Topics in Relativistic Astrophysics

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Part I: General relativistic perfect fluids

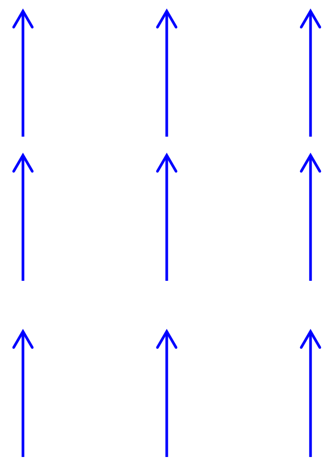
Mathematical prerequisites: Symmetries and Killing vectors

Symmetry under translations

A function f on flat space is symmetric under time translations if

$$\partial_t f = 0$$

The equation can be written in the form $t^\alpha \nabla_\alpha f = 0$, where t^α is the vector field with components $t^\mu = (1,0,0,0)$



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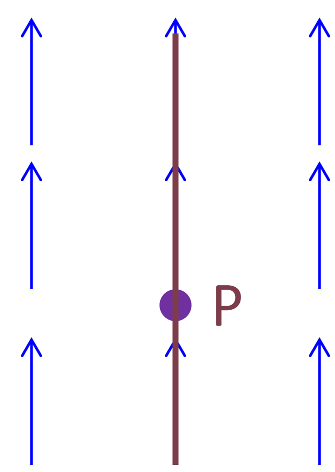


t^α generates a family of translations $T(t)$: It is tangent at each point P to the orbit $T(t)P$ of P. $T(t)(t_0, r_0, \theta_0, \phi_0) = (t_0 + t, r_0, \theta_0, \phi_0)$

Symmetry under rotations

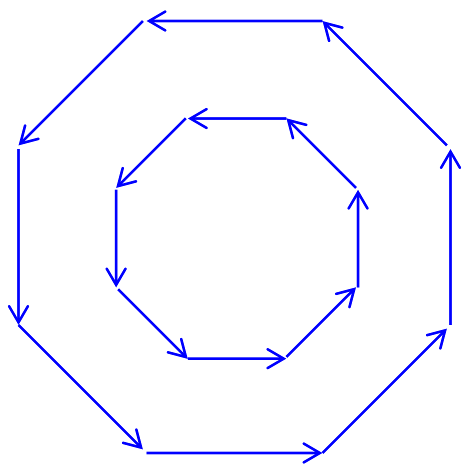
A function f on flat space is symmetric under rotations if

$$\partial_\phi f = 0$$

The equation can be written in the form $\phi^\alpha \nabla_\alpha f = 0$, where ϕ^α is the vector field whose components in coordinates (t, r, θ, ϕ) are

$$\phi^\mu = \delta^\mu_\phi \quad \text{or} \quad (\phi^\mu) = (0,0,0,1).$$

In terms of a Cartesian basis, $\boldsymbol{\phi} = x\,\boldsymbol{j} - y\,\boldsymbol{i}$ and $\partial_\phi = x\partial_y - y\partial_x$.



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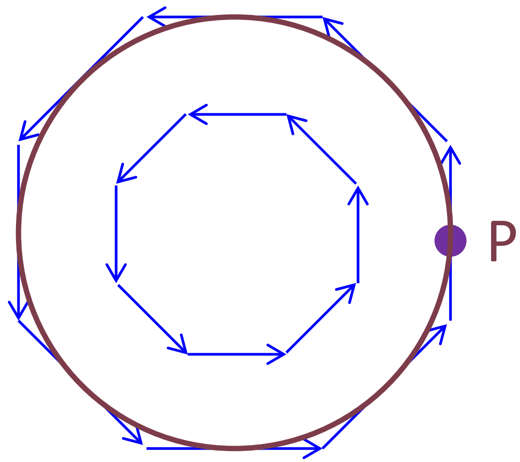
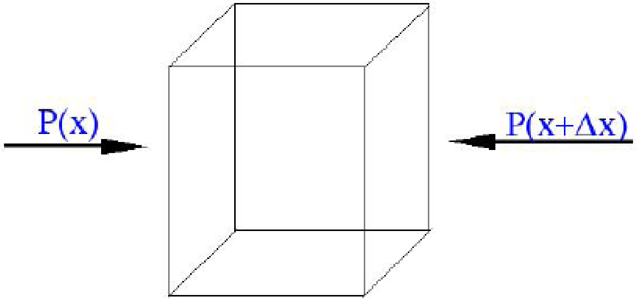


$\boldsymbol{\phi}$ generates a family of rotations $R(\phi)$: It is tangent at each point P to the circular orbit $R(\phi)P$ of P $\quad R(\phi)(t_0,r_0,\theta_0,\phi_0) = (t_0,r_0,\theta_0,\phi_0+\phi)$

The Euler equation

The Euler equation is just $\boldsymbol{F} = m\boldsymbol{a}$ written for a fluid element, shown here as a small box with density ρ and velocity $\boldsymbol{v}$.



