Lecture 1

Newtonian Fluid Spheres

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July 16, 2012

Overview of lectures

Equations of motion for a fluid ball

First warm up: homologous collapse

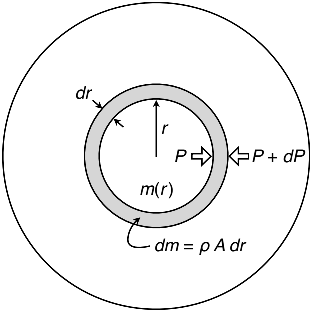
Second warm up: cosmological expansion

A simple stellar model

Overview of lectures

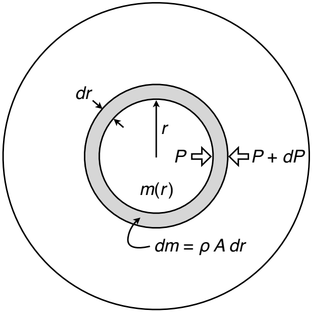
1. Newtonian Fluid Spheres
2. Relativistic Stars
3. Review of General Relativity
4. Gravitational Waves I
5. Gravitational Waves II

Equations of motion for a fluid ball



$$
\begin{aligned}
dm\,\frac{d^2 r}{dt^2} &= \text{gravity} + \text{buoyancy} \\
&= F_{\mathrm{g}} + dF_P \\
&= -\frac{Gm(r)}{r^2}\,dm - A\,dP \\
\rho\,\frac{d^2 r}{dt^2} &= -\frac{Gm(r)}{r^2}\,\rho - \frac{dP}{dr}
\end{aligned}
$$

Equations of motion for a fluid ball



$$\boxed{\begin{aligned} \rho \frac{d^2 r}{dt^2} &= -\frac{Gm(r)}{r^2}\,\rho - \frac{dP}{dr} \\ m(r) &= \int_0^r 4\pi r'^2 \, dr' \, \rho(r') \end{aligned}}$$

Homologous collapse of a dust cloud

Dust: pressure is negligible... no buoyancy!

$$\left|\frac{dP}{dr}\right| \ll \frac{Gm(r)}{r^2}\,\rho$$

$$\boxed{\frac{d^2 r}{dt^2} = -\frac{Gm(r)}{r^2}}$$

Assume mass within shell is constant as shell collapses.
"Homologous collapse": $m(r) =$ constant in time

Homologous collapse of a dust cloud

Solve:

$$\frac{d^2 r}{dt^2} = -\frac{Gm(r)}{r^2}$$

First multiply by integrating factor dr/dt:

$$\underbrace{\frac{dr}{dt}\frac{d^2 r}{dt^2}}_{\frac{1}{2}\frac{d}{dt}\left(\frac{dr}{dt}\right)^2} = \underbrace{-Gm(r)}_{\text{constant}} \times \underbrace{\frac{1}{r^2}\frac{dr}{dt}}_{\frac{d}{dt}\left(-\frac{1}{r}\right)}$$

Then integrate:

$$\boxed{\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \frac{Gm(r)}{r} - kc^2}$$

Homologous collapse of a dust cloud

Initial conditions are $r = r_0$, $dr/dt = 0$, $m(r_0) = \frac{4}{3}\pi r_0^3 \rho_0$, so

$$kc^2 = \frac{4}{3}\pi G \rho_0 r_0^2$$

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{8}{3}\pi G \rho_0\right)\left(\frac{r_0^3}{r} - r_0^2\right)$$

$$\frac{1}{r_0}\frac{dr}{dt} = -\sqrt{\left(\frac{8}{3}\pi G \rho_0\right)\left(\frac{r_0}{r} - 1\right)}$$

Homologous collapse of a dust cloud

Change of variables: $r = r_0 \cos^2 \theta$, $0 \leqslant \theta \leqslant \pi/2$
Also let: $\chi = (\frac{8}{3}\pi G\rho_0)^{1/2}$

$$\begin{aligned}
-2\sin\theta\cos\theta\frac{d\theta}{dt} &= -\chi\tan\theta \\
2\cos^2\theta\frac{d\theta}{dt} &= \chi \\
(1+\cos 2\theta)\frac{d\theta}{dt} &= \chi \\
\frac{d}{dt}(\theta + \tfrac{1}{2}\sin 2\theta) &= \chi
\end{aligned}$$

