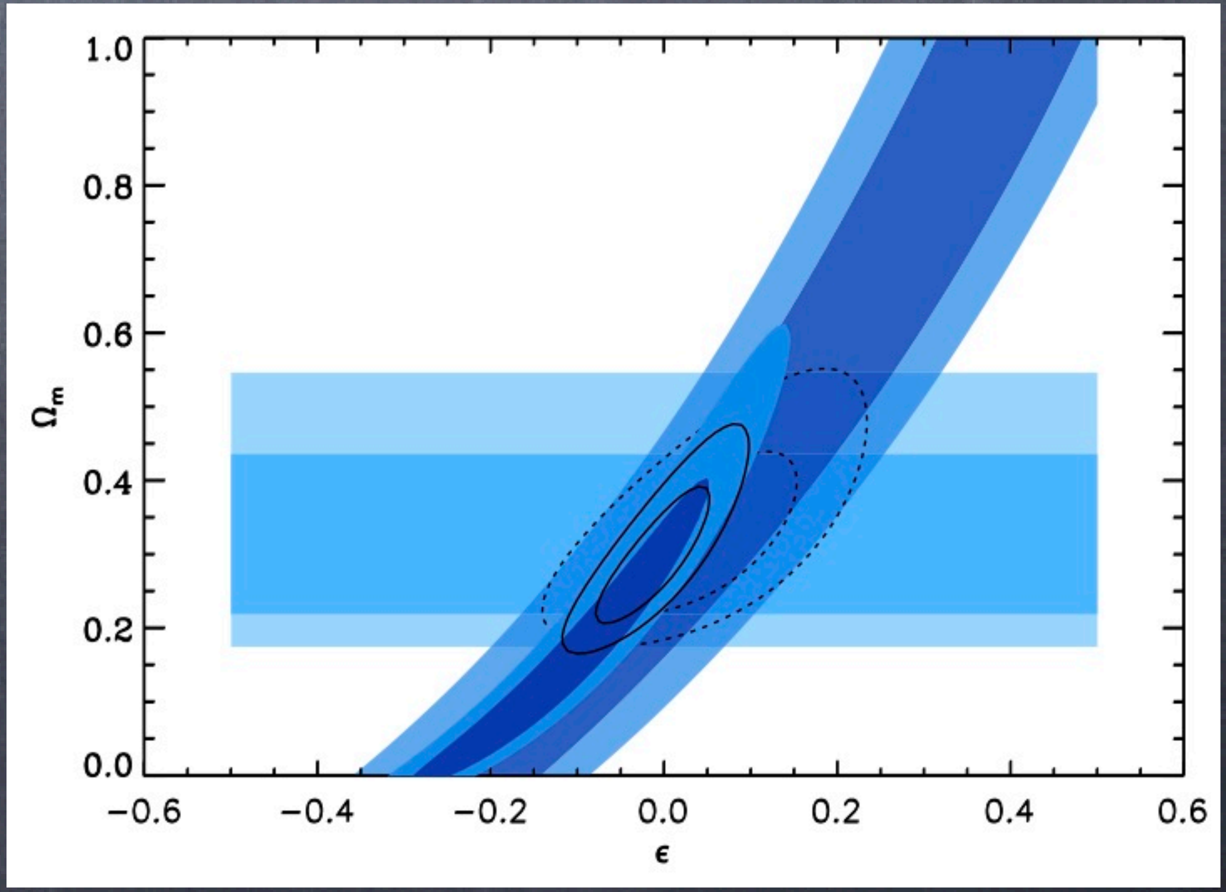
(b) metric theory of gravity

(c) photons travel on unique null geodesics

Check?...

(e.g., Avgoustidis, Jimenez & Verde, 2009)

$$\frac{d_L}{d_A} = (1+z)^{2+\epsilon}$$



Distance "consistency" relations

(Clarkson, Bassett & Hui-Ching Lu, PRL 2008)

Suppose we can measure independently $H(z)$ and, e.g., $d_L(z)$. Then:

$$d_L(z) = (1+z)\, r = \frac{(1+z)}{\sqrt{k}} \sin\left(\sqrt{k}\chi\right) = \frac{(1+z)}{\sqrt{k}} \sin\left[\sqrt{k} \int \frac{dt}{a(t)}\right]$$

$$= \frac{(1+z)}{\sqrt{k}} \sin\left[\sqrt{k} \int \frac{da}{a^2\, H}\right]$$

$$a = \frac{1}{1+z}$$

$$= -\frac{(1+z)}{\sqrt{k}} \sin\left[\sqrt{k} \int_z^0 \frac{dz'}{H(z')}\right]$$

$$da = -\frac{dz}{(1+z)^2}$$

$$= \frac{(1+z)}{\sqrt{k}} \sin\left[\sqrt{k} \int_0^z \frac{dz'}{H(z')}\right]$$

Therefore, it should be true that:

$$k = \frac{1 - [H(z)\chi'(z)]^2}{\chi^2(z)}$$

The derivative should be zero!

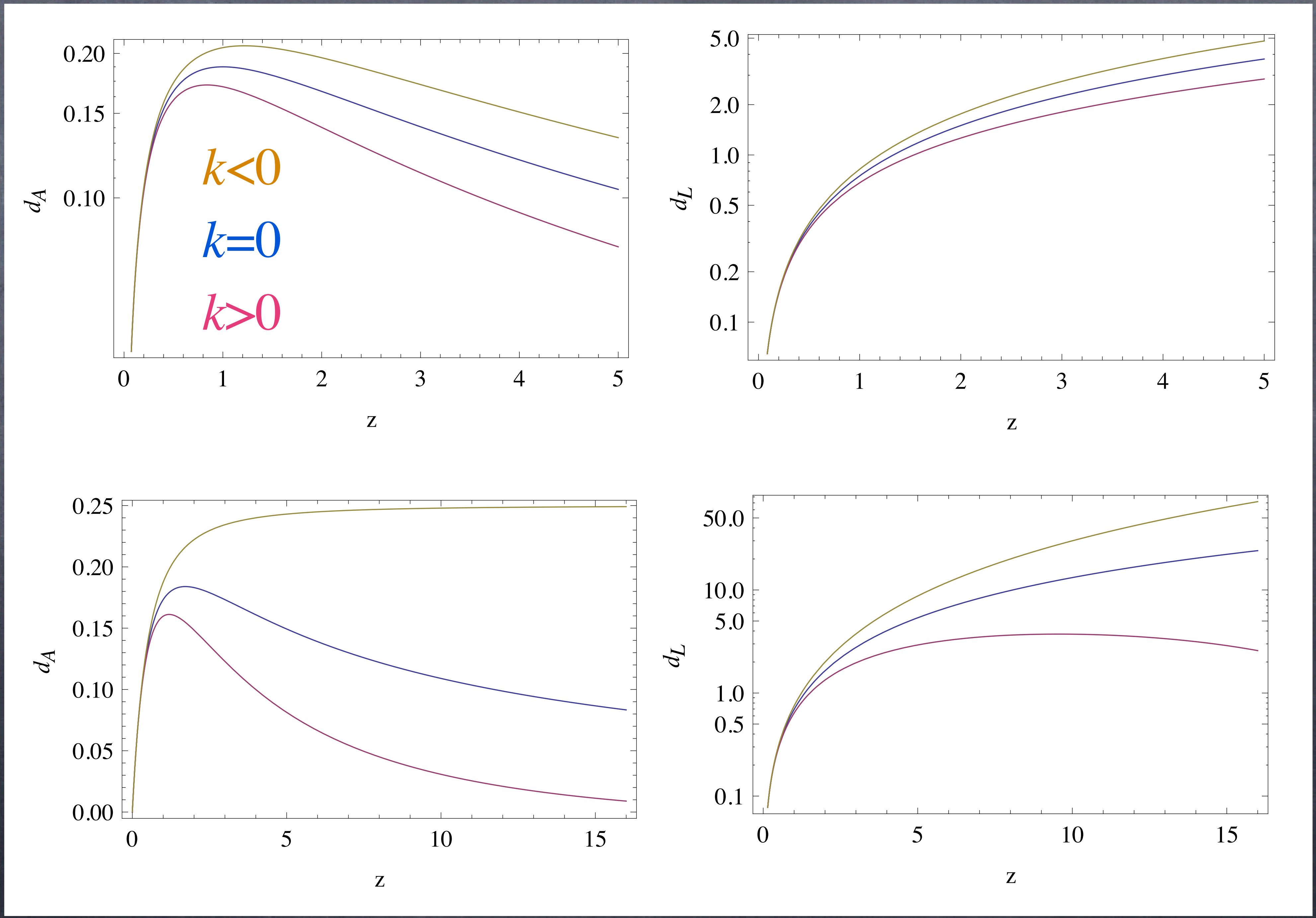
$$\mathcal{C}(z) = 1 + H^2(\chi\chi'' - \chi'^2) + HH'\chi\chi' \stackrel{?}{=} 0$$

Angular and Luminosity distances: examples

Using the null radial geodesic, $\chi = \int \frac{dt}{a(t)}$

$a \sim t^{2/3}$

$a \sim t^2$



Problem #13

Consider a universe where the scale factor is $a=(t/t_0)^2$. Take $t_0=1$ for simplicity.

(a) A light ray travels from an initial position χ_e at t_e, and arrives at the origin at t_0. Do all light rays arrive at the origin, independently of χ_e and of t_e ?

(b) Compute $\chi_e(t)$ in this model.

(c) Compute $d_L(z)$ in this model.

(d) Compute $d_A(z)$ in this model.

(e) Make (qualitative) plots for these three quantities.