The pressure on the left face is $P(x)$;
the pressure on the right face is $P(x+\Delta x)$.

With ΔA the area of each face of the box, The net force in the x-direction is

$$\begin{aligned} F_x &= P(x)\Delta A - P(x+\Delta x)\Delta A \\ &= -\frac{\partial P}{\partial x}\Delta V, \end{aligned}$$

where $\Delta V = \Delta x \Delta A$ is the volume of the fluid element. Thus

$$\mathbf{F} = -\nabla P\, \Delta V.$$

We want to write $\mathbf{F} = m\mathbf{a}$ or

$$-\nabla P \, \Delta V = \rho \Delta V \; \mathbf{a},$$

and we need to find $\mathbf{a}$ in terms of the velocity field $\mathbf{v}(t, \mathbf{x})$. The vector field $\mathbf{v}(t, \mathbf{x})$ has the meaning that at time t the fluid element at $\mathbf{x}$ has velocity $\mathbf{v}(t, \mathbf{x})$. Thus at time $t + \Delta t$ that same fluid element is at $\mathbf{x} + \mathbf{v}(t, \mathbf{x})\Delta t$ and has velocity $\mathbf{v}(t + \Delta t, \mathbf{x} + \mathbf{v}(t, \mathbf{x})\Delta t)$. The fluid element has changed its velocity by

$$\begin{aligned} \Delta \mathbf{v} &= \mathbf{v}(t + \Delta t, \mathbf{x} + \mathbf{v}(t, \mathbf{x})\Delta t) - \mathbf{v}(t, \mathbf{x}) \\ &= \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) \Delta t \end{aligned}$$

in time Δt, and its acceleration is therefore

$$\mathbf{a} = (\partial_t + \mathbf{v} \cdot \nabla) \, \mathbf{v}. \qquad (1)$$

In this way we obtain Euler's equation of motion

$$\rho\left(\partial_t + v^j \nabla_j\right) v^a = -\nabla^a P$$

In the presence of a gravitational field, with potential Φ satisfying $\nabla^2 \Phi = 4\pi G\rho$, there is an additional force $-\rho\Delta V\ \nabla\Phi$ on each fluid element; and the equation of motion becomes

$$\rho(\partial_t + v \cdot \nabla)\mathbf{v} = -\nabla P - \rho\nabla\Phi.$$

Conservation of mass may also be derived in this prosaic fashion: In the figure in your notes, the left face is at x at time t and at $x + v_x(t, \mathbf{x})\Delta t$ at $t + \Delta t$. The right face is at $x + \Delta x$ at t and at $x + \Delta x + v_x(t, x + \Delta x)\Delta t$ at $t + \Delta t$.

Then at $t + \Delta t$ the distance between left and right faces is

$$\begin{aligned} \overline{\Delta x} &= x + \Delta x + v_x(x + \Delta x)\Delta t - (x + v_x(x)\Delta t) \\ &= \Delta x \left(1 + \frac{\partial v_x}{\partial x}\Delta t\right). \end{aligned} \tag{2}$$

$$\begin{aligned} \text{Similarly,}\overline{\Delta y} &= \Delta y \left(1 + \frac{\partial v_y}{\partial y}\Delta t\right) \\ \overline{\Delta z} &= \Delta z \left(1 + \frac{\partial v_z}{\partial z}\Delta t\right). \end{aligned}$$

Thus the volume of the fluid element at $t + \Delta t$ is

$$\begin{aligned} \overline{\Delta V} &= \overline{\Delta x}\ \overline{\Delta y}\ \overline{\Delta z} = \Delta x\ \Delta y\ \Delta z(1 + \nabla \cdot \mathbf{v}\Delta t) \\ &= \Delta V(1 + \nabla \cdot \mathbf{v}\ \Delta t). \end{aligned} \tag{3}$$

