Lecture 3

# Review of General Relativity

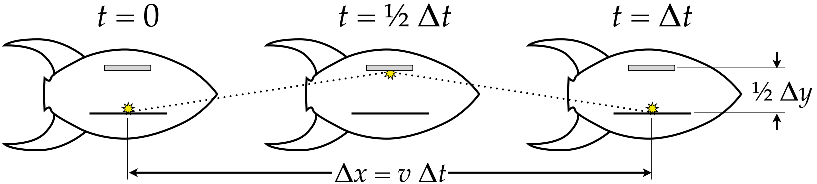
Jolien Creighton

University of Wisconsin–Milwaukee

July 16, 2012

- Whirlwind review of differential geometry
  - Coordinates and distances
  - Vectors and connections
  - Lie derivative
  - Curvature
- Formulation of General Relativity
  - Weak gravity and slow motion
  - Matter
  - Einstein field equations
- Linearized gravity
  - Newtonian limit
  - Plane wave solution

# Coordinates and distances




Measure the speed of light

- Observer in rocket: $c = \dfrac{\Delta y}{\Delta \tau}$
- Observer on Earth: $c^2 = \dfrac{(\Delta x)^2 + (\Delta y)^2}{(\Delta t)^2} = \dfrac{(\Delta x)^2 + (c\Delta \tau)^2}{(\Delta t)^2}$

$$c^2(\Delta \tau)^2 = c^2(\Delta t)^2 - (\Delta x)^2$$

# Coordinates and distances

Let $ds$ be the distance between neighboring events

- Curved spacetime:
$$ds^2 = g_{\mu\nu}(x^\alpha) dx^\mu dx^\nu$$

- Flat spacetime:
$$\begin{aligned} ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu \\ &= -c^2 dt^2 + dx^2 + dy^2 + dz^2 \\ &\qquad \text{(rectilinear coordinates)} \end{aligned}$$

- Equivalence principle:
A freely-falling frame defines locally-flat neighborhood of an event in spacetime.

# Coordinates and distances

- General coordinate transformation:

$$x'^{\alpha}(x^{\mu}) \Rightarrow dx'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} x^{\mu}$$

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = g'_{\alpha\beta} dx'^{\alpha} dx'^{\beta}$$

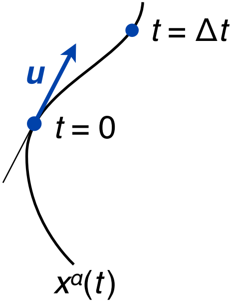
$$g'_{\alpha\beta} = g_{\mu\nu} \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}}$$

- Infinitesimal coordinate transformation:

$$x'^{\alpha} = x^{\alpha} + \xi^{\alpha}(x^{\mu}) \Rightarrow dx'^{\alpha} = dx^{\alpha} + \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} dx^{\mu}$$

$$g'_{\alpha\beta} = g_{\alpha\beta} - g_{\alpha\mu} \frac{\partial \xi^{\mu}}{\partial x^{\beta}} - g_{\mu\beta} \frac{\partial \xi^{\mu}}{\partial x^{\alpha}}$$

# Vectors and connections




On a curve $x^{\alpha}(t)$ define a function

$$f(t) = F\big(x^{\alpha}(t)\big)$$

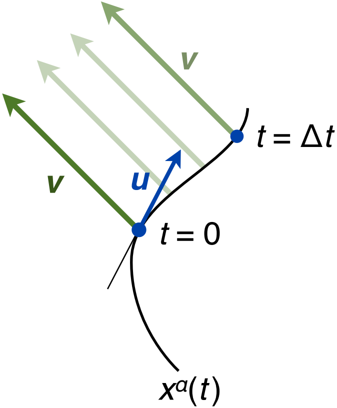
where $F(x^{\alpha})$ is some function on spacetime.

$$\frac{df}{dt} = \frac{dx^{\mu}}{dt}\frac{\partial F}{\partial x^{\mu}} u^{\mu}\frac{\partial F}{\partial x^{\mu}}$$