(f) A light ray observed today was emmitted at $z=1000$. At what time (relative to t_0) was it emmitted? And what was the comoving radius at the place where it was emmitted?

(g) A spherical galaxy at $z=1$ has proper radius R. What angle in the sky does it subtend, when the spatial section has curvature k=+1, -1 and 0?

Cosmography

Let's write the comoving radius as:

Invert limits of integration

$$\chi = \int \frac{dt}{a(t)} = \int \frac{dt}{da}\frac{da}{a} = \int \frac{da}{a^2 H} = \int \frac{dz}{H(z)}$$

$$a = \frac{1}{1+z}$$

$$da = -\frac{dz}{(1+z)^2}$$

$$a(t) = 1 + H_0(t-t_0) - \frac{1}{2}q_0 H_0^2 (t-t_0)^2 + \dots$$

$$H(t) = H_0\left[1 - H_0(1+q_0)(t-t_0) + \dots\right]$$

H_0: Hubble parameter now

q_0: deceleration parameter

Inverting, we obtain:

$$(t-t_0) = -H_0^{-1}\left[z - \left(1+\frac{1}{2}q_0\right)z^2 + \dots\right]$$

Substituting back into the first expression and integrating, we obtain:

$$\chi \simeq H_0^{-1}\left[z - \frac{1}{2}(1+q_0)\,z^2 + \dots\right]$$

$1/H_0$: cosmological length scales

The luminosity-distance is therefore expanded as:

$$d_L(z) = H_0^{-1}\left[z + \frac{1}{2}(1-q_0)z^2 + \ldots\right]$$

Or, in terms of the flux measured from an object of known luminosity, we have:

$$f = \frac{L}{4\pi\, d_L^2(z)} = \frac{L\, H_0^2}{4\pi\, z^2}\left[1 + \frac{1}{2}(q_0 - 1)z + \ldots\right]$$

The people that **actually measure** these fluxes (that would be the **astronomers**) have chosen (since Hipparchus, 2000 years ago!) to use a kind of **logarithmic scale**, called **magnitudes**. The **relative magnitude** (or **brightness**) of two objects is defined as:

$$\Delta m_{12} = m_1 - m_2 = -2.5\,\log_{10}\frac{f_1}{f_2}$$

As in the "decibel" scale, there is a "standard object" whose luminosity serves as the standard magnitude, relative to which all other magnitudes are measured.

This object is the star **Vega** (a very stable, very bright star).

Here are some of these "apparent magnitudes":

$$m \equiv -2.5 \log_{10} \frac{f}{f_{\rm Vega}}$$

Apparent magnitude (m)	Object
-27	Sun
-13	Full Moon
-5	Venus
0	Vega*, Saturn
6	Limit of human eye
8	Neptune
14	Pluto
20	Galaxy at z=1
27	Visible light limit of 8m telescope
32	Visible light limit of the HST telescope

* To be precise, the modern standard is a "theoretical" Vega. The real Vega was much better observed in recent times, and in terms of this modern definition, $m_{\rm VEGA} = 0.03$!

Absolute magnitude

We can also define an "absolute" brightness, by placing any object at a single distance from us. In Astronomy, the "standard" distance is **10 pc**, where **1 pc=3.26 light-years**.

We have, then:

$$M \equiv m + 2.5 \log_{10} \frac{f}{f_{10\,\mathrm{pc}}} = m - \underbrace{5\left[\log_{10} d_L(\mathrm{pc}) - 1\right]}_{\mu}$$

distance modulus

$$M = m - \mu$$

Hence, if the **distance is known** and the **relative magnitude** is **measured**, we can compute the **absolute magnitude** of an astronomical object.

Conversely, if the **absolute magnitude** of an object is **known**, and we **measure** its **relative magnitude**, we can infer its **distance modulus** - and its **luminosity distance**!

Cosmography in real life:

Standard candle: objects whose absolute luminosities are known (or can be calibrated with ancillary data, such as periodicity).

Ex.: Cepheid variable stars, Tully-Fisher relation, surface brightness fluctuations, fundamental plane, supernovas

Standard ruler: objects whose **sizes** are known (or can be calibrated with ancillary data, such as periodicity)

Ex.: Baryon Acoustic Oscillations, redshift distortions (see Caldwell's talk)

Cosmography example #1: The Hubble diagram and the HST Key Project

Hubble was the first (1929) to employ Cepheid variable stars to map redshift v. distance:

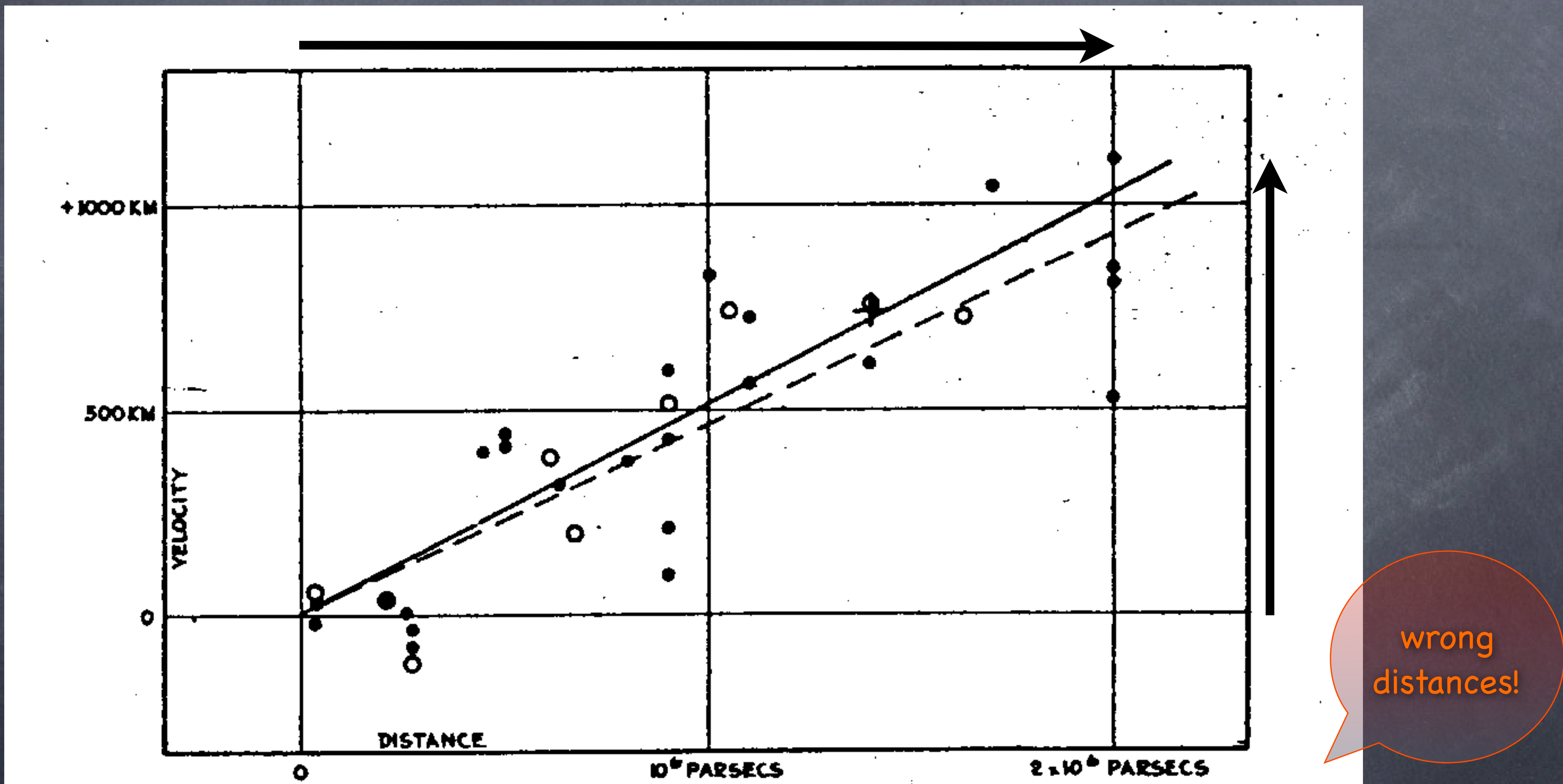
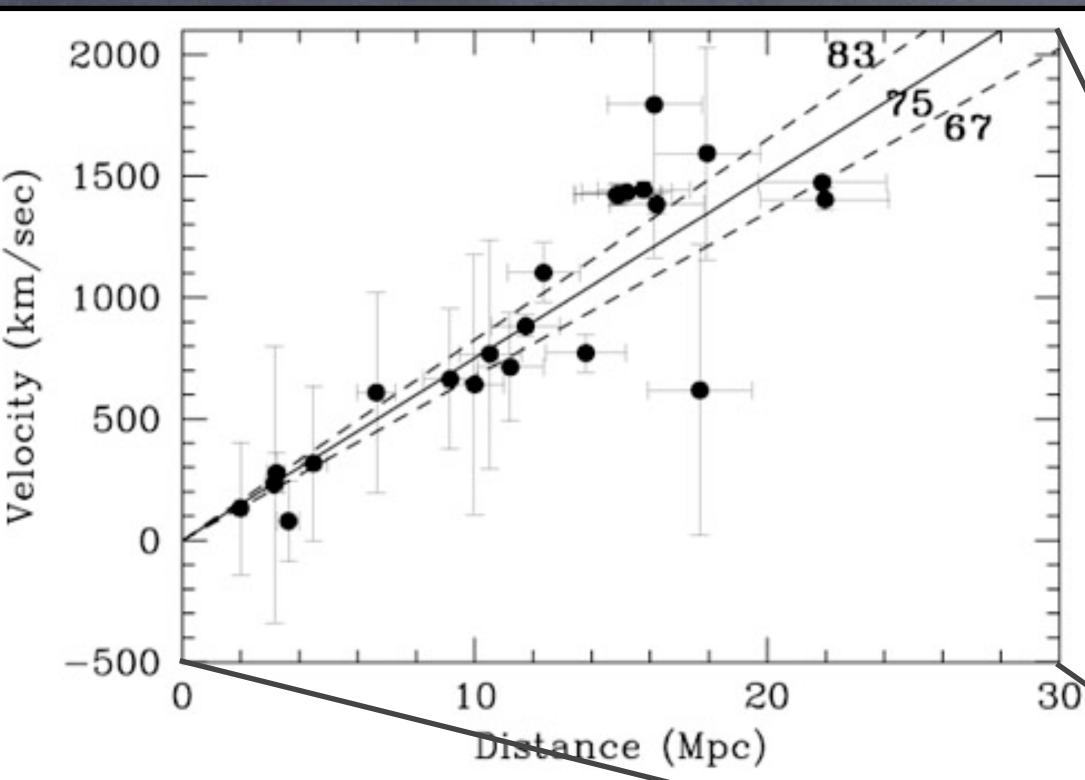