Now the statement that the mass of the fluid element is constant is

$$\bar{\rho}\,\overline{\Delta V} = \rho\,\Delta V, \tag{4}$$

where $\bar{\rho}$ is the density of the fluid element at $t + \Delta t$. At that time the fluid element is at $\mathbf{x} + \mathbf{v}\Delta t$; thus

$$\bar{\rho} = \rho(t + \Delta t, \mathbf{x} + \mathbf{v}\ \Delta t) = \rho(t, \mathbf{x}) + (\partial_t + v \cdot \nabla)\rho\ \Delta t \tag{5}$$

Finally from (3), (4), and (5)

$$[\rho + (\partial_t + \mathbf{v} \cdot \nabla)\rho\ \Delta t][1 + \nabla \cdot v\ \Delta t]\Delta V = \rho\Delta V$$

$$\partial_t\rho + \mathbf{v} \cdot \nabla\rho + \rho\nabla \cdot \mathbf{v} = 0, \quad \text{or}$$

$$\partial_t\rho + \nabla \cdot (\rho\mathbf{v}) = 0 \tag{6}$$

To summarize: in the absence of external forces, a Newtonian fluid is characterized by its pressure P, density ρ and 3-velocity $\mathbf{v}$ which satisfy

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0$$

$$\rho(\partial_t + v^b\nabla_b)v_a + \rho\nabla_a P + \nabla_a\Phi = 0. \tag{7}$$

General Relativistic Perfect Fluids

$\epsilon =$ energy density (including rest-mass density)

$\rho =$ baryon rest-mass density

$n =$ baryon number density

Then $\rho = m_B n$, where m_B is the rest mass per baryon.

Stress-energy tensor

A perfect fluid is a model for a large assembly of particles in which a continuous energy density ϵ can reasonably describe the macroscopic distribution of mass. One assumes that the microscopic particles collide frequently enough that their mean free path is short compared with the scale on which the density changes – that the collisions enforce a local thermodynamic equilibrium.

In particular, one assumes that a mean velocity field u^α and a mean stress-energy tensor $T^{\alpha\beta}$ can be defined in boxes –fluid elements – small compared to the macroscopic length scale but large compared to the mean free path; and that on scales large compared to the size of the fluid elements, the 4-velocity and thermodynamic quantities can be accurately described by continuous fields. An observer moving with the average velocity u^α of the fluid will see the collisions randomly distribute the nearby particle velocities so that the particles will look locally isotropic.

Because a comoving observer sees an anisotropic distribution of particles, the components of the fluid's stress-energy tensor in his frame must have no preferred direction:
$T^{\alpha\beta}u_\beta$ must be invariant under rotations that fix u^α.
Denote by

$$q^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta$$

the projection operator orthogonal to u^α.

$$q^\alpha{}_\beta u^\beta = (\delta^\alpha{}_\beta + u^\alpha u_\beta) u^\beta = u^\alpha - u^\alpha = 0$$

$$q^\alpha{}_\beta v^\beta = v^\beta, \quad \text{for} \quad v^\alpha \perp u^\alpha$$

A comoving observer's orthonormal frame $\{\mathbf{e}_\mu\}$ has $\mathbf{e}_0 = \boldsymbol{u}$ implying that the components of $\boldsymbol{u}$ and $\boldsymbol{q}$ in this frame are

$$\left(u^\mu\right) = (1,0,0,0)$$

$$\left(q^\mu{}_\nu\right) = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Because the momentum current $u_\beta q^\alpha_\gamma T^{\beta\gamma}$ is a vector in the 3-dimensional subspace orthogonal to u^α, it is invariant under rotations of that subspace only if it vanishes.
(That is, the spatial vector (T^{0i}) is invariant under rotations only if it vanishes.)

Similarly, the symmetric tracefree tensor

$$ {}^3T^{\alpha\beta} - \frac{1}{3} q^{\alpha\beta}\, {}^3T \equiv q^\alpha_\gamma q^\beta_\delta T^{\gamma\delta} - \frac{1}{3} q^{\alpha\beta} q_{\gamma\delta} T^{\gamma\delta} $$

transforms as a *j=2* representation of the rotation group and can be invariant only if it vanishes.

