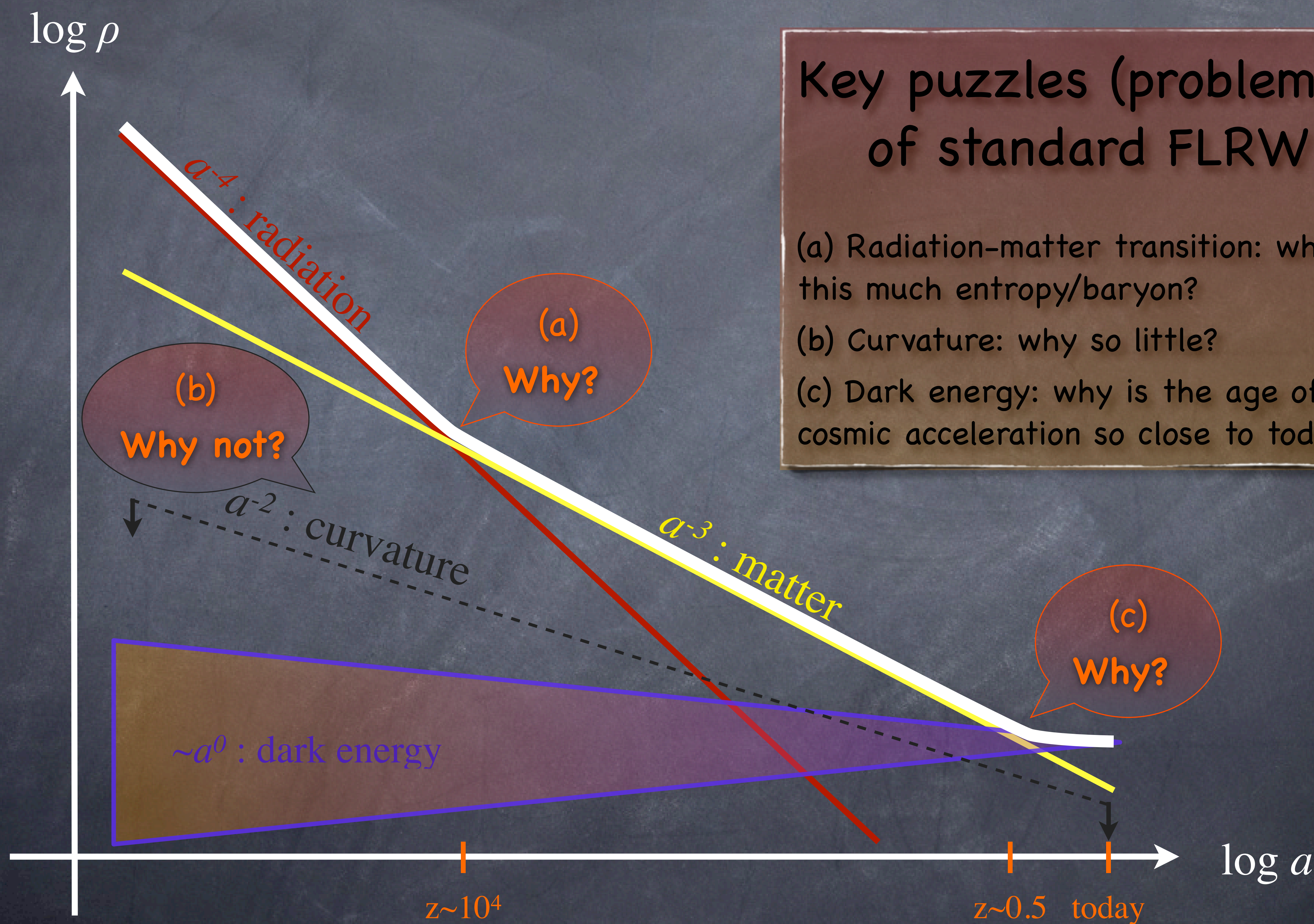
Continuation of Part 4: Dynamics of FLRW spacetimes

History of cosmic domination



Key puzzles (problems?) of standard FLRW

(a) Radiation-matter transition: why this much entropy/baryon?

(b) Curvature: why so little?

(c) Dark energy: why is the age of cosmic acceleration so close to today?

Puzzle #1: Why this amount of radiation relative to matter?

➪ Thermodynamics - coming up!

Puzzle #2: Why is the spatial curvature so small?

$$\rho_k = -\frac{3k}{8\pi\, G\, a^2}$$

$$\Omega_k = -\frac{k}{H^2\, a^2} \quad \leftrightarrow \quad \Omega_k^0 = -\frac{k}{H_0^2}$$

$$\rho_c = \frac{3H^2}{8\pi\, G}$$

Present observational limits (i.e., CMB) imply that the "**radius of curvature**" ($k^{-1/2}$) of the spatial section is much **larger** ($>$ 100 x) than the Hubble radius, 3 Gpc.