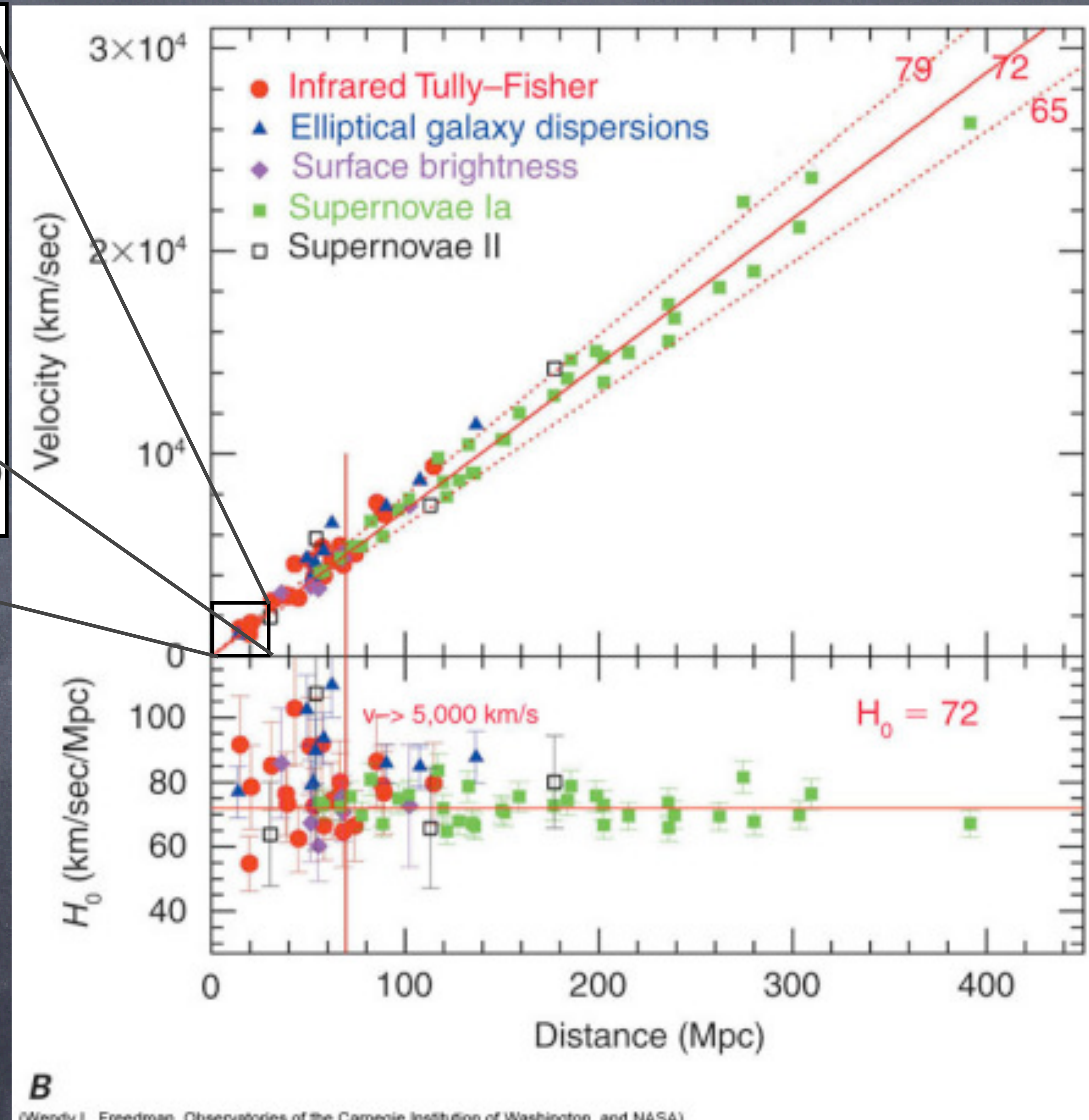
FIGURE 1

$$H_0 \rightarrow \frac{\Delta v}{\Delta \chi} \simeq \frac{cz}{\Delta \chi} = \frac{1000\,\mathrm{Km/s}}{2\,\mathrm{Mpc}} = 500\,\frac{\mathrm{Km}}{\mathrm{s\,Mpc}}\ !!!$$

More recently, the Hubble Space Telescope "Key Project" used basically all the available standard candles to draw the Hubble diagram with much higher precision.



(Cepheids only)



Freedman et al. 2001

Compare Hubble's original plot with Freedman's...