It follows that the only nonzero parts of $T^{\alpha\beta}$ are the rotational scalars

$$ \epsilon \equiv T^{\alpha\beta} u_\alpha u_\beta $$

and

$$ P \equiv \frac{1}{3} q_{\gamma\delta} T^{\gamma\delta} $$

More concretely, the components T^{0i} and $T^{ij} - \frac{1}{3}\delta^{ij}T_k^k$ must vanish, implying that $T^{\alpha\beta}$ has components

$$\left(T^{\mu\nu}\right) = \begin{pmatrix} \epsilon & & & \\ & P & & \\ & & P & \\ & & & P \end{pmatrix}$$

in the orthonormal frame of a comoving observer.

Summary:

The condition of local isotropy suffices to define a perfect fluid by enforcing a stress-energy tensor of the form

$$T^{\alpha\beta} = \epsilon u^\alpha u^\beta + P q^{\alpha\beta}$$

Departures from a perfect fluid

In neutron stars, departures from perfect-fluid equilibrium due to a solid crust are expected to be of order 10^{-3} or smaller, corresponding to the maximum strain that an electromagnetic lattic can support.

On a submillimeter scale, superfluid neutrons and protons in the interior of a neutron star have velocity fields that are curl-free outside a set of quantized vortices. On larger scales, however, a single, averaged, velocity field u^α accurately describes a neutron star (Baym and Chandler 1983; Sonin 1987; Mendell and Lindblom 1991). Although the approximation of uniform rotation is not valid on scales shorter than 1 cm, the error in computing the structure of the star on larger scales is negligible. In particular, with $T^{\alpha\beta}$ approximated by a value $\langle T^{\alpha\beta}\rangle$ averaged over several cm, the error in computing the metric is of order

$$\delta g_{\alpha\beta} \sim \left(\frac{1\mathrm{cm}}{R}\right)^2 \sim 10^{-11}$$

For equilibria, these are the main corrections. For dynamical evolutions -- oscillations, instabilities, collapse, and binary inspiral, however, one must worry about the microphysics governing, for example viscosity, heat flow, magnetic fields, superfluid modes, and turbulence.

B. The Einstein-Euler equations

A perfect-fluid spacetime is a spacetime $M, g_{\alpha\beta}$ whose source is a perfect fluid. That is, its metric satisfies

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

with $T^{\alpha\beta} = \epsilon u^\alpha u^\beta + P q^{\alpha\beta}$.

The Bianchi identities imply

$$\nabla_\beta T^{\alpha\beta} = 0,$$

and this equation, together with an equation of state, determines the motion of the fluid.
The projection of the equation $\nabla_\beta T^{\alpha\beta} = 0$ along u^α is an energy conservation law, while the projection orthogonal to u^α is the relativistic Euler equation. For an intuitive understanding of these equations, it is helpful to look first at conservation of baryons.

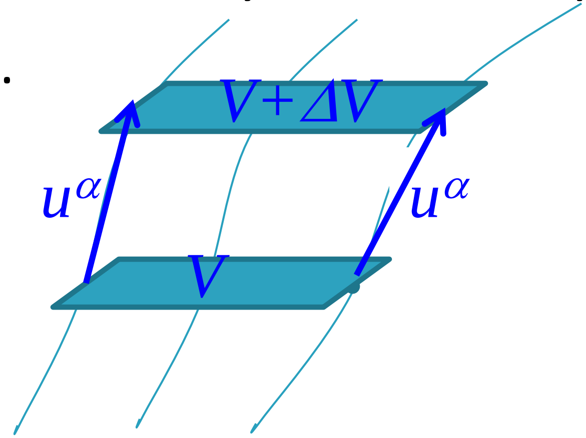
Conservation of baryons

The baryon mass M_0 of a fluid element is conserved by the motion of the fluid. The proper volume of a fluid element is the volume V of a slice perpendicular to u^α through the history of the fluid element; and conservation of baryons can be written in the form $0 = \Delta M_0 = \Delta(\rho V)$ or

$$0 = \frac{\Delta(\rho V)}{V} = \Delta\rho + \rho\frac{\Delta V}{V}. \tag{1.1}$$

First term of (1): With τ the proper time along each fluid trajectory, the change in ρ in a proper time $\Delta\tau$ is given by $\Delta\rho = \frac{d}{d\tau}\rho\ \Delta\tau = u^\alpha\nabla_\alpha\rho\,\Delta\tau$.

Second term: the fractional change in V in a proper time $\Delta\tau$ is given by the 3-dimensional divergence of the velocity, in the subspace orthogonal to u^α: $\frac{\Delta V}{V} = q^{\alpha\beta}\nabla_\alpha u_\beta \Delta\tau$.



