II. Rotating Relativistic Stars

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A collection of properties of observers and particles

An observer with 4-velocity u^α measures local quantities with respect to her orthonormal tetrad $(\mathbf{e}_{\hat{0}}, \mathbf{e}_{\hat{1}}, \mathbf{e}_{\hat{2}}, \mathbf{e}_{\hat{3}})$

$$e_{\hat{0}}^\alpha = u^\alpha \quad \Rightarrow \quad (u^{\hat{\mu}}) = (1,0,0,0).$$

Energy of a particle with 4-momentum p_α: $E = -p_\alpha u^\alpha$

A stationary, asymptotically flat spacetime has an asymptotically timelike Killing vector t^α.

A stationary observer at infinity has 4-velocity $u^\alpha = t^\alpha$.

In a spacetime with a Killing vector ξ^α, $p_\alpha u^\alpha$ is a constant of motion.

In a stationary axisymmetric spacetime with Killing vectors t^α and ϕ^α, the conserved quantities are $E_\infty = -p_\alpha t^\alpha$ and $J = p_\alpha \phi^\alpha$. (The subscript ∞ means energy measured by an observer at infinity.)

II. Relativistic stars: equilibria

Quick review of nonrotating stars.

A *static* spacetime is a spacetime with a timelike Killing vector t^α that is orthogonal to a spacelike hypersurface. Translating the hypersurface along t^α gives a family of hypersurfaces that we can label by a coordinate t for which $t^\alpha \nabla_\alpha t = 1$

Theorem (Masood-ul-Alam):

A static, asymptotically flat spacetime with a perfect-fluid source is spherically symmetric.

(1-parameter EOS, minimal assumptions)

Equilibria of nonrotating neutron stars are accurately modelled as static, spherically symmetric, asymptotically flat perfect-fluids. They satisfying the TOV equation with which you are all familiar. Here's a check that we get the equilibrium equation from the general form of the Euler equation

$$ds^2 = -e^{2\Phi}dt^2 + \frac{1}{1-2m(r)/r}dr^2 + r^2 d\Omega^2$$

$$u^\alpha = u^t t^\alpha,\ u^\alpha u_\alpha = -1 \Rightarrow u^t = e^{-\Phi}$$

$$\nabla_\alpha \ln u^t = \frac{\nabla_\alpha P}{\epsilon + P}$$ becomes

$$\Phi' = -\frac{P'}{\epsilon + P}$$

The $G^t{}_t = 8\pi T^t{}_t$ and $G^r{}_r = 8\pi T^r{}_r$ equations then give λ and Φ in the form

$$e^{2\lambda} = \frac{1}{1-\frac{2m}{r}},$$ with $m = \int_0^r \epsilon 4\pi r^2 dr,$

$$\Phi' = \frac{m + 4\pi P r^3}{r(r-2m)}.$$

Equating the two expressions for Φ 'we recover the TOV equation

$$\frac{dP}{dr} = -(\epsilon + P)\frac{m + 4\pi P r^3}{r(r-2m)} \quad .$$

One can obtain a star by integrating this equation, together with the defining equation for $m(r)$ and an equation of state, until the pressure drops to zero. Φ is fixed outside the star by

Φ=-λ , inside by

$$\frac{h}{u^t} = \mathscr{E} \Rightarrow \Phi = \Phi|_S - \ln h.$$

Rotating relativistic stars

The metric $g_{\alpha\beta}$ of a stationary axisymmetric rotating fluid has two commuting Killing vectors, t^α and ϕ^α, generating time translations and rotations.

Remark: Although in realistic stellar models, t^α is everywhere timelike, within a horizon or in an *ergosphere* of an exceptionally compact rotating star, t^α will be spacelike. The ergosphere is by definition the region in which an an asymptotically timelike Killing vector becomes spacelike.