$\mathbf{u} = \dfrac{d}{dt}$ is the **tangent vector** to the curve with components

$$u^{\alpha} = \frac{dx^{\alpha}}{dt} \quad \text{along curve}$$

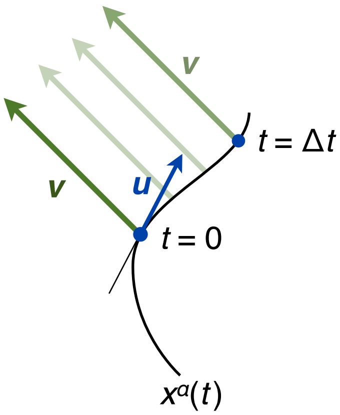
# Vectors and connections




Parallel transport **v** along curve defined by **u**:

1. Go into a locally flat reference frame
2. Take an infinitesimal step by parallel transport in the usual sense of flat space
3. Repeat steps 1 and 2 until final point is reached

# Vectors and connections

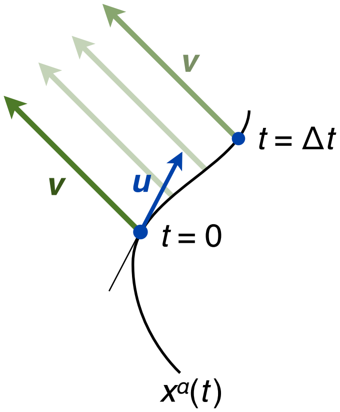



For each step, require

$$\begin{aligned} 0 &= \lim_{\Delta t \to 0} \frac{v^\alpha(t+\Delta t) - v^\alpha(t)}{\Delta t} \\ &= \frac{dv^\alpha}{dt} \\ &= u^\mu \nabla_\mu v^\alpha \end{aligned}$$

$\nabla_\mu$ is the **covariant derivative**

# Vectors and connections




- In locally inertial reference frame

$$\nabla_\mu = \frac{\partial}{\partial x^\mu}$$

- In other coordinates

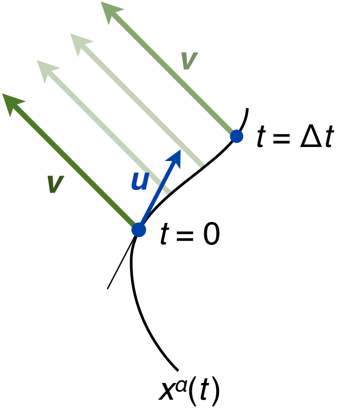
$$\nabla_\mu v^\alpha = \frac{\partial v^\alpha}{\partial x^\mu} + \Gamma^\alpha_{\mu\nu} v^\nu$$

$\Gamma^\alpha_{\mu\nu}$ are **connection coefficients**

$$\Gamma^\alpha_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\beta} \frac{\partial^2 x'^\beta}{\partial x^\mu \partial x^\nu}$$

where $x'^\alpha(x^\mu)$ is the transformation to an inertial reference frame

# Vectors and connections




Parallel transport preserves the inner product $\mathbf{v}\cdot\mathbf{w} = g_{\alpha\beta}v^\alpha w^\beta$

$$\begin{aligned} 0 &= \frac{d}{dt}(\mathbf{v}\cdot\mathbf{w}) \\ &= u^\mu \nabla_\mu (g_{\alpha\beta} v^\alpha w^\beta) \end{aligned}$$

so $\nabla_\mu g_{\alpha\beta} = 0$

This gives

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left( \frac{\partial g_{\beta\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\beta}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta} \right)$$

# Vectors and connections

A curve is a **geodesic** if its tangent vector is parallel-transports itself along the curve:

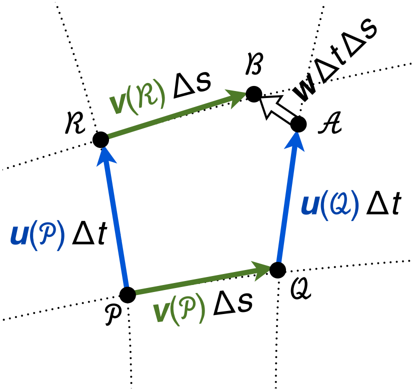
$$\begin{aligned} 0 &= u^{\mu} \nabla_{\mu} u^{\alpha} \\ &= \frac{dx^{\mu}}{dt} \left( \frac{\partial}{\partial x^{\mu}} \frac{dx^{\alpha}}{dt} + \Gamma^{\alpha}_{\mu\nu} \frac{dx^{\nu}}{dt} \right) \end{aligned}$$