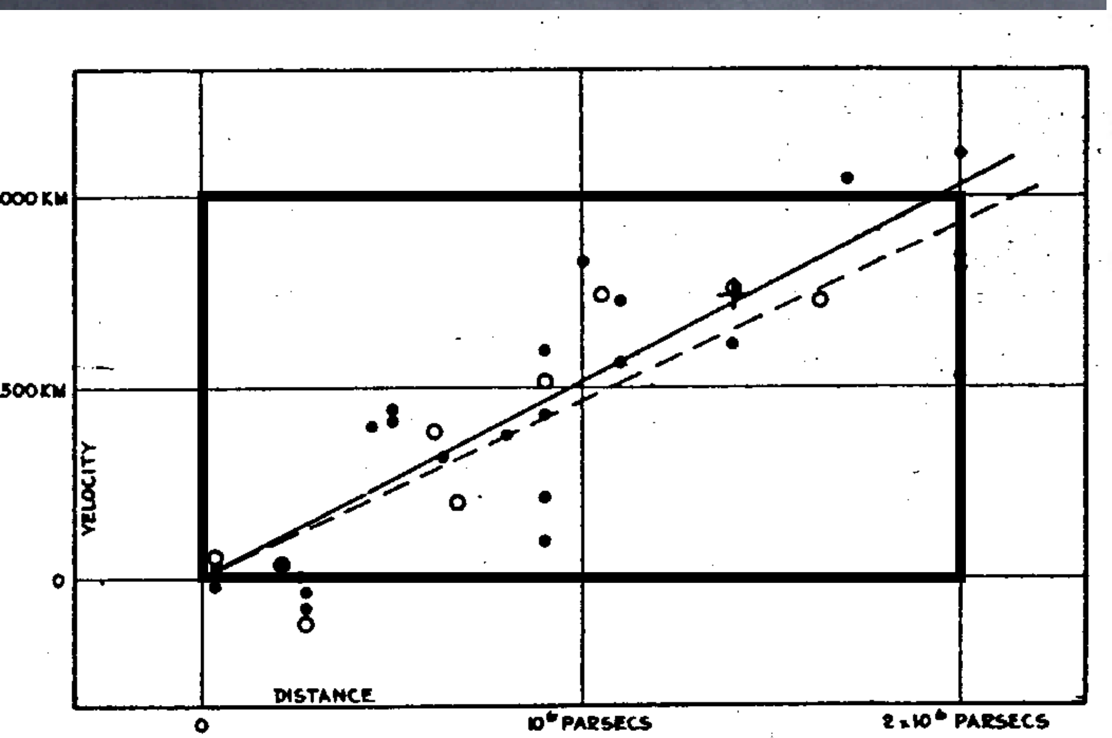
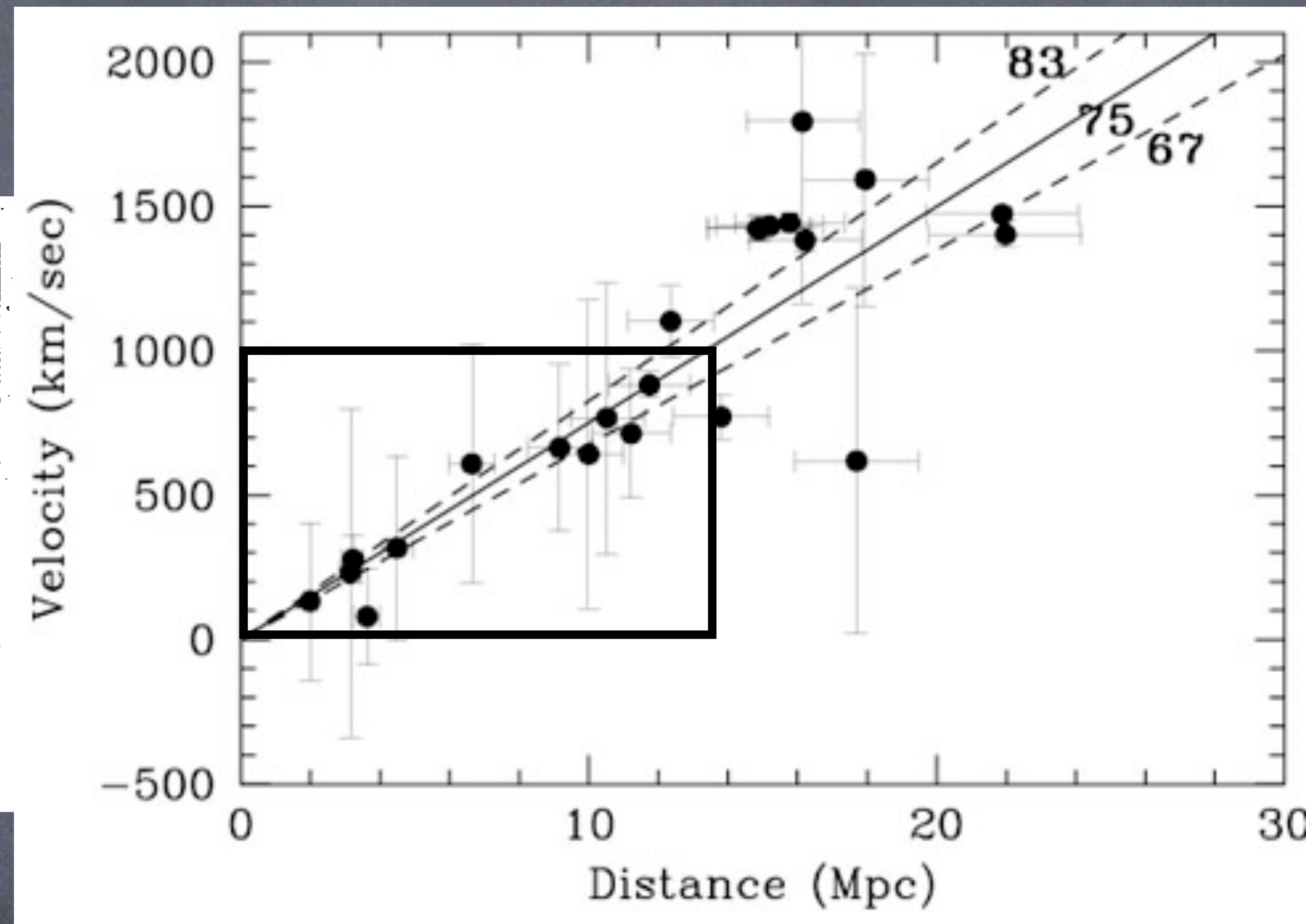
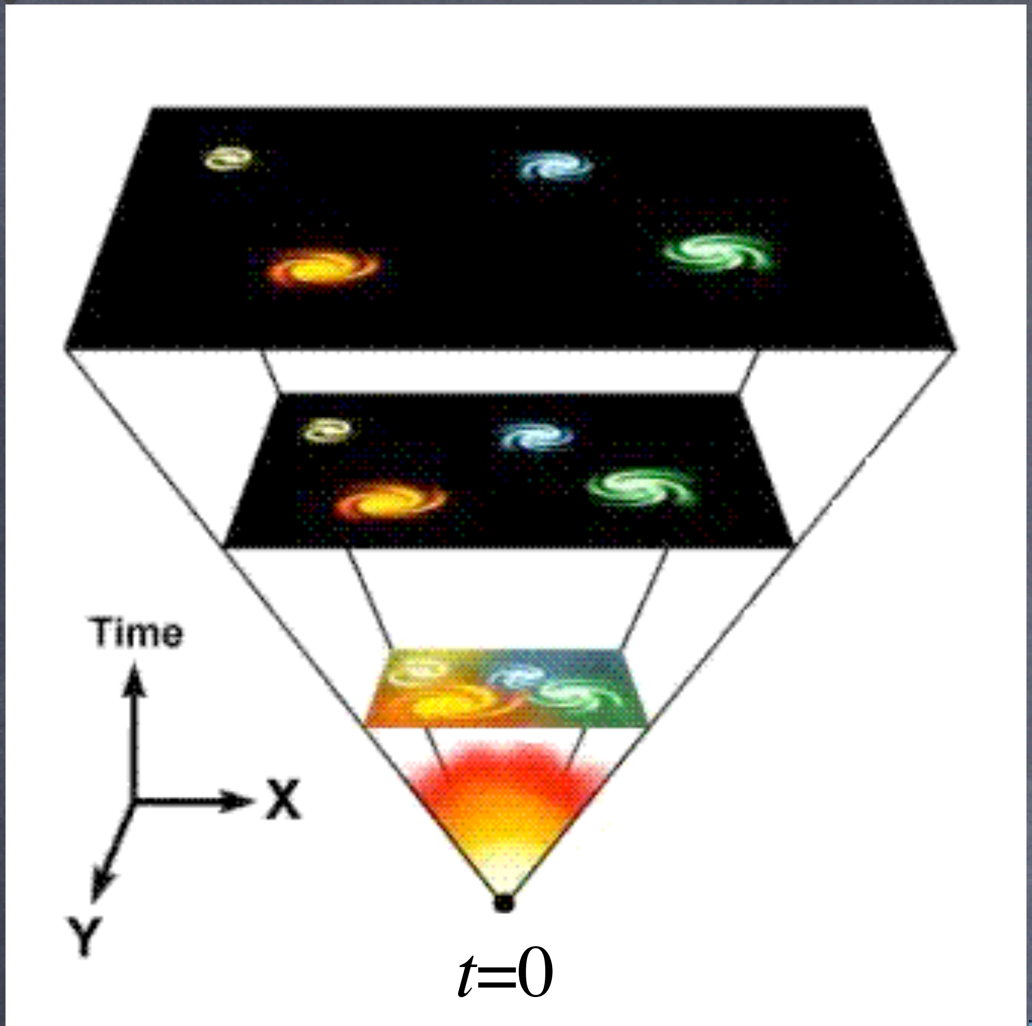


FIGURE 1



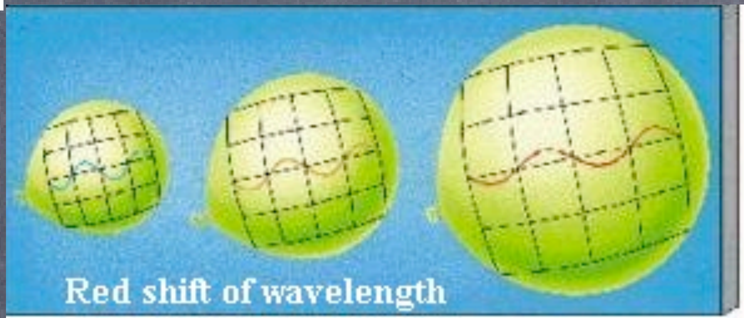
Present constraints on the Hubble parameter (HST Key Project only):

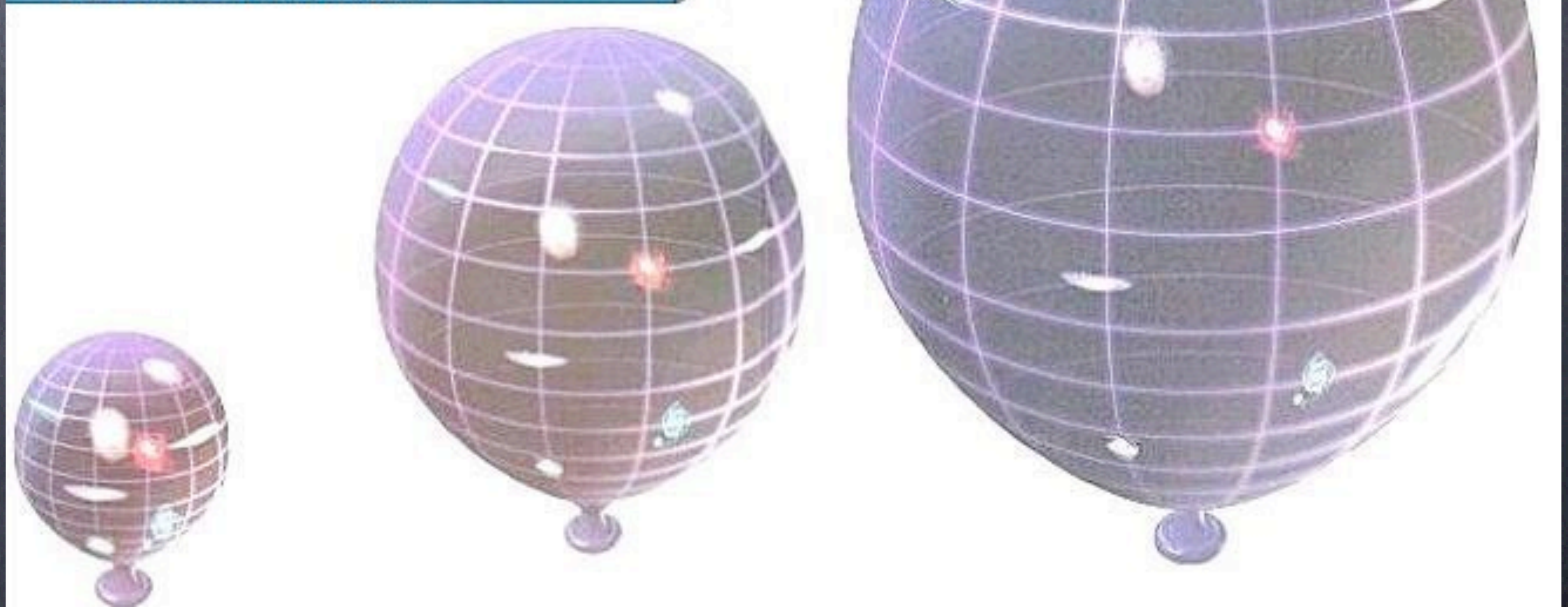
$$H_0 = 72 \pm 8\,\mathrm{Km/s/Mpc}$$



As everyone here knows, the Hubble diagram implies that the Universe is **expanding**!