As before, the fluid velocity has the form $u^\alpha = u^t(t^\alpha + \Omega\phi^\alpha)$, and the equation of hydrostatic equilibrium has the first integral

$$\frac{h}{u^t} = \mathscr{E}\,.$$

Geometry of a rotating star

The metric $g_{\alpha\beta}$ can be written in terms of dot-products of the Killing vectors, t^α and ϕ^α

$$t^\alpha t_\alpha, \qquad t^\alpha \phi_\alpha, \qquad \phi^\alpha \phi_\alpha,$$

and a conformal factor $e^{2\mu}$ that characterizes the geometry of the orthogonal 2-surfaces.

$$g^{tt} = \nabla_\alpha t \nabla^\alpha t = -e^{-2\nu}, \quad g_{\phi\phi} = \phi^\alpha \phi_\alpha = e^{2\psi}, \quad g_{t\phi} = t^\alpha \phi_\alpha = -\omega e^{2\psi}$$

Then

$$g_{tt} = t^\alpha t_\alpha = -e^{2\nu} + \omega^2 e^{2\psi}$$

and

$$ds^2 = -e^{2\nu} dt^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu}(d\varpi^2 + dz^2)$$

where ϖ (script pi) is a cylindrical coordinate (in flat space $\varpi = \sqrt{x^2 + y^2}$).

Because of the choice of an overall conformal factor $e^{2\mu}$ to describe the geometry of the surfaces, the exterior of a spherical star is the Schwarzschild geometry in isotropic coordinates,

$$e^{\nu} = \frac{1 - M/2r}{1 + M/2r}, \quad e^{\psi} = \varpi(1 + M/2r)^2, \qquad e^{\mu} = (1 + M/2r)^2$$

Asymptotically, the relations
hold for the potentials, because any stationary, asymptotically flat spacetime agrees with the Schwarzschild geometry to order r^{-1}.

$$e^{\psi} = \varpi\left(e^{-\nu} + O(r^{-2})\right), \quad e^{\mu} = e^{-\nu} + O(r^{-2}),$$

The angular velocity

$$\omega \equiv -t^\alpha \phi_\alpha / \phi^\beta \phi_\beta$$

measures the dragging of inertial frames in the sense that particles with zero angular momentum move along trajectories whose angular velocity relative to infinity is

$$\frac{d\phi}{dt} = \omega$$

In describing the fluid, it is helpful to introduce a family of zero-angular-momentum-observers (ZAMOs) \cite{Bardeen70,Bardeen73}, observers whose velocity has at each point the form (for circular, axisymmetric motion)

$$u^\alpha_{\text{ZAMO}} = u^t(t^\alpha + \omega\phi^\alpha) = e^{-\nu}(t^\alpha + \omega\phi^\alpha)$$

The worldlines of these observers are normal to the $t=$ constant hypersurfaces. Because spacetime is locally flat, the local observers can express velocities in the way one does in flat space, in terms of an orthonormal tetrad.

A natural tetrad is the frame of zero-angular-momentum-observers (ZAMOs), with basis covectors

$$\omega^{\hat{0}} = e^{\nu} dt, \quad \omega^{\hat{1}} = e^{\psi}(d\phi - \omega dt), \quad \omega^{\hat{2}} = e^{\mu} d\varpi, \quad \omega^{\hat{3}} = e^{\mu} dz,$$

and the corresponding contravariant basis vectors are

$$e_{\hat{0}} = e^{-\nu}(\partial_t + \omega\partial_\phi), \quad e_{\hat{1}} = e^{-\psi}\partial_\phi, \quad e_{\hat{2}} = e^{-\mu}\partial_\varpi, \quad e_{\hat{3}} = e^{-\mu}\partial_z.$$

The nonzero components of the four velocity u^α along these frame vectors can be written in terms of a fluid 3-velocity v in the manner

$$u^{\hat{0}} = \frac{1}{\sqrt{1-v^2}}, \qquad u^{\hat{1}} = \frac{v}{\sqrt{1-v^2}}.$$

$$u^t = u^\alpha \nabla_\alpha t = \frac{e^{-\nu}}{\sqrt{1-v^2}}, \qquad u^\phi = u^\alpha \nabla_\alpha \phi = \Omega u^t, \tag{1}$$

where Ω is the angular velocity of the fluid relative to infinity (measured by an asymptotic observer with 4-velocity along the asymptotically timelike Killing vector t^α). The 3-velocity v, written in terms of Ω, is

$$v = e^{\psi-\nu}(\Omega - \omega).$$

Note that $2\pi e^\psi$ is the circumference of a circle centered about the axis of symmetry (the z-axis); that is, e^ψ agrees for spherical stars with $r \sin\theta$, where r and θ are the usual Schwarzschild coordinates (not the isotropic coordinates introduced above).

