OUR UNIVERSE a simple model

RW spacetime : $ds^2 = -dt^2 + a^2(t)\, d\vec{x}^2$

Einstein Eq'ns : $\frac{3}{8\pi G} H^2 = \sum_i \rho_i$

conserve stress-energy : $\frac{d}{dt}\rho_i = -3H\rho_i(1+w_i)$

individually conserved species

i = matter, radiation, dark energy

$$\rho = \sum_i \rho_i = \rho_m^0 \left(\frac{a_0}{a}\right)^3 + \rho_r^0 \left(\frac{a_0}{a}\right)^4 + \rho_\Lambda$$

so $H(a) = H_0 \left[\Omega_m \left(\frac{a_0}{a}\right)^3 + \Omega_r \left(\frac{a_0}{a}\right)^4 + \Omega_\Lambda \right]^{1/2}$

where $\Omega_m + \Omega_r + \Omega_\Lambda = 1$

$$\rho_r = \frac{\pi^2}{30} g_\gamma T_\gamma^4 + N_{eff} \frac{7\pi^2}{120} T_\nu^4$$

$T_\gamma = 2.7\ K$ $\quad N_{eff} = 3.04$ (!)

$g_\gamma = 2$ $\quad T_\nu = (4/11)^{1/3} T_\gamma$

USING $h = 0.72$, $\Omega_r = 8.057 \times 10^{-5}$

$\Omega_M = 0.28$ (SANCHEZ et al 1203.6616)

BUILD A MODEL UNIVERSE !

COSMOLOGY

S. DODELSON "MODERN COSMOLOGY"

P. PETER & J. UZAN "PRIMORDIAL COSMOLOGY"

S. WEINBERG "COSMOLOGY"

L. AMENDOLA & S. TSUJIKAWA "DARK ENERGY"

E. KOLB & M. TURNER "THE EARLY UNIVERSE"

J. HARTLE "INTRODUCTION TO GENERAL RELATIVITY"

COSMOLOGICAL PERTURBATIONS

OUR UNIVERSE IS NOT PERFECTLY ROBERTSON-WALKER

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$
$$= a^2(\tau)\left[-d\tau^2 + d\vec{x}^2\right]$$

IT IS LUMPY! $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$

HOW TO PARAMETERIZE? ALL POSSIBLE DEFORMATIONS OF METRIC

$$ds^2 = a^2(\tau)\left[-(1+2A)\,d\tau^2 + 2B_i\, dx^i d\tau + (\gamma_{ij} + h_{ij})\,dx^i dx^j\right]$$

A : scalar fn

$B_i = \partial_i B + V_i$ so B is a scalar
V_i is a divergence-free vector

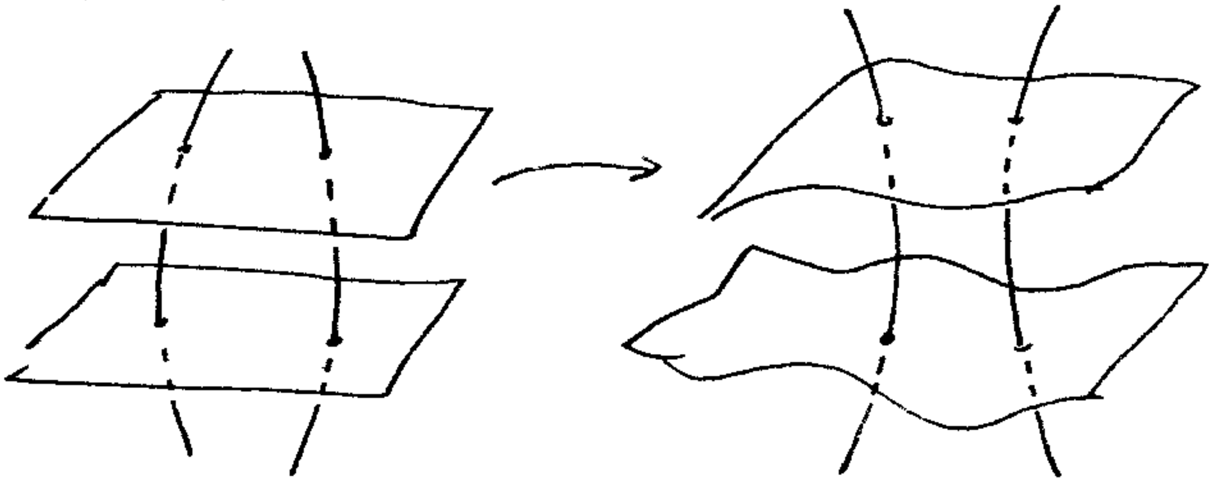
$h_{ij} = 2C\gamma_{ij} + 2\nabla_i\nabla_j E + \nabla_i K_j + \nabla_j K_i + 2H_{ij}$
so C, E are scalars
K_i is a vector, also div. free
H_{ij} is divergence-free, traceless tensor