Geodesic is defined by four coupled ordinary differential equations:

$$\frac{d^2 x^{\alpha}}{dt^2} = -\Gamma^{\alpha}_{\mu\nu} \frac{dx^{\mu}}{dt} \frac{dx^{\nu}}{dt}$$

Specify $dx^{\mu}/dt$ at a point $x^{\mu}$ and integrate to generate a geodesic

# Vectors and connections




Vector *fields* do not necessarily *commute*:

- Move $\Delta t$ along **u** and then $\Delta s$ along **v** to get to $\mathcal{B}$
- Move $\Delta s$ along **v** and then $\Delta t$ along **u** to get to $\mathcal{A}$

$$
\begin{aligned}
w^\alpha &= \lim_{\Delta s\to 0}\lim_{\Delta t\to 0} \frac{\{[u^\alpha(\mathcal{P})\Delta t + v^\alpha(\mathcal{R})\Delta s] - [v^\alpha(\mathcal{P})\Delta s + u^\alpha(\mathcal{Q})\Delta t]\}}{\Delta t\,\Delta s} \\
&= \underbrace{\lim_{\Delta t\to 0}\frac{v^\alpha(\mathcal{R}) - v^\alpha(\mathcal{P})}{\Delta t}}_{u^\mu\nabla_\mu v^\alpha} - \underbrace{\lim_{\Delta t\to 0}\frac{u^\alpha(\mathcal{Q}) - u^\alpha(\mathcal{P})}{\Delta t}}_{v^\mu\nabla_\mu u^\alpha}
\end{aligned}
$$

# Vectors and connections

$\mathbf{w} = [\mathbf{u}, \mathbf{v}]$ is the *commutator* of $\mathbf{u}$ and $\mathbf{v}$

$$w^\alpha = u^\mu \frac{\partial v^\alpha}{\partial x^\mu} - v^\mu \frac{\partial u^\alpha}{\partial x^\mu}$$

If two vectors commute, $[\mathbf{u}, \mathbf{v}] = 0$, then

- going $\Delta t$ along $\mathbf{u}$ and then going $\Delta s$ along $\mathbf{v}$

arrives at the same point as

- going $\Delta s$ along $\mathbf{v}$ and then going $\Delta t$ along $\mathbf{u}$

We can use $(s, t)$ to label the points ("coordinates"):
$\mathbf{u}$ and $\mathbf{v}$ are *coordinate basis* vectors

# Lie derivative

Consider a fluid flow with velocity field $\mathbf{u}(\mathbf{x})$

A scalar field $\rho(\mathbf{x})$ is *dragged along* by the flow if the value remains constant on a fluid element:

$$0 = \frac{d}{dt}\rho\big(\mathbf{x}(t)\big) = \mathbf{u}\cdot\boldsymbol{\nabla}\rho$$

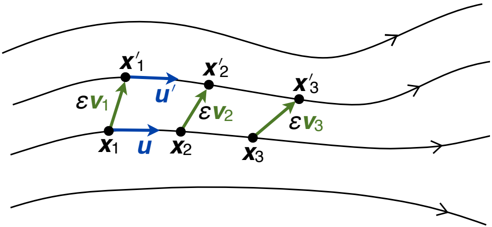
where $\mathbf{u} = d\mathbf{x}/dt$

The scalar field $\rho$ is then said to be *Lie-derived* by the vector field $\mathbf{u}$; the **Lie derivative** of $\rho$, defined by

$$\mathcal{L}_{\mathbf{u}}\rho = \mathbf{u}\cdot\boldsymbol{\nabla}\rho$$

is the rate of change of $\rho$ measured by a co-moving observer

# Lie derivative




Consider now a vector $\epsilon \mathbf{v}$ that joins two nearby fluid elements, $\epsilon \mathbf{v} = \mathbf{x}'(t) - \mathbf{x}(t)$. This vector is *dragged along* by the fluid flow.