The nonvanishing tetrad components of $T^{\alpha\beta}$ are

$$T^{\hat{0}\hat{0}} = \frac{\epsilon + pv^2}{1 - v^2}, \qquad T^{\hat{0}\hat{1}} = (\epsilon + p)\frac{v}{1 - v^2},$$

$$T^{\hat{1}\hat{1}} = \frac{\epsilon v^2 + p}{1 - v^2}, \qquad T^{\hat{2}\hat{2}} = T^{\hat{3}\hat{3}} = p.$$

The four potentials are determined by four components of the field equation

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}, \tag{2}$$

whose selection is a matter of taste. Following Bardeen and Wagoner (1971), Butterworth and Ipser (1976) and several subsequent authors based their code on the following four equations:pdfl

In these equations, ∇ is the *flat* 3-dimensional covariant derivative operator of the metric $d\varpi^2 + dz^2 + \varpi^2 d\phi^2$. The equations are simplified by writing $e^{\psi} = \varpi B e^{-\nu}$. This defines a potential B, and $B = 1$ to first post-Newtonian order.

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot(B\boldsymbol{\nabla}\nu) &= \frac{1}{2} r^2 \sin^2\theta B^3 e^{-4\nu} \boldsymbol{\nabla}\omega \cdot \boldsymbol{\nabla}\omega \\
&\quad + 4\pi B e^{2\mu-\nu}\left[\frac{(\epsilon+p)(1+v^2)}{1-v^2} + 2p\right], \\
\boldsymbol{\nabla}\cdot(r^2\sin^2\theta B^3 e^{-4\nu}\boldsymbol{\nabla}\omega) &= -16\pi r \sin\theta B^2 e^{2\mu-3\nu}\frac{(\epsilon+p)v}{1-v^2}, \\
\boldsymbol{\nabla}\cdot(r\sin\theta\boldsymbol{\nabla}B) &= 16\pi r\sin\theta B e^{2\mu-2\nu} p,
\end{aligned}
$$

(these are, respectively the $R_{\hat{0}\hat{0}}, R_{\hat{0}\hat{1}}$, and $R_{\hat{0}\hat{0}} - R_{\hat{1}\hat{1}}$ field equations), As a final equation, for μ, one can take (with $e^\beta = \varpi B$),

$$
\begin{aligned}
& e^{2\mu} G^{\hat{2}\hat{3}} = G_{\varpi z} : \\
& \mu_{,\varpi}\, \beta_{,z} + \mu_{,z}\, \beta_{,\varpi} \quad = \quad \beta_{,\varpi z} + \beta_{,\varpi}\, \beta_{,z} + 2\nu_{,\varpi}\, \nu_{,z} - \beta_{,z}\, \nu_{,\varpi} \\
& \qquad\qquad\qquad\qquad + \frac{1}{2} e^{2\beta - 4\nu - 2\mu} \omega_{,\varpi}\, \omega_{,z} \, .
\end{aligned}
$$

Alternatively can use a 4th elliptic equation for μ.

Living Reviews: Nick Stergioulas
Rotating relativistic stars, JF and Nick Stergioulas, Cambridge, in press.

Codes by Wilson; Bonazzola & Schneider; Butterworth & Ipser; JF, Ipser, Parker; Lattimer et al; Bonazzola, Gourgoulhon, Salgado, Marck; Ansorg, Kleinwachter, Meinel; Komatsu, Eriguchi, Hachisu; Cook, Shapiro, Teukolsky;
Stergioulas: *rns, a public domain code, available at*
http://www.gravity.phys.uwm.edu/rns.