Integrate:

$$\theta + \tfrac{1}{2}\sin 2\theta = \chi t$$

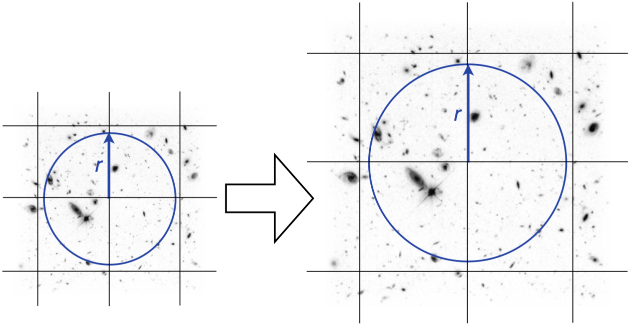
Homologous collapse of a dust cloud

Free-fall time: time to reach $r = 0$ or $\theta = \pi/2$

$$\boxed{t_{\mathrm{ff}} = \frac{\pi}{2\chi} = \sqrt{\frac{3\pi}{32}\frac{1}{G\rho_0}}}$$

Note: free-fall time independent of r_0 so all shells collapse in the same amount of time!
This justifies $m(r) = \text{constant in time}$

Cosmological expansion



Homogeneous universe: $dP/dr = 0$
Co-moving radial coordinate: $m(r)$ = constant in time
Again have:

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \frac{Gm(r)}{r} - kc^2$$

Cosmological expansion

Write: $m(r) = \frac{4}{3}\pi r^3 \rho$

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \frac{4\pi}{3}\, G\rho r^2 - kc^2$$

$$\boxed{\begin{aligned} \underbrace{3\left[\frac{1}{c^2}\left(\frac{1}{r}\frac{dr}{dt}\right)^2 + \frac{k}{r^2}\right]}_{G_{\hat{t}\hat{t}}} &= \underbrace{\frac{8\pi G}{c^2}\,\rho}_{} \\ &= \frac{8\pi G}{c^4}\, T_{\hat{t}\hat{t}} \end{aligned}}$$

Friedmann equation

Cosmological expansion

Multiply Friedmann equation by $r^3/3$...

$$\frac{r}{c^2}\left(\frac{dr}{dt}\right)^2 + kr = \frac{8\pi G}{3c^2}\,\rho r^3$$

...and take a time derivative:

$$\frac{1}{c^2}\left(\frac{dr}{dt}\right)^3 + \frac{2r}{c^2}\frac{dr}{dt}\frac{d^2r}{dt^2} + k\frac{dr}{dt} = \frac{8\pi G}{3c^2}\frac{d}{dt}(\rho r^3)$$

Assume expansion is adiabatic:

$$0 = đQ = dE + P\,dV = \frac{4}{3}\pi\left[d(\rho r^3)\,c^2 + P\,d(r^3)\right]$$

$$\frac{d}{dt}(\rho r^3) = -\frac{3}{c^2}Pr^2\frac{dr}{dt}$$

Cosmological expansion

$$\frac{1}{c^2}\left(\frac{dr}{dt}\right)^3 + \frac{2r}{c^2}\frac{dr}{dt}\frac{d^2r}{dt^2} + k\frac{dr}{dt} = \frac{8\pi G}{c^4}Pr^2\frac{dr}{dt}$$

Divide by $-r^2\,dr/dt$ to get

$$\boxed{\begin{aligned}\underbrace{-\left[\frac{1}{c^2}\frac{2}{r}\frac{d^2r}{dt^2} + \frac{1}{c^2}\left(\frac{1}{r}\frac{dr}{dt}\right)^2 + \frac{k}{r^2}\right]}_{G_{\hat{r}\hat{r}}} &= \underbrace{\frac{8\pi G}{c^4}\,P}_{} \\ &= \frac{8\pi G}{c^4}\,T_{\hat{r}\hat{r}}\end{aligned}}$$

acceleration equation

Cosmological expansion

We have two ordinary differential equations:

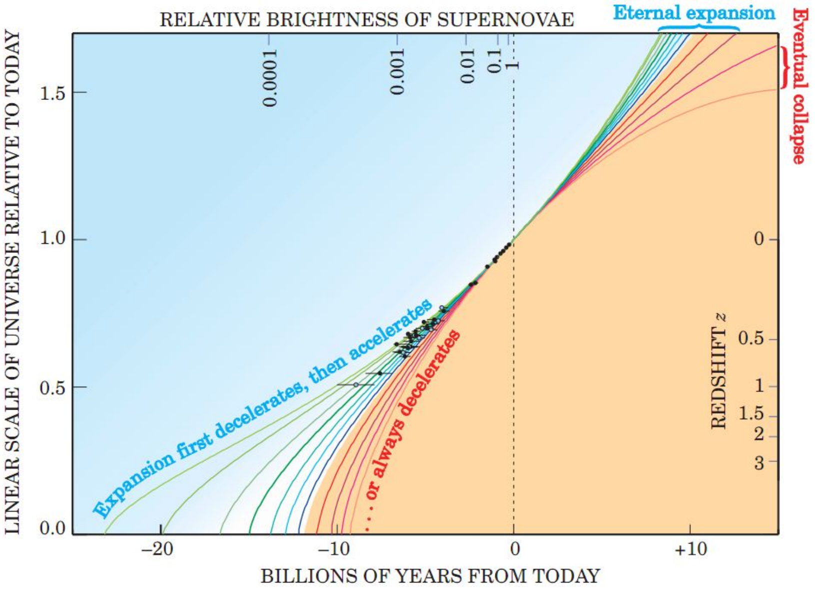
$$3\left[\frac{1}{c^2}\left(\frac{1}{r}\frac{dr}{dt}\right)^2+\frac{k}{r^2}\right]=\frac{8\pi G}{c^2}\,\rho$$

$$-\left[\frac{1}{c^2}\frac{2}{r}\frac{d^2r}{dt^2}+\frac{1}{c^2}\left(\frac{1}{r}\frac{dr}{dt}\right)^2+\frac{k}{r^2}\right]=\frac{8\pi G}{c^4}\,P$$

To solve, must specify an **equation of state**: $P = P(\rho)$

$$P(\rho)=\begin{cases} 0 & \text{“dust”: matter era} \\ \frac{1}{3}\rho c^2 & \text{“radiation”: radiation era} \\ -\rho c^2 & \text{“dark energy”: inflation; now} \end{cases}$$

Cosmological expansion



Simple stellar model

Recall:

$$\rho \frac{d^2 r}{dt^2} = -\frac{Gm(r)}{r^2}\,\rho - \frac{dP}{dr}$$

Equilibrium: $d^2 r/dt^2 = 0$

$$\boxed{\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\,\rho}$$

hydrostatic equilibrium

To solve, must specify an **equation of state**: $P = P(\rho, T)$

- **Barytropic** equation of state: $P = P(\rho)$
- **Polytropic** equation of state: $P = K\rho^\gamma$

Simple stellar model: polytropic equation of state

Rewrite hydrostatic equilibrium equation as

$$\begin{aligned}\frac{r^2}{\rho}\frac{dP}{dr} &= -Gm(r) \\ &= -4\pi G\int_0^r \rho(r')r'^2 dr'\end{aligned}$$

Take derivative with respect to r:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) + 4\pi G\rho = 0$$

Assume a polytropic equation of state, $P = K\rho^\gamma$:

$$\gamma K\frac{1}{r^2}\frac{d}{dr}\left(r^2\rho^{\gamma-2}\frac{d\rho}{dr}\right) + 4\pi G\rho = 0$$

Simple stellar model: polytropic equation of state

Change of variables: let

$$\rho = \rho_c \theta^n$$

where

- $0 \leqslant \theta \leqslant 1$, $\theta = 1$ at center of star, $\theta = 0$ at edge of star
- ρ_c is central density
- n is the **polytropic index** $n = 1/(\gamma - 1)$

Obtain:

$$\left(\frac{n+1}{n}\right) K \rho_c^{1/n} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \theta^{1-n} \frac{d}{dr} \theta^n \right) + 4\pi G \rho_c \theta^n = 0$$

$$\left[\frac{(n+1) K \rho_c^{(1-n)/n}}{4\pi G} \right] \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\theta}{dr} \right) + \theta^n = 0$$