How can this be?

$$\frac{d}{dt}\Omega_k = -\frac{d}{dt}\frac{k}{H^2\, a^2} = -\frac{d}{dt}\frac{k}{\dot{a}^2} = 2\frac{k}{H^2 a^2}\,\frac{1}{H}\,\frac{\ddot{a}}{a} = -2\Omega_k\,\frac{1}{H}\,(-4\pi G)(\rho + 3p)$$

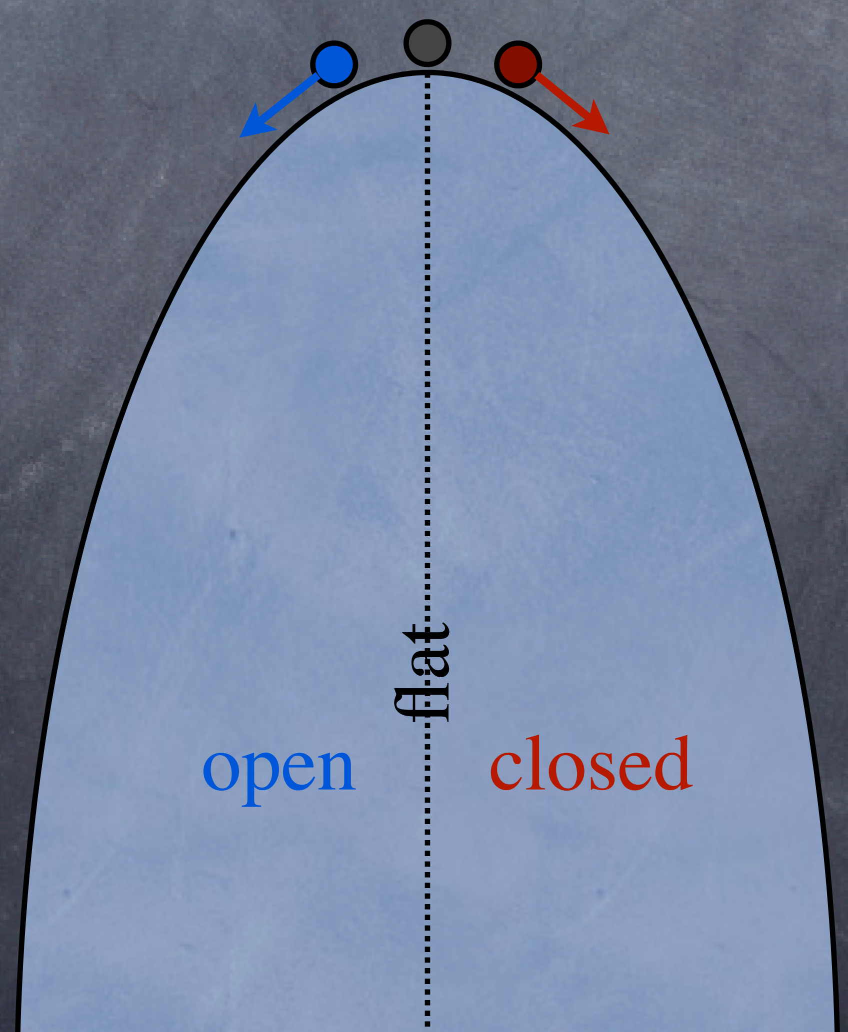
$$\Longrightarrow \frac{d}{da}\log\Omega_k = \frac{1}{H^2}\,8\pi G(\rho + 3p)$$

Ω_k always grows, unless $\rho + 3p < 0$!

With "normal" matter (dust and/or radiation), curvature should eventually dominate:

Curvature **small today** ⇨ Curvature **extremely small** in the early Universe!

Why?