$$\mathbf{u}' = \frac{d}{dt}\mathbf{x}'(t) = \frac{d}{dt}\Big(\mathbf{x}(t) + \epsilon \mathbf{v}\big(\mathbf{x}(t)\big)\Big) = \mathbf{u}(\mathbf{x}) + \epsilon \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{v}(\mathbf{x})$$

$$\mathbf{u}' = \mathbf{u}(\mathbf{x}') = \mathbf{u}(\mathbf{x} + \epsilon \mathbf{v}) = \mathbf{u}(\mathbf{x}) + \epsilon \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{u}(\mathbf{x}) + O(\epsilon^2)$$

so

$$[\mathbf{u}, \mathbf{v}] = \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{v} - \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{u} = 0$$

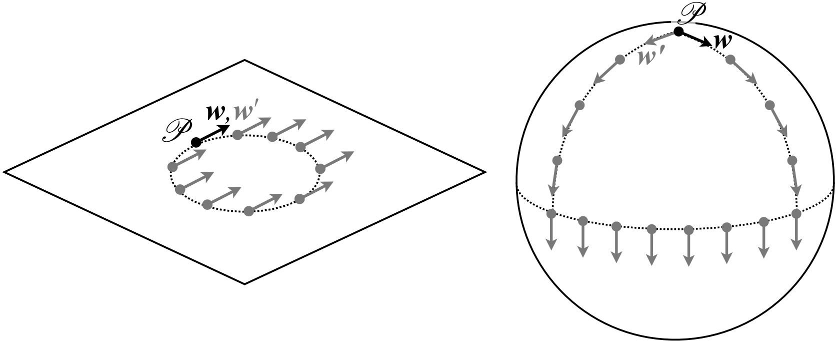
# Lie derivative

The Lie derivative of the vector $\mathbf{v}$ along the vector $\mathbf{u}$ is the commutator:

$$\mathcal{L}_{\mathbf{u}}\mathbf{v} = [\mathbf{u}, \mathbf{v}]$$

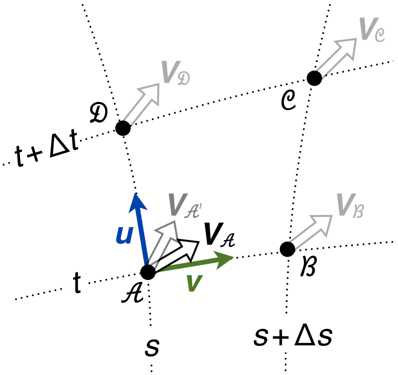
$$\mathcal{L}_{\mathbf{u}} v^{\alpha} = u^{\mu}\nabla_{\mu} v^{\alpha} - v^{\mu}\nabla_{\mu} u^{\alpha} = u^{\mu}\frac{\partial v^{\alpha}}{\partial x^{\mu}} - v^{\mu}\frac{\partial u^{\alpha}}{\partial x^{\mu}}$$

# Curvature




- In flat space: parallel transport of a vector about a closed path leaves the vector unchanged
- In curved space: vector is generally different after parallel transport about a closed path

# Curvature




Use: $u^\mu \dfrac{\partial}{\partial x^\mu} V^\alpha = -\Gamma^\alpha_{\mu\sigma} V^\sigma u^\mu$

0. Start at point $\mathcal{A}$ with vector $V^\alpha_{\mathcal{A}}$

1. $V^\alpha_{\mathcal{B}} = V^\alpha_{\mathcal{A}} - \displaystyle\int_{(s,t)}^{(s+\Delta s,t)} \Gamma^\alpha_{\mu\sigma} V^\sigma v^\mu ds$

2. $V^\alpha_{\mathcal{C}} = V^\alpha_{\mathcal{B}} - \displaystyle\int_{(s+\Delta s,t)}^{(s+\Delta s,t+\Delta t)} \Gamma^\alpha_{\mu\sigma} V^\sigma u^\mu dt$

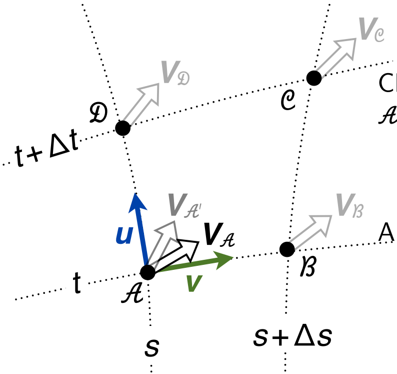
3. $V^\alpha_{\mathcal{D}} = V^\alpha_{\mathcal{C}} - \displaystyle\int_{(s+\Delta s,t+\Delta t)}^{(s,t+\Delta t)} \Gamma^\alpha_{\mu\sigma} V^\sigma v^\mu ds$