Code Comparison

	AKM	Lorene/ rotstar	SF (260x400)	BGSM	KEH
$\bar{p}_c$	1				
r_p/r_e	0.7			1e-3	
$\bar{\Omega}$	1.41170848318	9e-6	3e-4	1e-2	1e-2
$\bar{M}$	0.135798178809	2e-4	2e-5	9e-3	2e-2
$\bar{M}_0$	0.186338658186	2e-4	2e-4	1e-2	2e-3
$\bar{R}_{circ}$	0.345476187602	5e-5	3e-5	3e-3	1e-3
$\bar{J}$	0.0140585992949	2e-5	4e-4	2e-2	2e-2
Z_p	1.70735395213	1e-5	4e-5	2e-2	6e-2
Z_{eq}^f	−0.162534082217	2e-4	2e-3	4e-2	2e-2
Z_{eq}^b	11.3539142587	7e-6	7e-5	8e-2	2e-1
$\lvert\mathrm{GRV3}\rvert$	$4e-13$	3e-6	3e-5	4e-3	1e-1

Table: Table adapted from Stergioulas et al. 03.

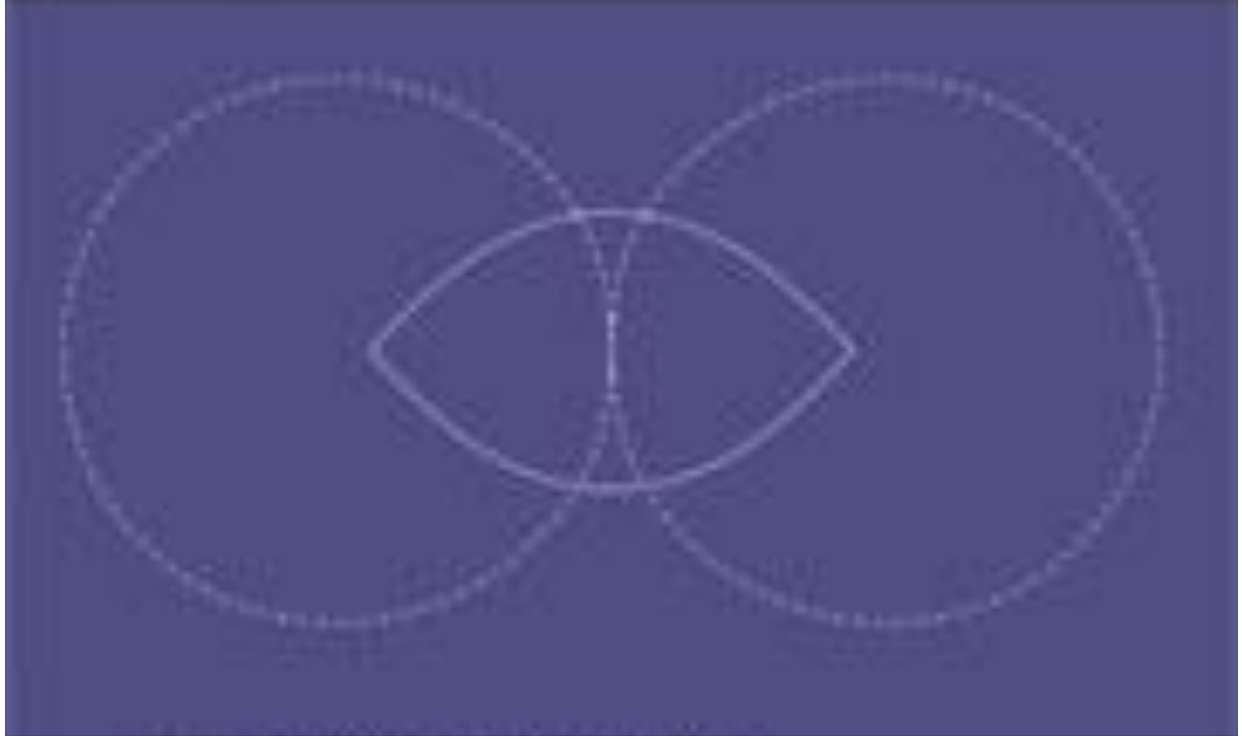
Method:

1. Start with a guessed solution (e.g., for a spherical configuration). Solve the 4 field equations by Newton-Raphson, putting the linearized operator on the left side and the nonlinear terms on the right. (KEH solve by keeping only a flat-space laplacian on the each left side and solving by using the known Green's function).
2. Update h from the first integral of the equation of hydrostatic equilibrium, and use the EOS to find P, ϵ.
3. Find the new surface of the star.
4. Use the updated ϵ, P and the updated potentials to recompute the right-hand sides of the field equations.