Simple stellar model: polytropic equation of state

Now let

$$r = \lambda \xi$$

where

- ξ is a dimensionless radial variable
- λ is a constant with dimensions of length

$$\lambda = \sqrt{\frac{(n+1) K \rho_c^{(1-n)/n}}{4\pi G}} = \sqrt{\frac{(n+1) P_c}{4\pi G \rho_c^2}}$$

Result is

$$\boxed{\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) + \theta^n = 0}$$

Lane-Emden equation

Simple stellar model: Lane-Emden equation

Solve Lane-Emden equations with boundary conditions:

- $\theta = 1$ at $\xi = 0$
- $d\theta/d\xi = 0$ at $\xi = 0$

Let ξ_1 be the first zero of $\theta(\xi)$. This is the surface of the star.

- Radius of star is $\boxed{R = \lambda\xi_1}$
- Mass of star is

$$M = 4\pi\lambda^3\rho_c \int_0^{\xi_1} \theta^n \xi^2 d\xi$$

but since

$$\theta^n \xi^2 = -\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right)$$

$$\boxed{M = 4\pi\lambda^3\rho_c \left(-\xi_1^2 \left.\frac{d\theta}{d\xi}\right|_{\xi_1}\right)}$$

Simple stellar model: Lane-Emden equation

Recall that $\lambda \sim \rho_c^{(1-n)/2n}$

$$R \sim \lambda \sim \rho_c^{(1-n)/2n}$$

$$M \sim \lambda^3 \rho \sim \rho_c^{(3-n)/2n}$$

$$M \sim R^{(3-n)/(1-n)}$$

- When $n = 0$, $\rho = \rho_c = \text{const}$ and $M \sim R^3$ (of course!)
- When $n = 1$, R is independent of M and ρ_c
- When $n = 3$, M is independent of R and ρ_c

Simple stellar model: incompressible star

Solve Lane-Emden equation with $n = 0$: $\rho = \rho_c \theta^n = \text{const.}$
Lane-Emden equation becomes:

$$\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\xi^2$$

Integrate:

$$\xi^2 \frac{d\theta}{d\xi} = a - \frac{1}{3}\xi^3$$

Boundary condition: $d\theta/d\xi = 0$ at $\xi = 0$ so $a = 0$, and

$$\frac{d\theta}{d\xi} = -\frac{1}{3}\xi$$

Integrate:

$$\theta = b - \frac{1}{6}\xi^2$$

Boundary condition: $\theta = 1$ at $\xi = 0$ so $b = 1$, and

$$\boxed{\theta = 1 - \frac{1}{6}\xi^2}$$

Simple stellar model: incompressible star

$$\theta = 1 - \frac{1}{6}\xi^2 \implies \begin{cases} \theta|_{\xi_1} = 0 \quad \text{for} \quad \xi_1 = \sqrt{6} \\ \left.\frac{d\theta}{d\xi}\right|_{\xi_1} = -\frac{1}{3}\xi_1 = -\frac{\sqrt{6}}{3} \end{cases}$$

$$\begin{aligned} R &= \lambda\xi_1 = \sqrt{6}\lambda \\ M &= 4\pi\lambda^3\rho_c\left(-\xi_1^2 \left.\frac{d\theta}{d\xi}\right|_{\xi_1}\right) = 8\sqrt{6}\pi\rho_c\lambda^3 \\ \implies M &= \tfrac{4}{3}\pi R^3\rho_c \end{aligned}$$

$$\lambda = \sqrt{\frac{P_c}{4\pi G\rho_c^2}} \implies P_c = \frac{3}{8\pi}\frac{GM^2}{R^4}$$

Simple stellar model: white dwarfs

Other interesting equations of state:

- Non-relativistic degenerate matter:

  $$P \propto \rho^{5/3} \implies n = \frac{3}{2}$$

  E.g., low-mass white dwarfs
- Relativistic degenerate matter:

  $$P \propto \rho^{4/3} \implies n = 3$$

  E.g., high-mass white dwarfs

These need to be solved numerically.