∇ is derivative on background spacetime

COUNT UP degrees of freedom
A, B, C, E (4)
V_i (2), K_i (2), H_{ij} (2)

AMBIGUITY!

text: Peter & Uzan



HOW TO CONNECT $Q_0(\tau, \bar{x})$ WITH $Q(\tau, \bar{x})$?

HOW DOES A "POINT" MOVE WHEN A SPACETIME IS PERTURBED?

$$x^\mu \to x'^\mu = x^\mu - \xi^\mu$$

SINCE AN ARBITRARY COORD. TRANSF. ALSO "PERTURBS" THE METRIC

UNDER $\xi^\mu = (T, \nabla^i L + N^i)$ $\qquad \nabla_i N^i = 0$

$$A \to A + \frac{\partial}{\partial \tau} T + \mathcal{H} T \qquad ' = \frac{\partial}{\partial \tau}, \quad \mathcal{H} = \frac{a'}{a}$$

$$B_i \to B_i - \partial_i T + (\partial_i L + N_i)'$$

$$h_{ij} \to h_{ij} + \partial_i(\partial_j L + N_j) + \partial_j(\partial_i L + N_i) + 2\mathcal{H} T \gamma_{ij}$$

THIS MEANS $H_{ij} \to H_{ij}$ indep. of coord. shift.

$$\left.\begin{array}{l} V_i \to V_i + N_i' \\ K_i \to K_i + N_i \end{array}\right\} \Phi_i = K_i' - V_i$$

AND SINCE

$$\left.\begin{aligned} A &\to A + T' + \mathcal{H}T \\ B &\to B - T + L' \\ C &\to C + \mathcal{H}T \\ E &\to E + L \end{aligned}\right\} \quad \begin{aligned} Q_1 &= C + \mathcal{H}(B - E') \\ Q_2 &= A + \mathcal{H}(B - E') + (B - E')' \end{aligned}$$

Q_1, Q_2, $\mathcal{D}_i$, H_{ij} are all gauge-invariant

Using 4 gauge degrees of freedom (ξ^μ) we reduce $10 \to 6$ degrees of freedom

In general, we will focus on the scalars.

$$ds^2 = a^2(\tau)\left[-(1+2Q_2)d\tau^2 + (1+2Q_1)\,d\vec{x}^2\right]$$

MANY DIFFERENT NAMING SCHEMES...

I prefer $\quad Q_2 = \Psi$

$Q_1 = -\phi$

same as Ma + Bertschinger 1995

$$ds^2 = a^2(\tau)\left[-(1+2\Psi)d\tau^2 + (1-2\phi)\,d\vec{x}^2\right]$$

In the case $a(\tau) = 1$, these labels say that

Poisson Eq'n: $\quad \nabla^2\phi = 4\pi G\rho$

acceleration: $\quad \vec{a} = -\vec{\nabla}\Psi$

ANOTHER GAUGE: SYNCHRONOUS $A = B = 0$

$$ds^2 = a^2(\tau)\left[-d\tau^2 + (\gamma_{ij} + h_{ij})\,dx^i dx^j\right]$$

$$h_{ij} = 2C\gamma_{ij} + 2\nabla_i\nabla_j E$$

conventional names: $h = 6C + 2\nabla^2 E$

$\eta = -C$

FT: $$h_{ij}(\tau,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}} \left\{ \hat{k}_i \hat{k}_j (h + 6\eta) - 2\eta\,\delta_{ij} \right\}$$

fcns of $\tau, \vec{k}$ here

Relationship between SYNCHRONOUS + GAUGE-INVARIANT VARIABLES

α: $\nabla^2\alpha = -\frac{1}{2}(h + 6\eta)'$

so $\Psi = \alpha' + \mathcal{H}\alpha$

$\Phi = \eta - \mathcal{H}\alpha$

Note: CONFORMAL-NEWTONIAN gauge coincides with Gauge-Inv.