Because $u^\alpha u_\alpha = -1$, we have $u^\beta \nabla_\alpha u_\beta = \frac{1}{2}\nabla_\alpha\left(u^\beta u_\beta\right) = 0$, implying

$$q^{\alpha\beta}\nabla_\alpha u_\beta = \nabla_\alpha u^\alpha, \quad \frac{\Delta V}{V} = \nabla_\alpha u^\alpha \Delta\tau,$$

and conservation of baryons takes the form

$$0 = \Delta\rho + \rho\frac{\Delta V}{V} = (u^\alpha \nabla_\alpha \rho + \rho\nabla_\alpha u^\alpha)\Delta\tau$$

or

$$\nabla_\alpha(\rho u^\alpha) = 0. \qquad (1.2)$$

A more formal derivation is given below, in part to introduce a perturbation formalism that one needs to discuss the Hamiltonian formalism, stellar oscillations and stability, the virial theorem, and thermodynamics of neutron stars and black holes.

Conservation of energy

$$u_\alpha \nabla_\beta T^{\alpha\beta} = 0$$

$$\begin{aligned} 0 &= u_\alpha \nabla_\beta T^{\alpha\beta} \\ &= u_\alpha \nabla_\beta T^{\alpha\beta} = u_\alpha \nabla_\beta [\epsilon u^\alpha u^\beta + P q^{\alpha\beta}] \\ &= -\nabla_\beta(\epsilon u^\beta) + P u_\alpha \nabla_\beta (g^{\alpha\beta} + u^\alpha u^\beta) \\ &= -\nabla_\beta(\epsilon u^\beta) - P \nabla\cdot u \end{aligned}$$

$$\nabla_\beta(\epsilon u^\beta) = -P\nabla\cdot u \quad \text{or } u^\beta \nabla_\beta \epsilon = -(\epsilon + P)\nabla\cdot u \qquad (1.3)$$

The equation means that the energy ϵV of a fluid element decreases by the work,

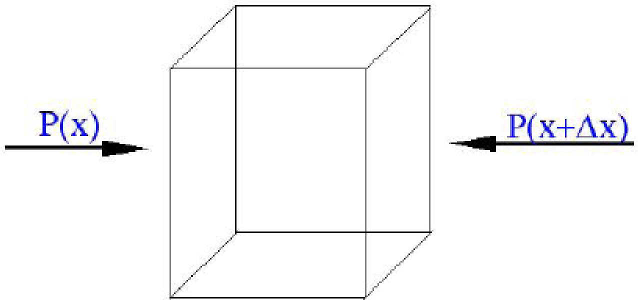
$$P\, dV = P\, \nabla\cdot u\, V d\tau$$

it does in proper time $d\tau$.

Prob. 1: Check that this *is* the meaning of Eq. (1.3), by following steps analogous to those in the heuristic derivation of baryon conservation.

Relativistic Euler equation

$$q^{\alpha}{}_{\gamma}\nabla_{\beta}T^{\beta\gamma}=0$$



$$\begin{aligned}0&=q^{\alpha}{}_{\gamma}\nabla_{\beta}[\epsilon u^{\beta}u^{\gamma}+Pq^{\beta\gamma}]\\&=q^{\alpha}{}_{\gamma}\epsilon u^{\beta}\nabla_{\beta}u^{\gamma}+q^{\alpha\beta}\nabla_{\beta}P+q^{\alpha}{}_{\gamma}P\nabla_{\beta}(u^{\beta}u^{\gamma})\\&=\epsilon u^{\beta}\nabla_{\beta}u^{\alpha}+q^{\alpha\beta}\nabla_{\beta}P+Pu^{\beta}\nabla_{\beta}u^{\alpha}\end{aligned}$$

$$(\epsilon+P)u^{\beta}\nabla_{\beta}u^{\alpha}=-q^{\alpha\beta}\nabla_{\beta}P. \qquad (1.3)$$

Newtonian limit:

Let ε be a small parameter of order v/c or v_{sound}/c, whichever is larger.
Then

$$u^\mu = (1, \mathrm{v}^i) + 0(\varepsilon^2)$$

$$P/\epsilon = 0(\varepsilon^2)$$

$$\epsilon = \rho + 0(\varepsilon^2)$$

Conservation of energy coincides to lowest order with conservation of baryons:

$$\partial_t(\rho u^t) + \partial_i(\rho u^i) = -P(\partial_t u^t + \partial_i u^i)$$

$$\partial_t \rho + \partial_i(\rho \mathrm{v}^i) = 0 + 0(e^2)$$

To recover the Euler equation, we need

$$g_{tt} = -(1 + 2\Phi)$$

$$\Gamma^i{}_{tt} = -\frac{1}{2}\partial_i g_{tt} = \nabla_i \Phi.$$

Then $\rho u^\mu \nabla_\mu u^i = -\nabla^i P$ becomes

$$\rho(\partial_t + \mathrm{v}^j \nabla_j)\mathrm{v}_i + \rho \nabla_i \Phi = -\nabla_i P$$

C. Barotropic flows: enthalpy, the Bernoulli equation, injection energy, and conservation of circulation

A fluid with a one-parameter EOS is called barotropic. Neutron-star matter is accurately described by a one-parameter EOS because it is approximately *isentropic: It has nearly constant* (nearly zero) entropy per baryon. (There is, however, a composition gradient in neutron stars, with the density of protons and electrons ordinarily increasing outward, and this dominates a departure from a barotropic equation of state in stellar oscillations).

Recall that, that enthalpy H is defined by $H=E+PV$. The specific enthalpy – enthalpy per unit rest mass— is then

$$h=\frac{E+PV}{M_0}=\frac{\epsilon+P}{\rho}$$

In the Newtonian approximation, $h-1=\frac{\epsilon-\rho}{\rho}+\frac{P}{\rho}=u+\frac{P}{\rho}$,

with $u = e/\rho$ the internal energy per unit rest mass.

A stationary flow is described by a spacetime with a timelike Killing vector, t^α, the generator of time-translations that leave the metric and the fluid variables fixed:

$$\pounds_{\mathrm{t}} g_{\alpha\beta} = \pounds_{\mathrm{t}} u^\alpha = \pounds_{\mathrm{t}} \epsilon = \pounds_{\mathrm{t}} P = 0 \qquad (1.4)$$

Bernoulli's law

In the Newtonian approximation, Bernoulli's law is conservation of enthalpy for a stationary flow, and its relativistic form is

$$\pounds_{\mathrm{u}}\left(h u_\alpha t^\alpha\right) = \pounds_{\mathrm{u}}\left(\frac{\epsilon + P}{\rho} u_\alpha t^\alpha\right) = 0. \qquad (1.5)$$

or

$$u^\mu \partial_\mu \left(\frac{\epsilon + P}{\rho} u_t\right) = 0.$$

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To derive this equation, use the relation

$$u^\alpha \nabla_\alpha h = \frac{u^\alpha \nabla_\alpha P}{\rho},$$

which follows from conservation of energy and baryon number:

That is, from

$$\frac{u^\alpha \nabla_\alpha \epsilon}{\epsilon + P} = -\nabla_\alpha u^\alpha = \frac{u^\alpha \nabla_\alpha \rho}{\rho},$$

we have

$$u^\alpha \nabla_\alpha \left(\frac{\epsilon + P}{\rho} \right) = \frac{1}{\rho}(u^\alpha \nabla_\alpha \epsilon + u^\alpha \nabla_\alpha P) - \frac{\epsilon + P}{\rho^2} u^\alpha \nabla_\alpha \rho$$

$$= \frac{u^\alpha \nabla_\alpha P}{\rho}$$

Because $\mathcal{L}_{\mathrm{u}} u_\alpha = u^\beta \nabla_\beta u_\alpha + u_\beta \nabla_\alpha u^\beta = u^\beta \nabla_\beta u_\alpha$,

the Euler equation, (1.3), becomes

$$\mathcal{L}_{\mathrm{u}}(h u_\alpha) = -\frac{\nabla_\alpha P}{\rho}. \tag{1.6}$$

Contracting this form of the Euler equation with t^α and using Eq. (1.4), we obtain the relativistic Bernoulli equation, (1.5).