Not only that: the Universe appears to have **started** at a **certain point in time in the past** (~13.7 Billion years ago)!

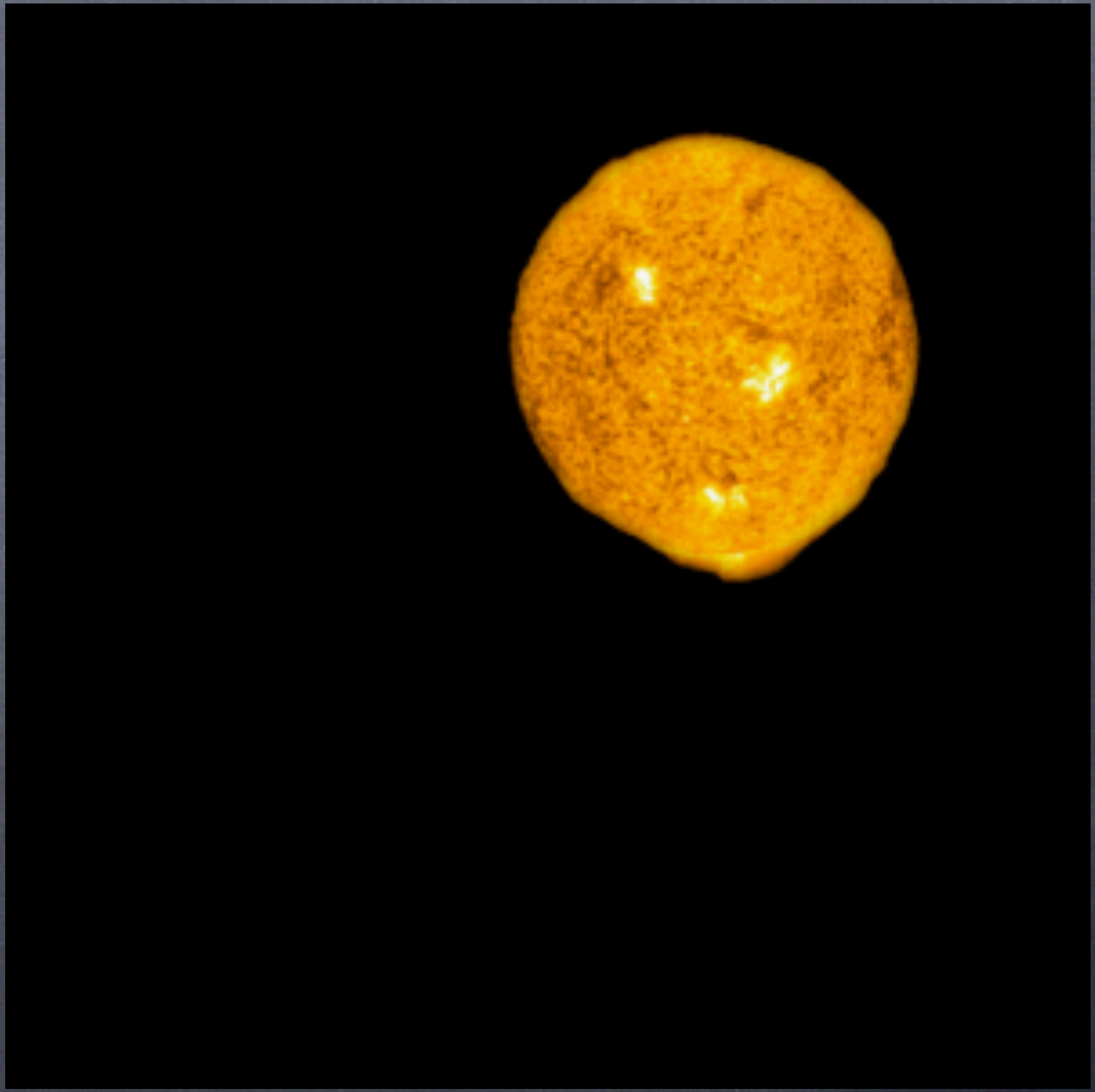




In fact, the Universe could not be **both infinite and eternal**, otherwise the sky would be infinitely bright! This is known as **Olbers' paradox** (1823) - although the idea dates back to Thomas Digges (16th century) and J. Kepler himself (c. 1610). Ref.: Wikipedia

The way out was provided by, e.g., **Lord Kelvin**, and also by **E. A. Poe**, in his essay **Eureka** (1848):

> "Were the succession of stars endless, then the background of the sky would present us a uniform luminosity, like that displayed by the Galaxy - since there could be absolutely no point, in all that background, at which would not exist a star. The only mode, therefore, in which, under such a state of affairs, we could comprehend the voids which our telescopes find in innumerable directions, would be by supposing the distance of the invisible background so immense that no ray from it has yet been able to reach us at all."



Cosmography example #2: Type Ia supernovas

Supernovas are the largest stellar explosions in the Universe.

The light from these explosions can easily outshine an entire galaxy!