$B = E = 0$

Description of Matter

$$T^{\mu\nu} = (\rho+p)u^\mu u^\nu + p g^{\mu\nu} \qquad \text{perfect fluid}$$

$$T^{\mu\nu} \to T^{\mu\nu} + \delta T^{\mu\nu}$$

$$\rho \to \rho + \delta\rho \;,\quad p \to p + \delta p \;,\quad u^\mu \to u^\mu + \delta u^\mu$$

UNDER A GAUGE TRANSFORMATION

$$\delta T^0{}_0 = -\delta\rho \quad \to \quad -(\delta\rho + \rho' T)$$

so physical quantities $\delta\rho, \delta p$ etc depend on observer's coordinate system.

Construct gauge-invariant combinations

$$\delta T^0{}_0{}^{(GI)} = \delta T^0{}_0 + (T^0{}_0)'(B-E') \qquad = \delta\rho^{GI}$$

$$\delta T^i{}_j{}^{(GI)} = \delta T^i{}_j + (T^i{}_j)'(B-E') \qquad = -\delta p^{GI}\delta^i_j + \sigma_{,ij}$$

$$\delta T^0{}_i{}^{(GI)} = \delta T^0{}_i + (T^0{}_0 - \tfrac{1}{3}T^k{}_k)(B-E')_{,i}$$
$$= (\rho+p)\frac{1}{a}\,\delta u_i^{GI}$$

Note: σ = anisotropic stress

Anisotropic Stress tensor

$$\delta T_{ij} \ni a^2 P\, \Pi_{ij}$$

$$\Pi_{ij} = \left(\nabla_i \nabla_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\Pi^S + \nabla_{(i}\Pi^V_{j)} + \Pi^T_{ij}$$

$$\sigma: \quad \nabla^2 \sigma = -\frac{2}{3}\frac{P}{\rho+p}\Pi^S$$

already gauge-invariant

Fluid velocity Divergence

$$\nabla^i \delta T^0{}_i = \frac{1}{a^2}(\rho+p)\,\theta \quad \text{or} \quad \nabla^i \delta u_i = \frac{1}{a}\,\theta$$

Convert between Conformal-Newtonian and Synchronous gauges

$$\delta_S = \delta_C - \alpha \frac{\rho'}{\rho}$$

$$\delta p_S = \delta p_C - \alpha p'$$

$$\theta_S = \theta_C + \nabla^2 \alpha$$

$$\sigma_S = \sigma_C$$

Fourier Transform

$$f(\tau, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \, e^{i\vec{k}\cdot\vec{x}} \, f(\tau, \vec{k})$$

$$f(\tau, \vec{k}) = \int d^3x \, e^{-i\vec{k}\cdot\vec{x}} \, f(\tau, \vec{x})$$

Einstein Eq'ns

$$k^2\phi + 3\mathcal{H}(\phi' + \mathcal{H}\psi) = -4\pi G a^2 \delta\rho$$

$$k^2(\phi' + \mathcal{H}\psi) = 4\pi G a^2(\rho+p)\theta$$

$$\phi'' + \mathcal{H}(\psi' + 2\phi') + \left(2\frac{a''}{a} - \mathcal{H}^2\right)\psi + \frac{1}{3}k^2(\phi - \psi) = 4\pi G a^2 \delta p$$

$$k^2(\phi - \psi) = 12\pi G a^2(\rho+p)\sigma$$

in C-N (gauge-inv.) coords.

$$k^2\eta - \frac{1}{2}\mathcal{H}h' = -4\pi G a^2 \delta\rho$$

$$k^2\eta' = 4\pi G a^2(\rho+p)\theta$$

$$h'' + 2\mathcal{H}h' - 2k^2\eta = -24\pi G a^2 \delta p$$

$$(h'' + 6\eta'') + 2\mathcal{H}(h' + 6\eta') - 2k^2\eta = -24\pi G a^2(\rho+p)\sigma$$

in S coordinates.