$5 \equiv 1.$

Use spherical harmonics (Legendre polynomials) or Chebyshev polynomials for the θ dependence. For r dependence, directly specify function on the grid, using finite differences for radial derivatives, or use spectral decomposition with Chebyshev polynomials.

Instead of representing a function of r by its values on the grid, on can, as in the case of the angular dependence, write the function as a sum of orthogonal polynomials (typically Chebyshev polynomials) and soving numerically for the coefficients. This is called a *spectral method*. The accuracy of spectral methods was initially limited by the Gibbs phenomenon at the stellar surface, but the most recent spectral codes by the Meudon group and by Ansorg et al. overcome the problem by using two or three domains fitted to the stellar surface. Ansorg et al. obtain near-machine accuracy with two domains and a Chebyshev expansion for both r and θ.



Meinel, Ansorg, Kleinwachter, Neugebauer, Petroff

The solid curve shows the surface of a uniformly rotating, uniform-density star rotating at maximum angular velocity Ω_K: The star rotates at the angular velocity of a satellite in circular orbit at the equator. The two dotted lobes mark the boundaries of the ergosphere. Uniformly rotating stars with realistic equations of state, however, reach Ω_K before an ergosphere appears.

The set of equilibrium configurations of a uniformly rotating star is two-dimensional, specified, for example, by M_0 and Ω. The 2-dimensional surface of equilibria shown on the next page is ruled by lines of constant J and M_0. For fixed J, the maximum mass configuration accurately approximates the onset of instability to collapse. As we discuss in the next part of these notes, it is exact only for spherical stars.
(JF, Ipser, Sorkin; Katami, Rezzolla, Yoshida.)

Although not shown in the figure, at low density there is a similar line of minimum mass configurations. Below the minimum mass, configurations are unstable to explosion - they are unbound.
Candidates for realistic equations of state typically have maximum masses for uniform rotation below $2.5M_\odot$. (See, e.g. Cook, Shapiro, Teukolsky, for a representative sample of candidate EOSs).
A hard upper limit on the mass of uniformly rotating, self-gravitating stars is found by using the stiffest EOS consistent with causality ($v_{sound} = dP/d\epsilon = 1$), matching at a density ϵ_m to a known low-density EOS.

$$M < 6.1M_\odot \left(\frac{2 \times 10^{14}\text{g/cm}^3}{\epsilon_m} \right)^{1/2} \tag{6}$$

JF,Ipser; Koranda, Stergioulas, JF

M and J for a rotating star

For an axisymmetric system the angular momentum current $j^\alpha = T^\alpha_\beta \phi^\beta$ is conserved: Killing's equation implies

$$\nabla_\alpha j^\alpha = 0.$$

The angular momentum of an asymptotically flat, axisymmetric spacetime is then given by

$$J = \int_V j^\alpha dS_\alpha = \int_V T^\alpha_\beta \phi^\beta dS_\alpha$$

For a Killing vector ξ^α, $\nabla_\beta \xi^\beta = 0 \Rightarrow$

$$\nabla_\beta \nabla^\alpha \xi^\beta = (\nabla_\beta \nabla^\alpha - \nabla^\alpha \nabla_\beta)\xi^\beta = R_\beta{}^\alpha{}_{\beta\gamma}\xi^\gamma = R^\alpha_\beta \xi^\beta.$$

Now the Einstein equation implies

$$J = \frac{1}{8\pi}\int_V R^\alpha_\beta \phi^\beta dS_\alpha = \frac{1}{8\pi}\int_V \nabla_\beta \nabla^\alpha \phi^\beta dS_\alpha$$

and by Gauss's theorem we have

$$J = \frac{1}{8\pi}\int_\infty \nabla^\alpha \phi^\beta dS_{\alpha\beta}$$

where $dS_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta} dS^{\gamma\delta}$

The expression for the mass is similar. One of today's problems is a short calculation that gives

$$M = -\frac{1}{4\pi}\int_\infty \nabla^\alpha t^\beta dS_{\alpha\beta},$$

for a stationary, axymptotically flat spacetime. This is the Komar expression for the mass. Then

$$\begin{aligned} M &= -\frac{1}{4\pi}\int_V \nabla_\beta \nabla^\alpha t^\beta dS_\alpha = -\frac{1}{4\pi}\int_V R^\alpha_\beta t^\beta dS_\alpha \\ &= -\int_V (2T^\alpha_\beta - \frac{1}{2}\delta^\alpha_\beta T) t^\beta dS_\alpha \end{aligned}$$

Injection Energy (Thorne, Carter)

The quantity $\mathcal{E}$ has a natural physical interpretation, as the energy per unit mass needed to inject matter into the star, with the injected fluid in the same local state (same composition, density, and entropy per baryon as the surrounding star). We will compute the initial energy δM needed to inject a ring of fluid into a rotating star, after dropping it to its new location, a circle C about the axis of symmetry.