Simple stellar model: numerical solution

Lane-Emden

$$\frac{d^2\theta}{d\xi^2} + \frac{2}{\xi}\frac{d\theta}{d\xi} + \theta^n = 0$$

Introduce auxiliary variable $\omega = d\theta/d\xi$ and write as a first-order system

$$\begin{aligned} \frac{d\theta}{d\xi} &= \omega \\ \frac{d\omega}{d\xi} &= -\theta^n - 2\omega/\xi \end{aligned}$$

Boundary conditions: near $\xi = 0$, $\theta = 1$ and $\omega = 0$

Simple stellar model: module `odeint.py`

```python
def rk4(f, y, x, h):
        """ Fourth order Runge-Kutta integration step. """
        k1 = f(y, x) * h
        k2 = f(y + 0.5 * k1, x + 0.5 * h) * h
        k3 = f(y + 0.5 * k2, x + 0.5 * h) * h
        k4 = f(y + k3, x + h) * h
        return y + k1 / 6.0 + k2 / 3.0 + k3 / 3.0 + k4 / 6.0
```

Integrates

$$\frac{dy}{dx} = f(y, x)$$

from x to $x + h$ using the 4th-order Runge-Kutta

Returns: $y(x + h)$

Simple stellar model: program `lanemden.py`

```python
import pylab, odeint

# the Lane-Emden ODE
def lanemden(y, xi):
    global n
    theta = y[0]
    omega = y[1]
    domega = -theta**n - 2.0*omega/xi
    return pylab.array([omega, domega])
```

to be continued...

First-order form of Lane-Emden equations with auxiliary variable $\omega = d\theta/d\xi$ and $\mathbf{y} = [\theta, \omega]$

Returns: $$\frac{d\mathbf{y}}{d\xi} = \begin{bmatrix} d\theta/d\xi \\ d\omega/d\xi \end{bmatrix} = \begin{bmatrix} \omega \\ -\theta^n - 2\omega/\xi \end{bmatrix}$$

Simple stellar model: program `lanemden.py`

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```
# perform integration for various values of n
dx = 0.1
xi = pylab.arange(dx, 10.0, dx)
for n in range(6):
    y = pylab.array([1.0, 0.0])
    theta = []
    for x in xi:
        y = odeint.rk4(lanemden, y, x, dx)
        theta = theta + [y[0]]
    pylab.plot(xi, theta, label='n=%d'%n)
```

to be continued...

Simple stellar model: program `lanemden.py`

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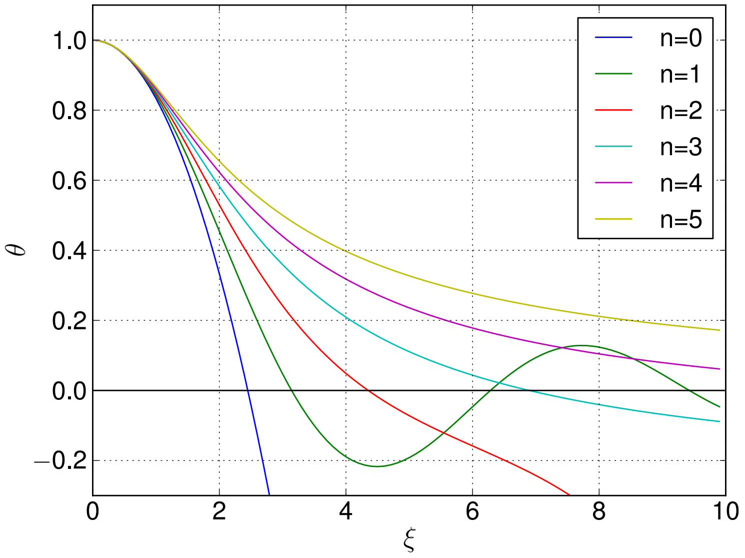
```
# plot results
pylab.axhline(color='black')
pylab.legend()
pylab.xlabel('xi')
pylab.ylabel('theta')
pylab.ylim(ymin=-0.3, ymax=1.1)
pylab.grid()
pylab.show()
```

Simple stellar model: program `lanemden.py`



Simple stellar model: zeros of the Lane-Emden equation

n	ξ_1	$-\xi_1^2 \left. \dfrac{d\theta}{d\xi} \right\|_{\xi_1}$
0.0	2.449 490	4.898 983
1.5	3.653 753	2.714 058
3.0	6.896 845	2.018 236