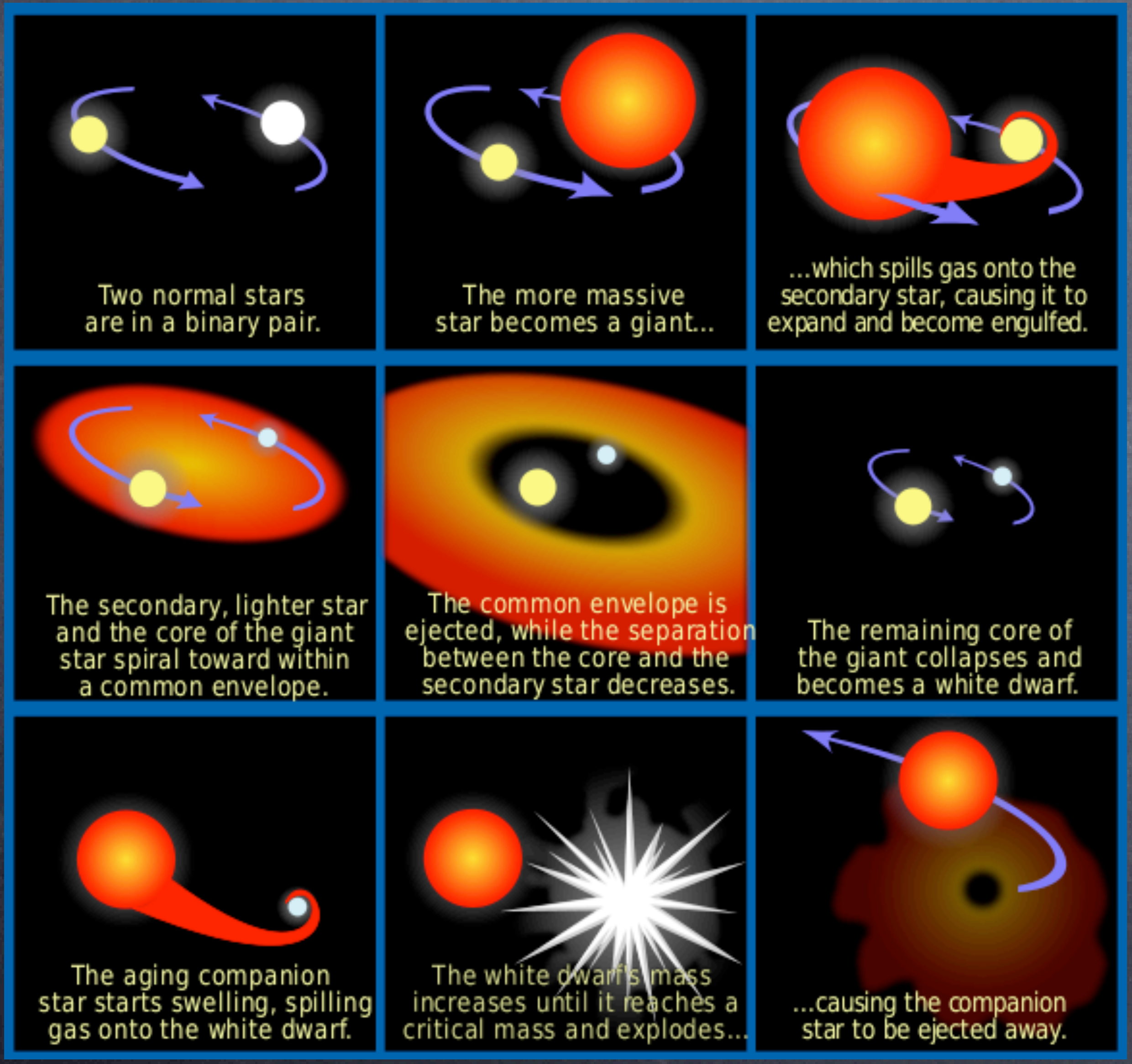
NGC 4526 →

SN 1994D →

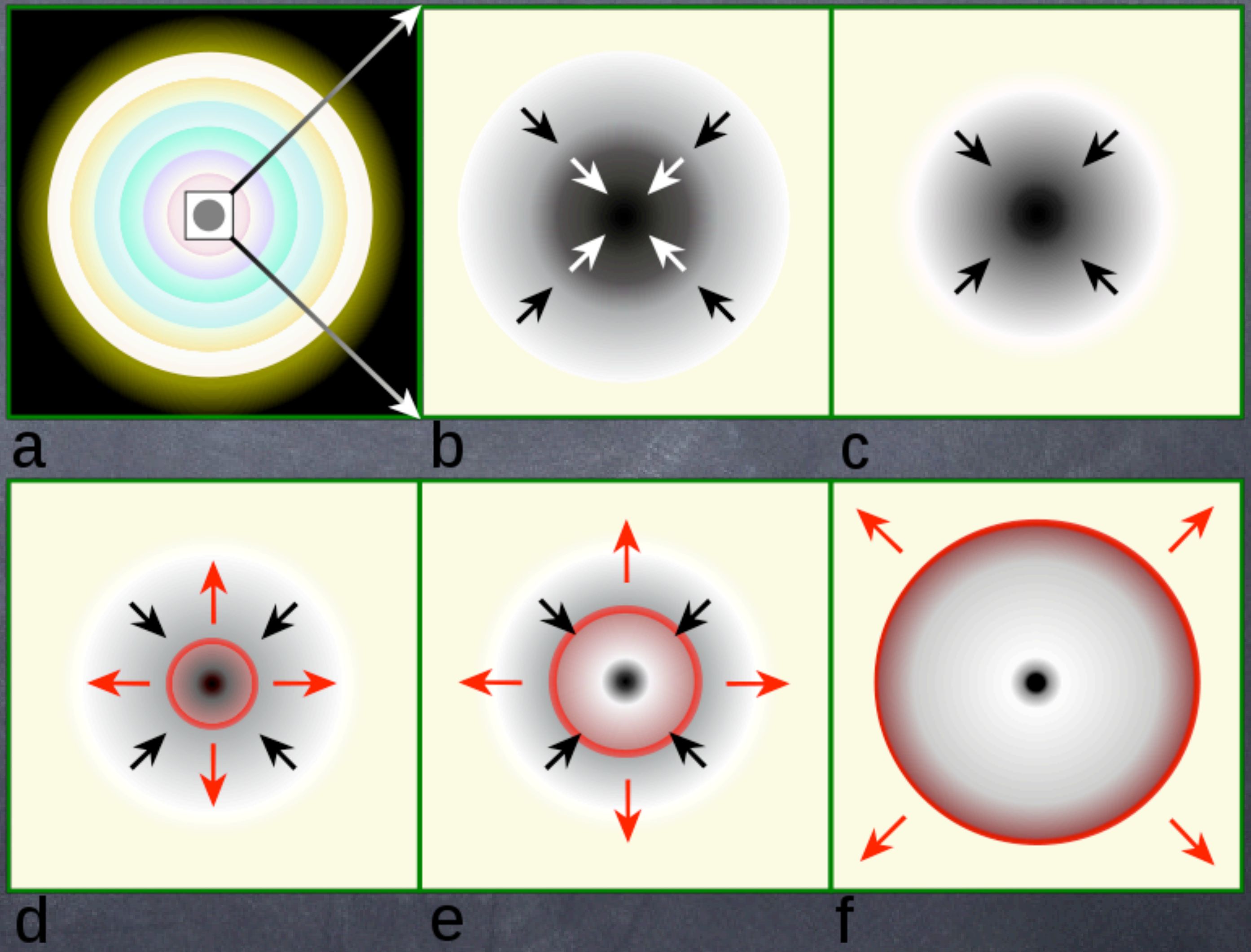
Supernovas come in several types, but basically:

Type I - explosion of a CO white dwarf

(Explossion triggered at approx. same masses)

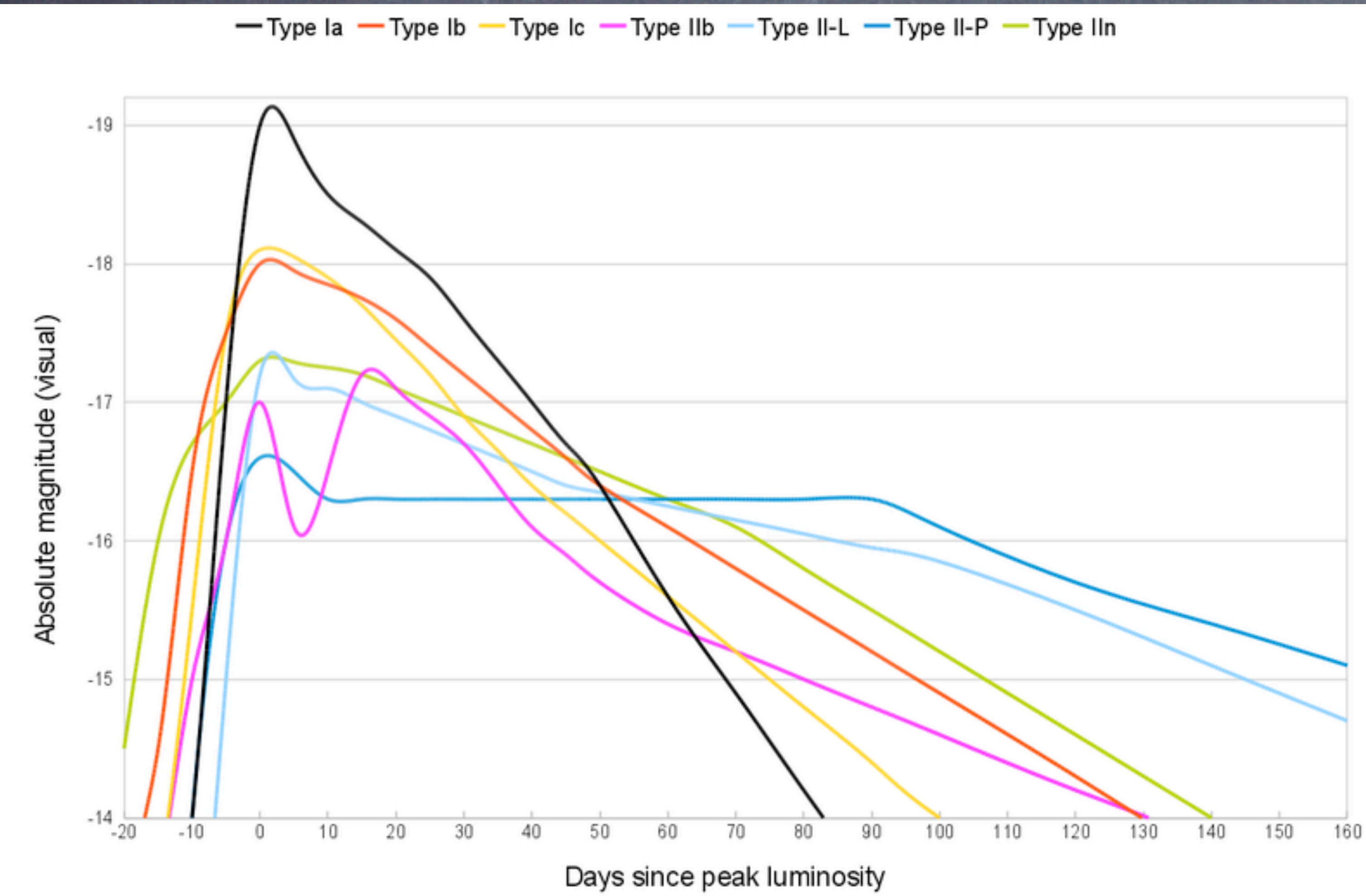


Type II - core-collapse

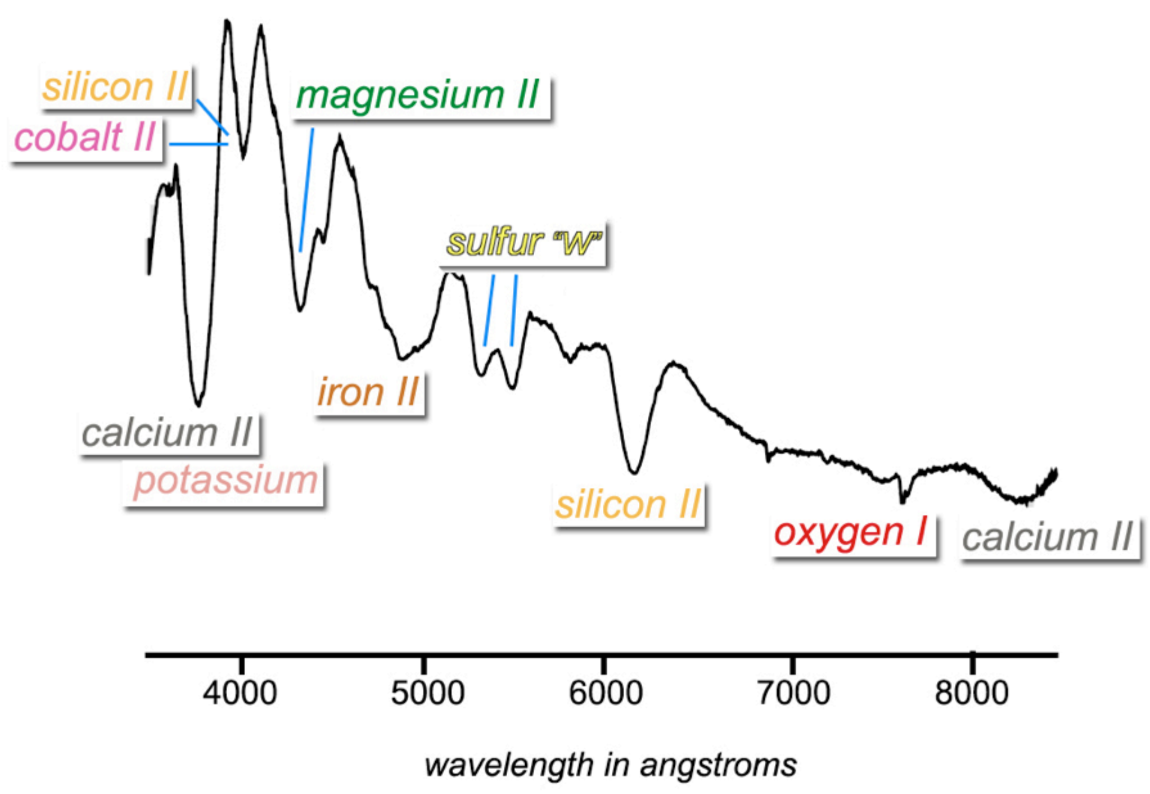


Supernovas are further categorized in terms of their **light curves** (i.e., how the explosion happens in time), and also in terms of their **spectra** (or SEDs - Spectral Energy Distributions).