4. $V^\alpha_{\mathcal{A}'} = V^\alpha_{\mathcal{D}} - \displaystyle\int_{(s,t+\Delta t)}^{(s,t)} \Gamma^\alpha_{\mu\sigma} V^\sigma u^\mu dt$

# Curvature

$$
\begin{aligned}
\Delta V^{\alpha} &= V^{\alpha}_{\mathcal{A}'} - V^{\alpha}_{\mathcal{A}} \\
&= \int_{(s,t)}^{(s,t+\Delta t)} \Gamma^{\alpha}_{\mu\sigma} V^{\sigma} u^{\mu}\, dt - \int_{(s+\Delta s,t)}^{(s+\Delta s,t+\Delta t)} \Gamma^{\alpha}_{\mu\sigma} V^{\sigma} u^{\mu}\, dt \\
&\quad + \int_{(s,t+\Delta t)}^{(s+\Delta s,t+\Delta t)} \Gamma^{\alpha}_{\mu\sigma} V^{\sigma} v^{\mu}\, ds - \int_{(s,t)}^{(s+\Delta s,t)} \Gamma^{\alpha}_{\mu\sigma} V^{\sigma} v^{\mu}\, ds \\
&\approx -\int_{t}^{t+\Delta t} \Delta s\, v^{\nu} \frac{\partial}{\partial x^{\nu}} (\Gamma^{\alpha}_{\mu\sigma} V^{\sigma})\, u^{\mu}\, dt + \int_{s}^{s+\Delta t} \Delta t\, u^{\nu} \frac{\partial}{\partial x^{\nu}} (\Gamma^{\alpha}_{\mu\sigma} V^{\sigma})\, v^{\mu}\, ds \\
&\approx \Delta s \Delta t \left[ -u^{\mu} v^{\nu} \frac{\partial}{\partial x^{\nu}} (\Gamma^{\alpha}_{\mu\sigma} V^{\sigma}) + v^{\mu} u^{\nu} \frac{\partial}{\partial x^{\nu}} (\Gamma^{\alpha}_{\mu\sigma} V^{\sigma}) \right] \\
&= \Delta s \Delta t \underbrace{\left[ -\frac{\partial}{\partial x^{\nu}} \Gamma^{\alpha}_{\mu\sigma} + \Gamma^{\alpha}_{\mu\rho} \Gamma^{\rho}_{\nu\sigma} + \frac{\partial}{\partial x^{\mu}} \Gamma^{\alpha}_{\nu\sigma} - \Gamma^{\alpha}_{\nu\rho} \Gamma^{\rho}_{\mu\sigma} \right]}_{-R_{\mu\nu\sigma}{}^{\alpha}} u^{\mu} v^{\nu} V^{\sigma}
\end{aligned}
$$

# Curvature




Change in $V^\alpha$ parallel transported $\mathcal{A} \to \mathcal{B} \to \mathcal{C} \to \mathcal{D} \to \mathcal{A}'$ is

$$\Delta V^\alpha = -(\Delta s \Delta t) R_{\mu\nu\sigma}{}^{\alpha} u^\mu v^\nu V^\sigma$$

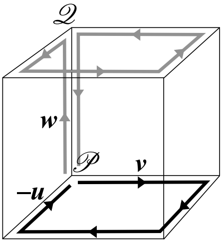
Also:

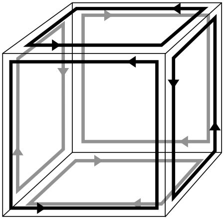
$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\alpha = -R_{\mu\nu\sigma}{}^{\alpha} V^\sigma$$

$$\boxed{R_{\mu\nu\sigma}{}^{\alpha} = -\frac{\partial}{\partial x^\mu}\Gamma^\alpha_{\nu\sigma} + \frac{\partial}{\partial x^\nu}\Gamma^\alpha_{\mu\sigma} - \Gamma^\alpha_{\mu\rho}\Gamma^\rho_{\nu\sigma} + \Gamma^\alpha_{\nu\rho}\Gamma^\rho_{\mu\sigma}}$$