The derivation holds for any Killing vector that Lie-derives the fluid variables, and, for an axisymmetric flow, yields conservation of a fluid element's angular momentum in the form

$$\pounds_{\boldsymbol{u}}(hu_\beta \phi^\beta) = 0. \tag{1.7}$$

Check of Newtonian limit of the relativistic Bernoulli equation:
Here's an outline of the steps. We need u_t to order v^2/c^2.

$$-1 = g_{\mu\nu}u^\mu u^\nu = -(1+2\Phi)\left(u^t\right)^2 + \mathrm{v}^2 + O(\varepsilon^4) \Rightarrow \left(u^t\right)^2 = \frac{1}{\left(1-\mathrm{v}^2\right)(1+2\Phi)}$$

$$u^t = 1 + \frac{1}{2}\mathrm{v}^2 - \Phi$$

Next, $$u_t = g_{tt}u^t = -\left(1 + \frac{1}{2}\mathrm{v}^2 + \Phi\right)$$

Finally, (1.5) becomes

$$h_{\text{Newt}} = \frac{\epsilon - \rho + P}{\rho} = h - 1.$$

$$(\partial_t + \pounds_{\mathrm{v}})(h_{\text{Newt}} + \frac{1}{2}\mathrm{v}^2 + \Phi) = 0$$

$$\pounds_{u^t k} u_\alpha = u^t \cancel{\pounds_k u_\alpha} + u_\beta k^\beta \nabla_\alpha u^t$$

$$= -\nabla_\alpha \ln u^t$$

Because $u^\beta \nabla_\beta P = u^t k^\beta \nabla_\beta P = 0,$

we have $$q_\alpha^{\ \beta} \nabla_\beta P = \nabla_\beta P$$

$$\frac{\nabla_\beta P}{\epsilon + P} = \nabla_\beta \log h.$$

Euler's equation thus has the form,

$$-\nabla_\alpha \log u^t = -\nabla_\beta \log h \Rightarrow \nabla_\alpha \log \frac{h}{u^t} = 0$$

with first integral $$\frac{h}{u^t} = \mathscr{E}, \qquad (1.8)$$

$\mathscr{E}$ constant throughout the star.

$\frac{h}{u^t} = \mathscr{E}$ is the (first-integral of the) equation of hydro-equilibrium for a uniformly rotating barotropic star. $\mathscr{E}$ is the *injection energy per baryon*, the energy needed to lower a collection of baryons at zero temperature from infinity, expand a volume to accommodate them, add kinetic energy to match the rotation of the star, and insert them in the star.

Bernoulli's law:
hu_t is conserved along the fluid worldlines when entropy per baryon is conserved by the flow. Newtonian limit $h_{\text{Newt}} + \frac{1}{2}\text{v}^2 + \Phi$

Hydro equilibrium (Poincare-Wavre):
h/u^t is constant throughout a uniformly rotating barotropic star.
Newtonian limit $h_{\text{Newt}} - \frac{1}{2}\text{v}^2 + \Phi$

This first integral of hydro-equilibrium is sometimes mistakenly called Bernoulli's law in the relativity literature.

Conservation of vorticity and circulation: Newtonian approximation

In the Newtonian approximation, the vorticity of a fluid is defined by

$$\omega^a = \nabla \times \mathrm{v} \quad \text{or} \quad \omega_{ab} = \nabla_a \mathrm{v}_b - \nabla_b \mathrm{v}_a$$

As we will quickly show in the full theory, it is preserved by the fluid flow:

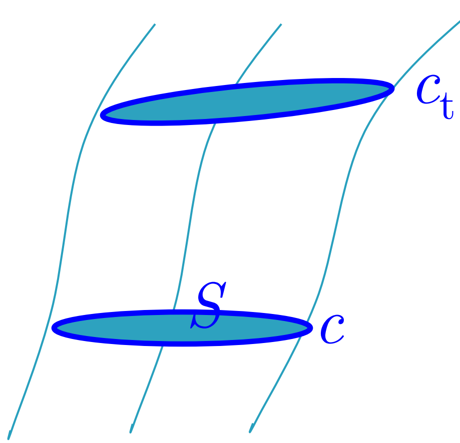
$$(\partial_t + \pounds_{\boldsymbol{v}})\nabla_{[a} v_{b]} = 0.$$

That is, vorticity is conserved in this Lie-derived sense.

The meaning of the conservation law is easier to understand using the integral form of vorticity: circulation. If S is a surface bounded by a curve c, then the circulation of the fluid about c is

$$\text{circulation} := \int_c v_a d\ell^a = \int_S (\nabla_a v_b - \nabla_b v_a) dS^{ab}.$$

Let c_t be the curve c moved along with the fluid flow for a time t: Move each point of c for a time t along the fluid trajectory through that point.



Then

$$\frac{d}{dt}\int_{c_t} v_a\, d\ell^a = 0.$$

It turns out to be quicker and, I hope, clearer, to derive these laws in the full theory, where the derivative along the 4-velocity is not split into $\partial_t + \pounds_{\boldsymbol{v}}$.