Consequences of a curvature-dominated Universe

Suppose there is **only radiation**, and that the spatial curvature is **negative**.

$$H(t) = \frac{1}{a}\frac{da}{dt} = H_0 \sqrt{\Omega_r^0\, a^{-4} + \Omega_k^0\, a^{-2}} \qquad \Omega_r^0 + \Omega_k^0 = 1$$

Defining: $a_* = \sqrt{\frac{\Omega_r^0}{-\Omega_k^0}} = \sqrt{\frac{\Omega_k^0 - 1}{\Omega_k^0}}$

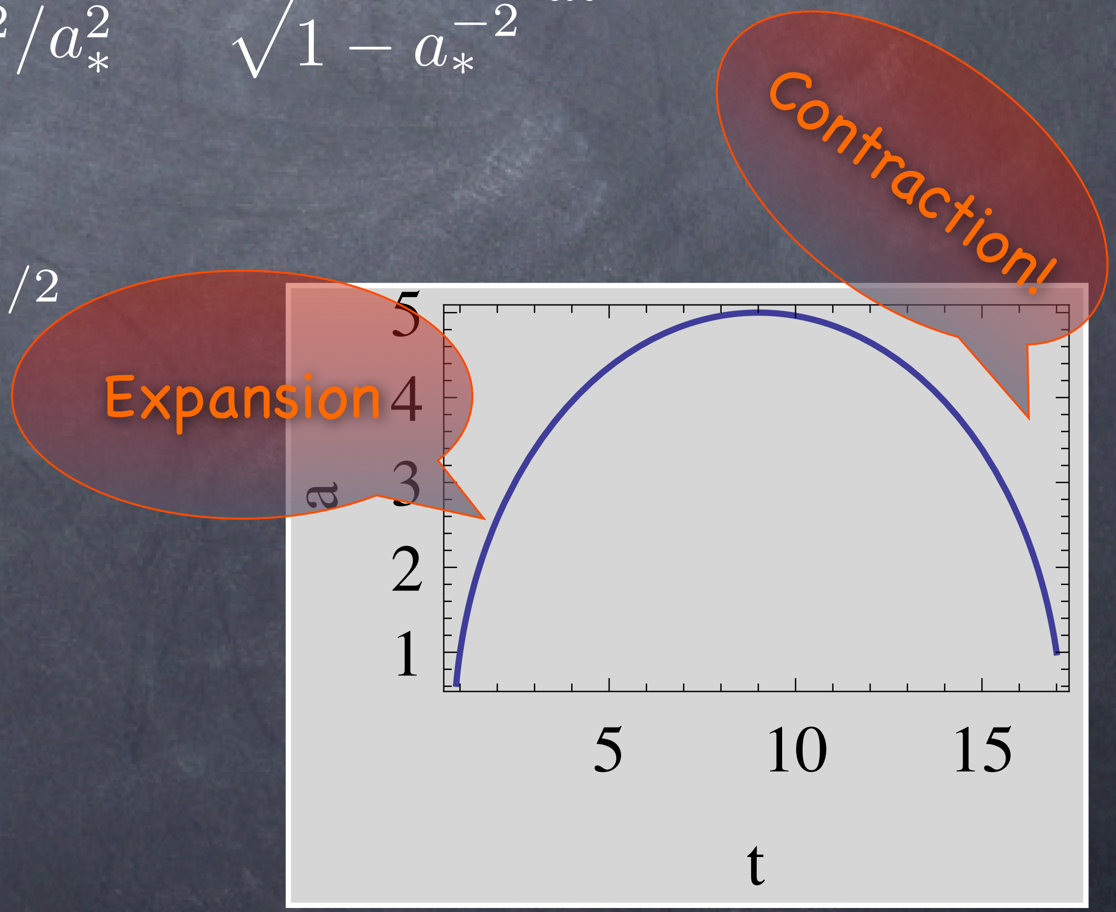
we rewrite this equation as: $$\frac{da}{a}\frac{a^2}{\sqrt{1 - a^2/a_*^2}} = \frac{H_0}{\sqrt{1 - a_*^{-2}}}\, dt$$

The solution is:

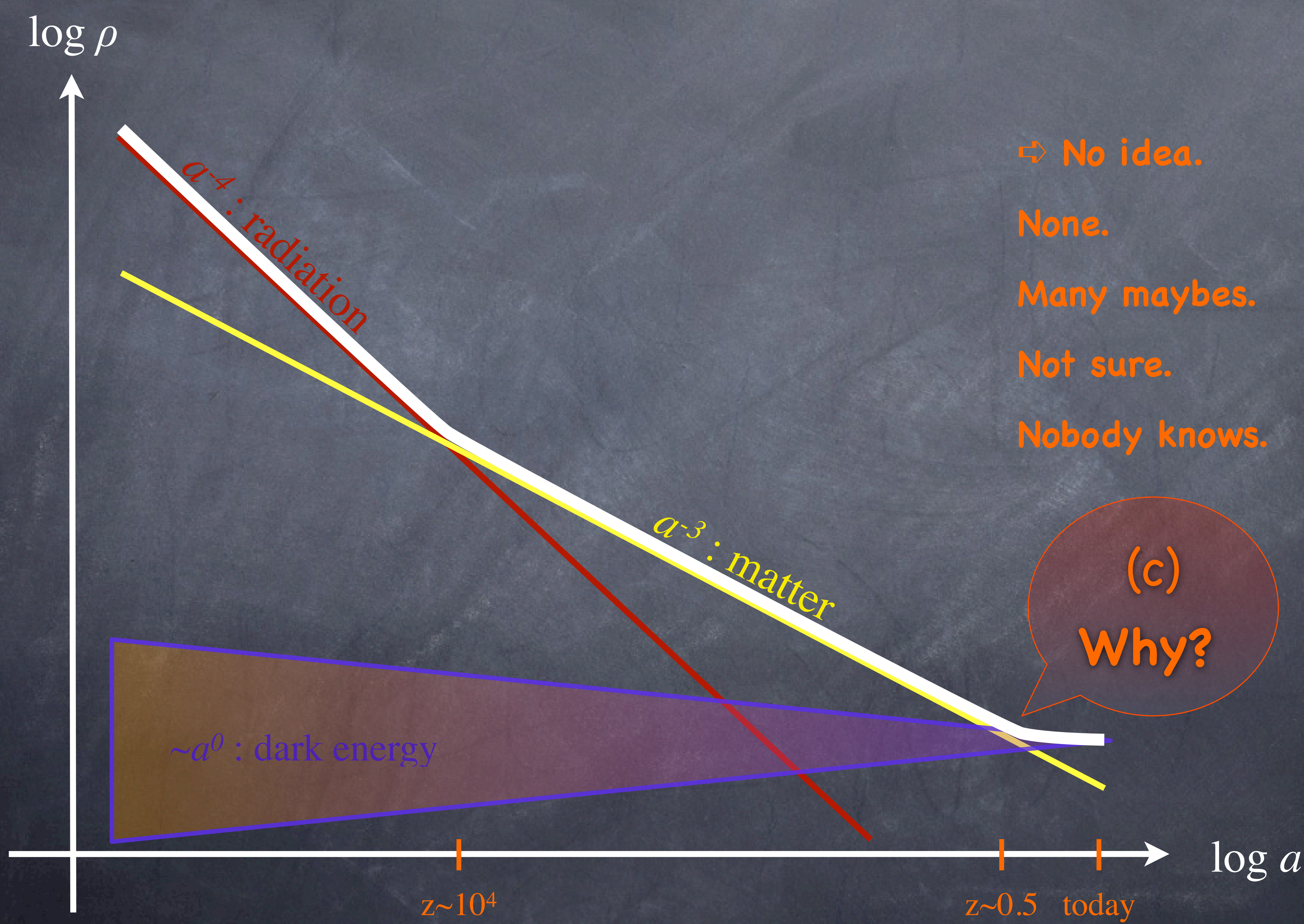
$$a = \left[1 + 2\, H_0\,(t - t_0) - \frac{1}{a_*^2 - 1}\, H_0^2\,(t - t_0)^2\right]^{1/2}$$

This function has a **maximum** at:

$$t_* = t_0 + (a_*^2 - 1) H_0^{-1} \qquad a(t_*) = a_*$$



Puzzle #3: Why has the Universe just recently started to accelerate?



➪ No idea.

None.

Many maybes.

Not sure.

Nobody knows.

Useful numbers

- Age of the Universe today: $T_0 = (13.7 \pm 0.2)$ Gy
- Density: $\rho_0 = (1.9 \pm 0.15)\, h^2 \times 10^{-29}$ g cm^{-3}
- Hubble parameter: $H_0 = 100\, h$ Km s^{-1} Mpc^{-1}

  $h = \mathbf{0.70 \pm 0.03}$

  1 pc = 2.36 l.yr.
  1 Mpc = 3.1 x 10^{24} cm

  Hubble radius: $c/H_0 = 3\, h$ Gpc
- Baryon fraction: $\Omega_b h^2 = (\rho_b / \rho_{tot})\, h^2 = 0.024 \pm 0.003$
- Radiation

  (photons and massless neutrinos): $\Omega_r = 2.5 \times 10^{-5}\, h^{-2}$
- Matter (dark matter + baryons) $\Omega_m = 0.2 - 0.3$
- Curvature $|\Omega_k| < 0.01$

With these we can compute, e.g., matter-radiation equality:

$$\frac{\rho_m}{\rho_r} = \frac{\rho_{0m}\, a^{-3}}{\rho_{0r}\, a^{-4}} = \frac{\Omega_m}{\Omega_r} \frac{1}{(1+z)} \qquad \boxed{\Rightarrow z_{eq} = \frac{\Omega_m}{\Omega_r} - 1 \simeq 0.8 - 1.2 \times 10^4}$$