**Riemann curvature tensor**

# Bianchi identity



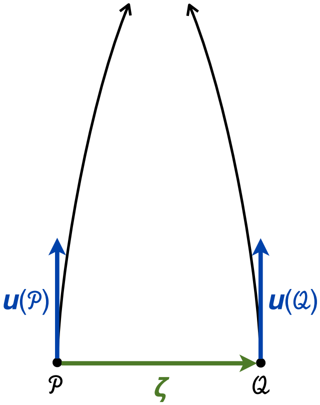
$$\Delta V^\alpha = \underbrace{\epsilon^2 R_{\mu\nu\sigma}{}^\alpha(\mathcal{Q}) u^\mu v^\nu V^\sigma}_{\text{top face}}$$
$$\underbrace{-\epsilon^2 R_{\mu\nu\sigma}{}^\alpha(\mathcal{P}) u^\mu v^\nu V^\sigma}_{\text{bottom face}}$$
$$= \epsilon^3 (w^\rho \nabla_\rho R_{\mu\nu\sigma}{}^\alpha) u^\mu v^\nu V^\sigma$$

$$0 = \Delta V^\alpha \quad \text{(all faces)}$$
$$= \epsilon^3 (\nabla_\rho R_{\mu\nu\sigma}{}^\alpha + \nabla_\mu R_{\nu\rho\sigma}{}^\alpha$$
$$+ \nabla_\nu R_{\rho\mu\sigma}{}^\alpha) u^\mu v^\nu w^\rho V^\sigma$$

$$\boxed{\nabla_\rho R_{\mu\nu\sigma}{}^\alpha + \nabla_\mu R_{\nu\rho\sigma}{}^\alpha + \nabla_\nu R_{\rho\mu\sigma}{}^\alpha = 0}$$

**Bianchi identity**

# Geodesic deviation




Let $\zeta = d/dx$ be the separation vector between two geodesics

Relative velocity of the two bodies

$$\mathbf{v} = \frac{d\boldsymbol{\zeta}}{dt} = \mathbf{u} \cdot \boldsymbol{\nabla} \boldsymbol{\zeta} = \boldsymbol{\zeta} \cdot \boldsymbol{\nabla} \mathbf{u}$$

Relative acceleration of the two bodies

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \mathbf{u} \cdot \boldsymbol{\nabla} (\boldsymbol{\zeta} \cdot \boldsymbol{\nabla} \mathbf{u})$$

Use geodesic equation and definition of Riemann tensor to obtain

$$\boxed{a^{\alpha} = -R_{\mu\sigma\nu}{}^{\alpha} u^{\mu} \zeta^{\sigma} u^{\nu}}$$

**geodesic deviation equation**

- Whirlwind review of differential geometry
  - Coordinates and distances
  - Vectors and connections
  - Lie derivative
  - Curvature
- Formulation of General Relativity
  - Weak gravity and slow motion
  - Matter
  - Einstein field equations
- Linearized gravity
  - Newtonian limit
  - Plane wave solution

# Weak gravity and slow motion

Linear perturbation to flat spacetime:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} + O(h^2)$$

4-velocity of slowly moving particle, $v \ll c$:

$$\mathbf{u} = \frac{d\mathbf{x}}{dt} = [1, 0, 0, 0] + O(v/c)$$

Geodesic equation:

$$\frac{d^2 x^i}{dt^2} \approx -\Gamma^i_{00} \approx \frac{1}{2}\frac{\partial}{\partial x^i} h_{00}$$

where we assume a nearly stationary background, $\partial h_{\alpha\beta}/\partial t \approx 0$
Geodesic deviation equation:

$$\frac{d^2 \zeta^i}{dt^2} \approx -R_{0i0j}\xi^j \approx -\frac{1}{2}\frac{\partial^2 h_{00}}{\partial x^i \partial x^j}\,\zeta^j$$

Identify $h_{00} = -2\Phi$

# Matter

Perfect fluid **stress energy tensor**
Locally-inertial frame:

$$\mathbf{T} = \begin{bmatrix} -\rho c^2 & & & \\ & P & & \\ & & P & \\ & & & P \end{bmatrix}$$