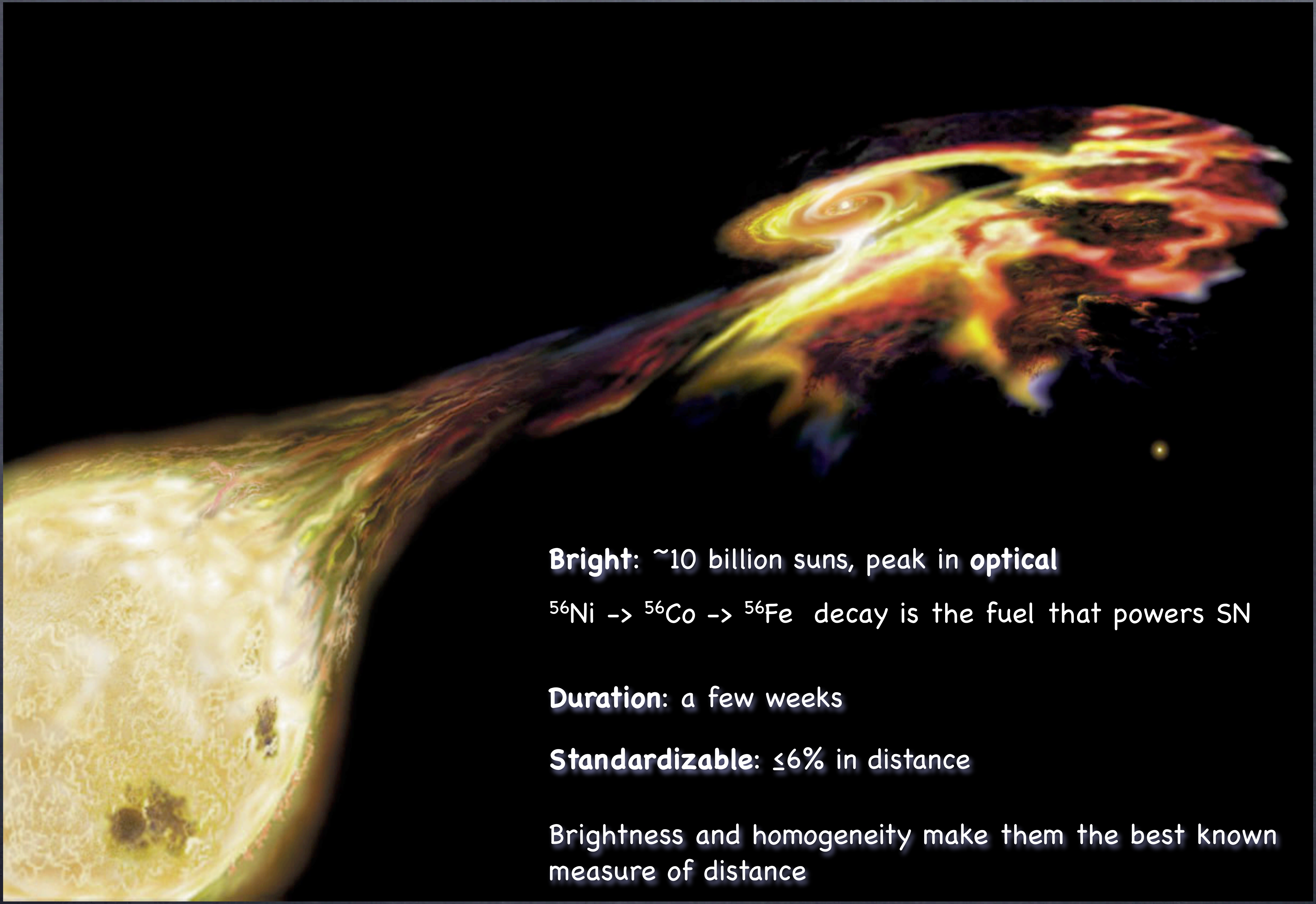
Light curves of several types of SNs



Typical SED of a Type Ia SN



Type Ia supernovas are especially interesting: they seem to have very **similar brightnesses**, indicating that the mass of the progenitors are similar



Bright: ~10 billion suns, peak in **optical**

^{56}Ni -> ^{56}Co -> ^{56}Fe decay is the fuel that powers SN

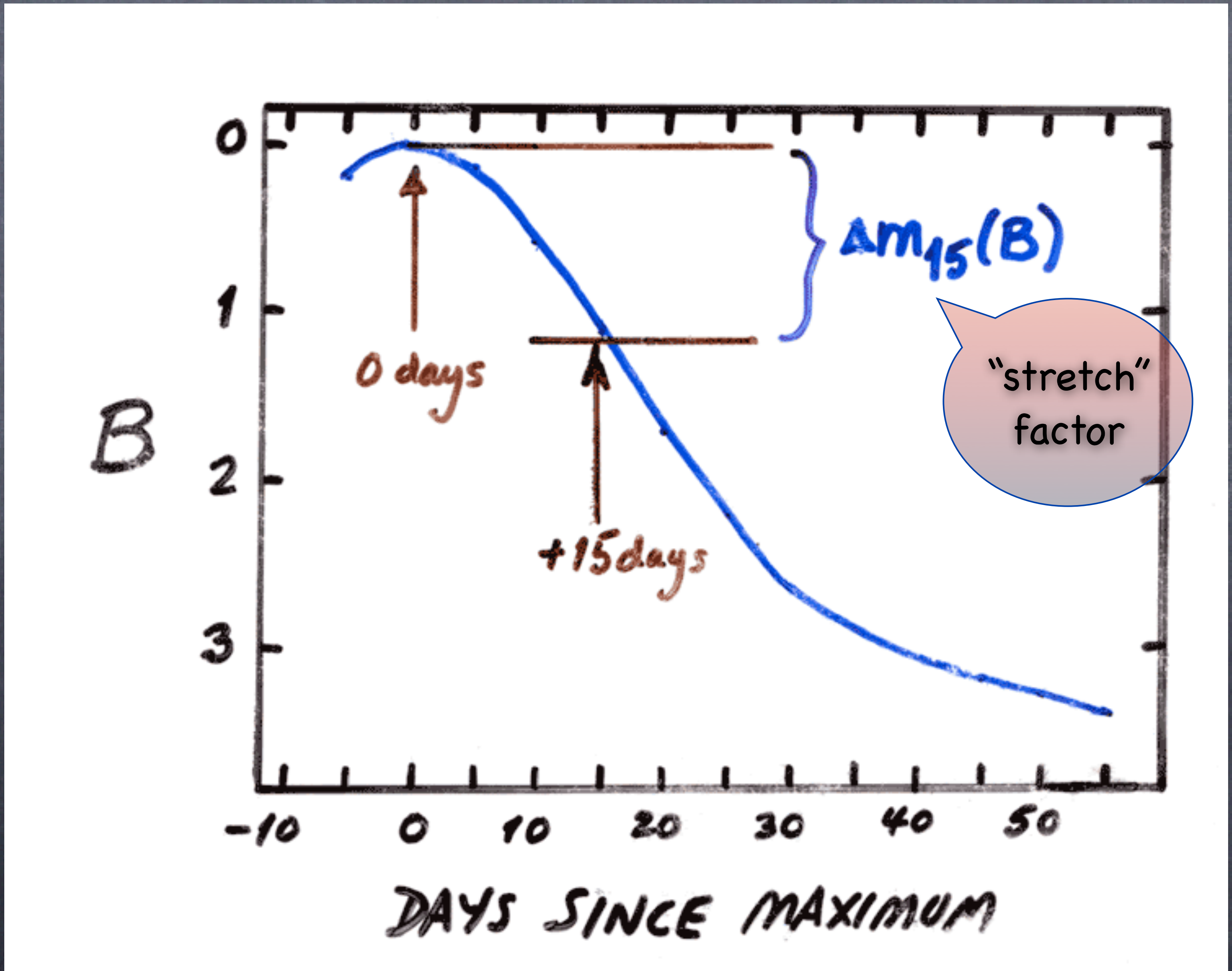
Duration: a few weeks

Standardizable: ≤6% in distance

Brightness and homogeneity make them the best known measure of distance

In 1993, after observing several nearby Type Ia SNs, **M. Phillips** (Ap. J. Lett. 413: L105-L108) noticed a curious relationship between their **light curves** and their **absolute magnitudes**. According to this relationship:

$$M_B = -21.726 + 2.698\,\Delta m_{15}(B)$$



Phillips' original diagram

The Phillips relation was further refined, with **corrections** depending on the "**color**" of the supernova.