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When the freely falling box reaches a point P in the star, its energy measured by a comoving observer (at rest with respect to the fluid) is

$$\delta E_1 = -p_{1\alpha} u^\alpha,$$

where $u^\alpha = u^t(t^\alpha + \Omega\phi^\alpha)$ is the fluid four-velocity.

$$\begin{aligned}\delta E_1 &= -u^t(p_{1\alpha}t^\alpha + \Omega p_{1\alpha}\phi^\alpha) \\ &= u^t(\delta M_1 - \Omega\delta J),.\end{aligned}$$

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Now, following Thorne, we suppose the observer catches the box at P and reversibly injects its contents into the fluid, imparting to the fluid an energy

$$\delta E = T\delta S + \mu\delta N.$$

Because the initial entropy per baryon was already s, not all of the available energy is used: Our active observer uses the remaining energy to throw the empty box back up to infinity, on a trajectory with zero angular momentum, so that the angular momentum δJ is retained by the fluid. The returning box has energy $\delta E_2 = \delta E_1 - \delta E = p_{2\alpha}u^\alpha$, with $p_{2\alpha}$ the momentum with which it is thrown. Because its free trajectory conserves $p_{2\alpha}t^\alpha$, the box reaches infinity with redshifted energy
$\delta M_2 = p_{2\alpha}t^\alpha = \dfrac{\delta E_2}{u^t}$.

The change in mass of the star is then related to the change in baryon number, entropy, and angular momentum of the fluid by

$$\begin{aligned} \delta M &= \delta M_1 - \delta M_2 = \frac{1}{u^t}(\delta E_1 - \delta E_2) + \Omega\delta J = \frac{\delta E}{u^t} + \Omega\delta J \\ &= \frac{\mu}{u^t}\delta N + \frac{T}{u^t}\delta S + \Omega\delta J \qquad (7) \\ &= \mathcal{E}\delta M_0 + \frac{T}{u^t}\delta S + \Omega\delta J. \qquad (8) \end{aligned}$$

The coefficient of δM_0 is the energy $\mathcal{E}$, the injection energy per unit rest mass of matter with zero initial entropy and angular momentum. Eq. (7) (or 8) is the first law of thermodynamics for relativistic stars (in two equivalent forms).

Why is $\mathcal{E}$ is constant in a star with constant entropy per baryon and constant angular velocity Ω? An equilibrium configuration is an extremum of mass at fixed angular momentum, entropy and baryon number: Small changes in the structure of the star leave the mass fixed. In particular, suppose one moves a ring of fluid from one location to another in a uniformly rotating white dwarf or neutron star, stars that are approximately barotropic because T is approximately zero (that is, $kT << \varepsilon_F$).

Changing the location of the ring is equivalent to moving it out to infinity and back in to a new location in the star. According to Eq. (8) with $T = 0$

$$\delta M = (\mathcal{E}_2 - \mathcal{E}_1)\delta M_0 + (\Omega_2 - \Omega_1)\delta J.$$

Uniform rotation: $\Omega_1 = \Omega_2$ Then $\delta M = 0 \quad \Longrightarrow$

$$\mathcal{E}_2 = \mathcal{E}_1, \tag{9}$$

and we conclude that $\mathcal{E}$ is constant throughout the star.

When the spacetime includes a black hole – e.g., for a black hole and a disk – the first law includes terms associated with the area and angular momentum of the black hole:

$$\delta M = \mathcal{E}\delta M_0 \quad + \frac{T}{u^t}\delta S + \Omega \delta J$$
$$+ \kappa \; \delta A \; + \Omega_{\rm BH}\delta J_{\rm BH}.$$