Generally:

$$T^{\alpha\beta} = (\rho + P/c^2) u^\alpha u^\beta + P g^{\alpha\beta}$$

# Einstein field equations

$$\boxed{G_{\alpha\beta} = \frac{8\pi G}{c^4}\, T_{\alpha\beta}}$$

**Einstein field equations**

where the Einstein tensor, Ricci tensor, and Ricci scalar are

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \qquad R_{\alpha\beta} = R_{\alpha\mu\beta}{}^{\mu} \qquad R = g^{\mu\nu} R_{\mu\nu}$$

The Bianchi identity implies $\nabla_\mu G^{\mu\alpha} = 0$ which yields the equations of motion for matter

$$\nabla_\mu T^{\mu\alpha} = 0$$

# Linearized gravity

Define trace-reversed metric perturbation

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h$$

Choose Lorenz gauge (harmonic coordinates)

$$\frac{\partial}{\partial x^{\mu}} \bar{h}^{\mu\alpha} = 0$$

via gauge transformation $x^{\alpha}_{\mathrm{new}} = x^{\alpha}_{\mathrm{old}} + \xi^{\alpha}$ where

$$\Box \xi^{\alpha} = \frac{\partial}{\partial x^{\mu}} \bar{h}^{\mu\alpha}_{\mathrm{old}}$$

$$\boxed{\Box \bar{h}_{\alpha\beta} = -\frac{16\pi G}{c^4} T_{\alpha\beta}}$$

**linearized field equations**

# Newtonian limit

Leading order in $1/c^2$:

$$\mathbf{T} \approx \begin{bmatrix} -\rho c^2 & & & \\ & \cancel{P} & & \\ & & \cancel{P} & \\ & & & \cancel{P} \end{bmatrix} \qquad \text{and} \qquad \Box = \cancel{-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}} + \nabla^2$$

Non-trivial field equation is

$$\nabla^2 \bar{h}_{00} = 16\pi G\rho$$

Identify $\bar{h}_{00} = -4\Phi$ where $\Phi$ is the Newtonian potential

$$\boxed{\nabla^2 \Phi = 4\pi G\rho}$$

**Poisson equation**

# Newtonian limit

Newtonian metric: $\mathbf{g} = \begin{bmatrix} -c^2 - 2\Phi & & & \\ & 1 - \dfrac{2\Phi}{c^2} & & \\ & & 1 - \dfrac{2\Phi}{c^2} & \\ & & & 1 - \dfrac{2\Phi}{c^2} \end{bmatrix}$

Geodesic equation: $\dfrac{d^2 x^i}{dt^2} = -\dfrac{\partial \Phi}{\partial x^i}$

Geodesic deviation equation: $\dfrac{d^2 \zeta^i}{dt^2} = -\dfrac{\partial^2 \Phi}{\partial x^i \partial x^j}\, \zeta^j$

# Plane wave solution

Linearized *vacuum* field equations:

$$\Box \bar{h}_{\alpha\beta} = 0$$

Plane wave solution travelling in $z$-direction is

$$\bar{h}_{\alpha\beta} = \bar{h}_{\alpha\beta}(t - z/c)$$

Lorenz gauge condition imposes four constraints:

$$\bar{h}_{00} = -c\bar{h}_{03} = c^2\bar{h}_{33}, \quad \bar{h}_{01} = -c\bar{h}_{31}, \quad \bar{h}_{02} = -c\bar{h}_{32}$$

and thereby reduces independent degrees of freedom to six

$$\bar{h}_{00},\ \bar{h}_{01},\ \bar{h}_{02},\ \bar{h}_{11},\ \bar{h}_{12},\ \bar{h}_{22}$$

# Plane wave solution

Remaining freedom within Lorenz gauge: $x^\alpha_{\text{new}} = x^\alpha_{\text{old}} + \xi^\alpha$ where

$$\Box \xi^\alpha = 0$$

is used to set $\bar{h}_{00} = \bar{h}_{01} = \bar{h}_{02} = 0$ and $\bar{h}_{11} = -\bar{h}_{22}$
Resulting metric perturbation has two degrees of freedom:

$$\mathbf{h} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $h_+ = h_+(t - z/c)$ and $h_\times = h_\times(t - z/c)$ are the two transverse polarizations of the plane wave