This relationship is **entirely phenomenological** - there is **no well-understood theory** behind it.

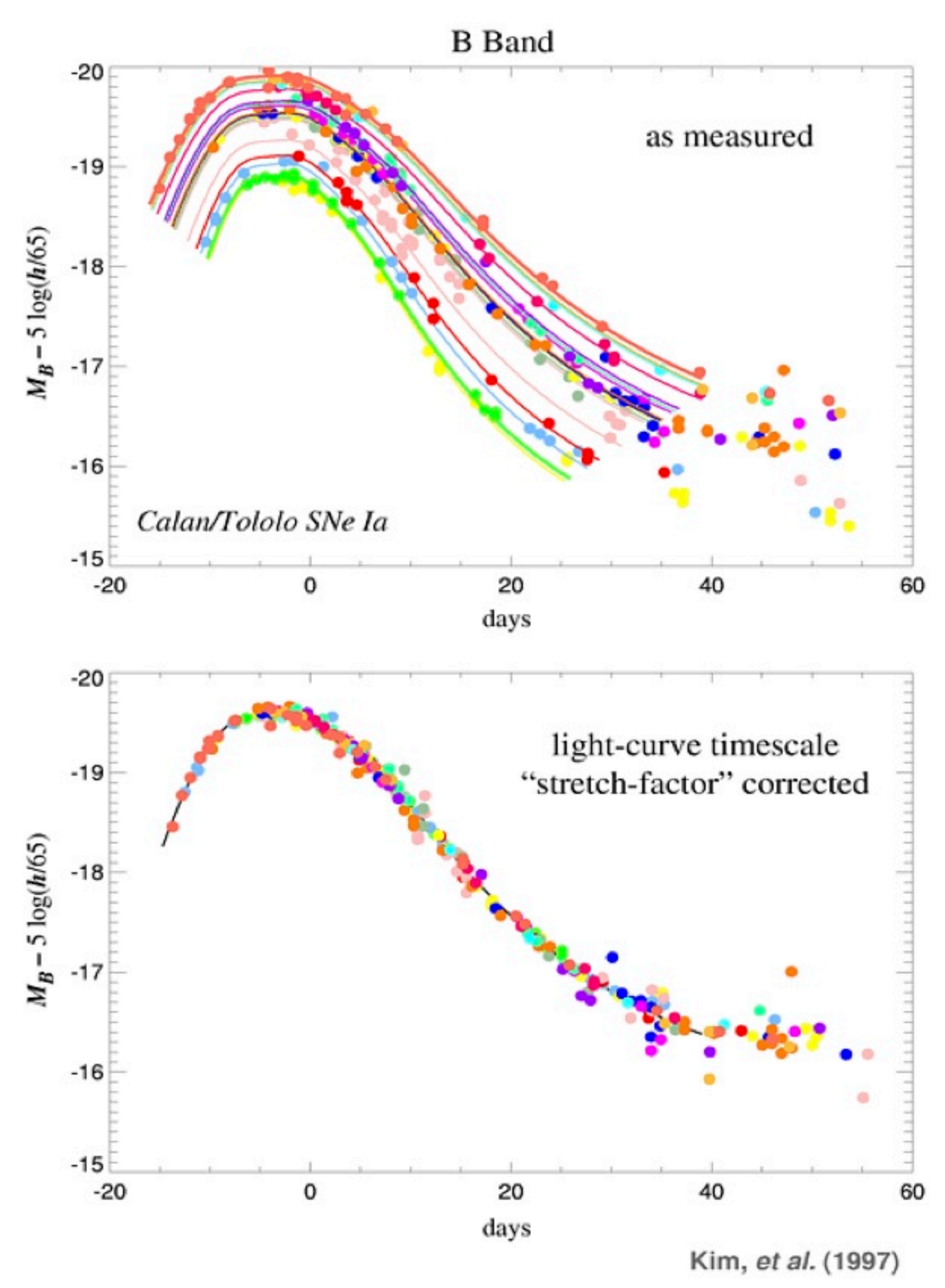
However, it seems to work incredibly well!

Therefore, we can define a "**distance modulus**" for Type Ia supernovas, which is a measure of the **luminosity distance**:

$$\mu_B = m_b - M_B + \alpha\, \Delta m_{15} + \beta\, c$$

$$\mu_B \to 5\, \log_{10} \frac{d_L}{10\,\mathrm{pc}}$$

$$d_L(z) = (1+z)\,\chi = H_0 \left[z + \frac{1}{2}(1-q_0) z^2 + \dots \right]$$

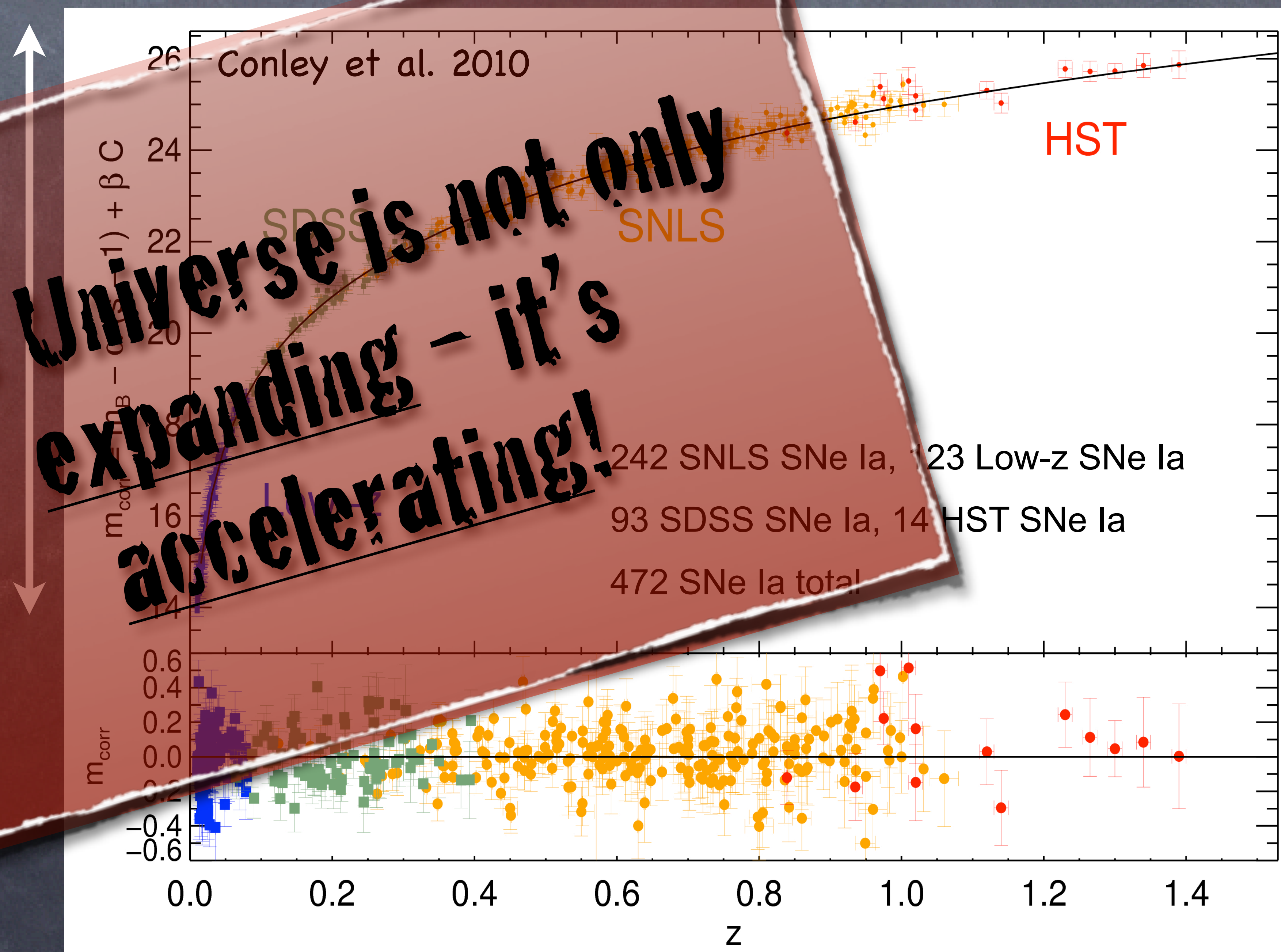


So, we can use Type Ia supernovas to write a diagram of **distance v. redshift** - or, better still, of the **(corrected) distance modulus v. redshift**:

$$\mu_B \to 5\,\log_{10}\frac{d_L}{10\,\mathrm{pc}} \qquad d_L(z) = (1+z)\,\chi = H_0^{-1}\left[z + \frac{1}{2}(1-q_0)z^2 + \ldots\right]$$

Offset: M_B - 5 $\log_{10}$ ($c/H_0/10$pc)

-> **Shape** of the curve can determine q_0 !



The Universe is not only expanding - it's accelerating!

Observations:

$$q_0 = -0.5 \pm 0.2$$

However, a simple measurement of q_0 is **far from ideal**, since the **next-order term** ("**jerk**", j_0) has already started to become important as the **data** became more **abudant**, and more **accurate**, over the past ~7 years.

Although there are ways to improve cosmographical methods, we really need to have a **dynamical description of the expansion**, which can allow us to **check** the **expansion law** against **other observables**, like the **matter/energy content** of the Universe, **the evolution of its large-scale structure**, etc.

-> Next class: Dynamics of FLRW models