Conservation of vorticity and circulation in GR

The vorticity $\omega_{\alpha\beta}$ is

$$\omega_{\alpha\beta} = \nabla_\alpha (h u_\beta) - \nabla_\beta (h u_\alpha),$$

the differential conservation law is the curl of the Euler equation,

$$\pounds_u \omega_{\alpha\beta} = 0.$$

Again, because the vorticity is the curl (exterior derivative) of the vector field hu_α, Stokes' theorem relates the integral of vorticity over a 2-surface S to the line integral of hu_α around its boundary c.

$$\text{circulation} := \int_c hu_\alpha d\ell^\alpha = \int_S [\nabla_\alpha (hu_\beta) - \nabla_\beta (hu_\alpha)] dS^{\alpha\beta}.$$

From the form (1.6) $\pounds_{\mathrm{u}}(hu_\alpha) = -\nabla_\alpha \ln h$ of the Euler equation and the relation $[\pounds_u, d] = 0,$ the differential form of the conservation law is immediate.

The curl of the Euler equation is

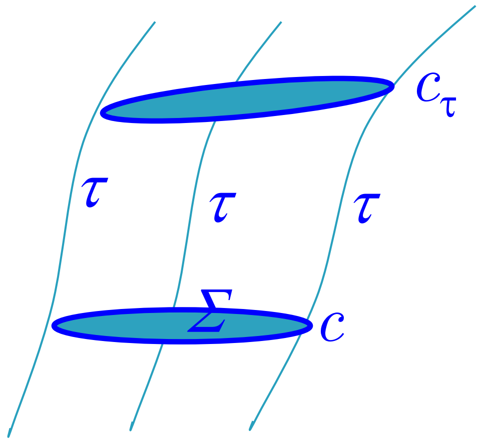
$$\nabla_\alpha \pounds_{\mathrm{u}}(hu_\beta) - \nabla_\beta \pounds_{\mathrm{u}}(hu_\alpha) = -\nabla_\alpha \nabla_\beta \ln h + \nabla_\beta \nabla_\alpha \ln h = 0$$

$$\pounds_{\mathrm{u}}[\nabla_\alpha (hu_\beta) - \nabla_\beta (hu_\alpha)] = 0$$

or

$$\pounds_{\mathrm{u}} \omega_{\alpha\beta} = 0.$$

The corresponding integral law again involves the circulation along a curve moving with the fluid. Let c be a closed curve in the fluid, bounding a 2-surface S; and let c_τ be the curve obtained by moving each point of c a proper time τ along the fluid trajectory through that point.



Claim: $$\frac{d}{d\tau}\int_{c_\tau} h\, u_\alpha\, d\ell^\alpha = 0.$$

The proof involves two of the main properties of integrals: invariance of an integral under a diffeo (active version of invariance under change of coordinates) and Stokes' theorem. And it uses the geometrical definition of Lie derivative.

Here it is:

Start with the curl of the Euler equation in its original form:

$$\nabla_\alpha \pounds_{\rm u}(h u_\beta) - \nabla_\beta \pounds_{\rm u}(h u_\alpha) = 0.$$

Integrate over the surface S_τ , and write the integral of the curl as a line integral over c_τ:

$$\int_{S_\tau} [\nabla_\alpha \pounds_{\rm u}(h u_\beta) - \nabla_\beta \pounds_{\rm u}(h u_\alpha)] dS^{\alpha\beta} = \int_{c_\tau} \pounds_{\rm u}(h u_\alpha)\, d\ell^\alpha$$

Use the geometrical definition of Lie derivative of a vector along u as the rate of change of a vector dragged along by the fluid flow

$$\pounds_{\rm u}(h u_\alpha) = \frac{d}{d\tau} \psi_{-\tau}(h u_\alpha)$$

Finally, use the invariance of an integral under a diffeo

$$0 = \frac{d}{d\tau} \int_c \psi_{-\tau}(h u_\alpha) d\ell^\alpha = \frac{d}{d\tau} \int_{c_\tau} h u_\alpha\, d\ell^\alpha$$

$$\left[\int_{\psi_\tau c} \psi_\tau \sigma_\alpha d\ell^\alpha = \int_c \sigma_\alpha d\ell^\alpha \Rightarrow \int_c \psi_{-\tau} \sigma_\alpha d\ell^\alpha = \int_{\psi_\tau c} \sigma_\alpha d\ell^\alpha \right]$$