Part 5: Thermal history

Suggested reading:

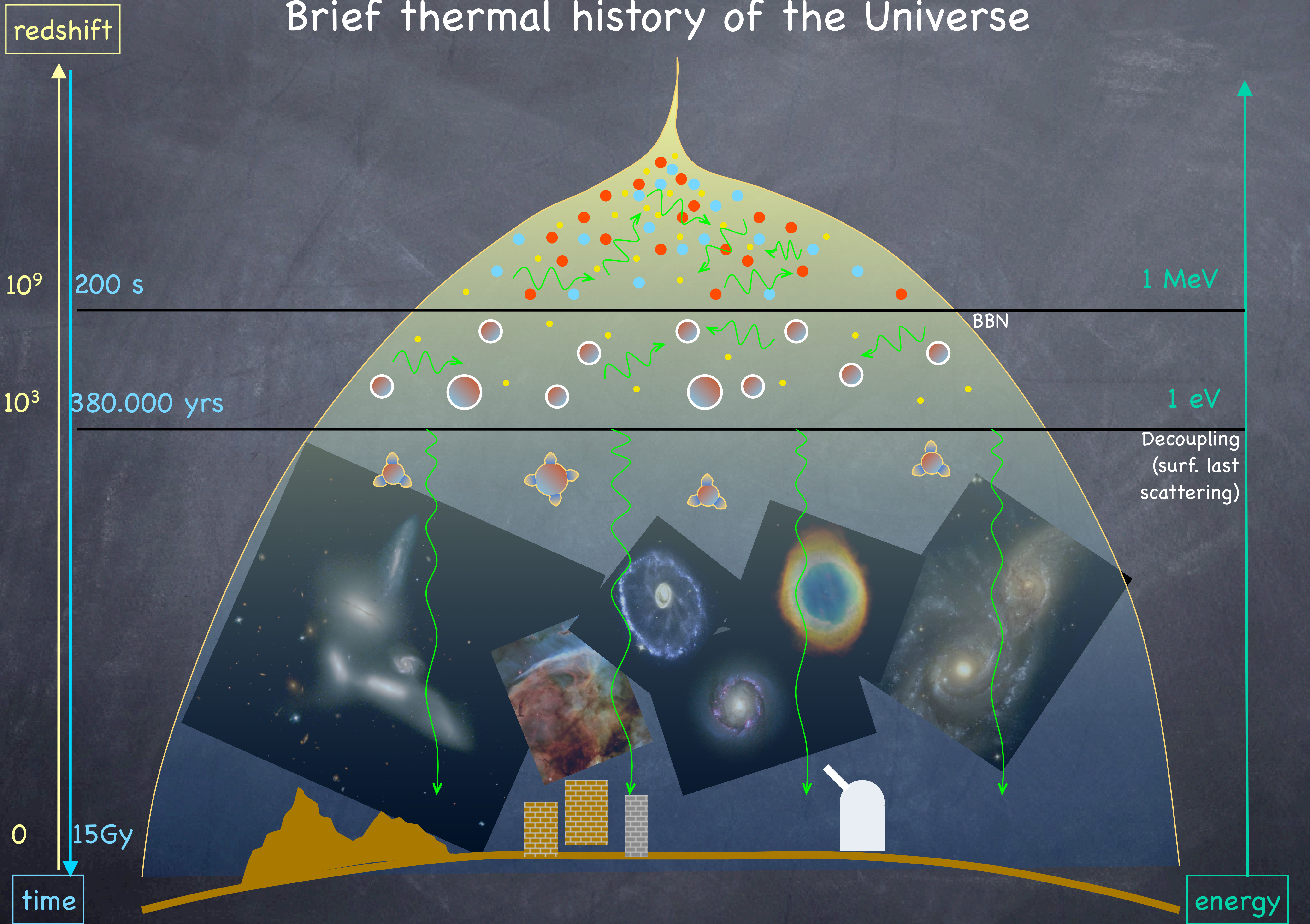
V. Mukhanov, "Physical Foundations of Cosmology"

P. Peter and J.-P. Uzan, "Primordial Cosmology"

E. Kolb & M. Turner, "The Early Universe"

[Original works by Zwicky; Gamow, Alpher, Herman; Bethe; Penzias, Wilson; Peebles; Schramm; ...]

Brief thermal history of the Universe



Thermodynamics in an FLRW Universe

As the Universe **expands** and becomes **less dense**, it also **cools down**.

Hence the behavior of the energy density and pressure of the matter content is tied up with its **Thermodynamics** and **Statistical Mechanics**.

Elementary particles come in two types:

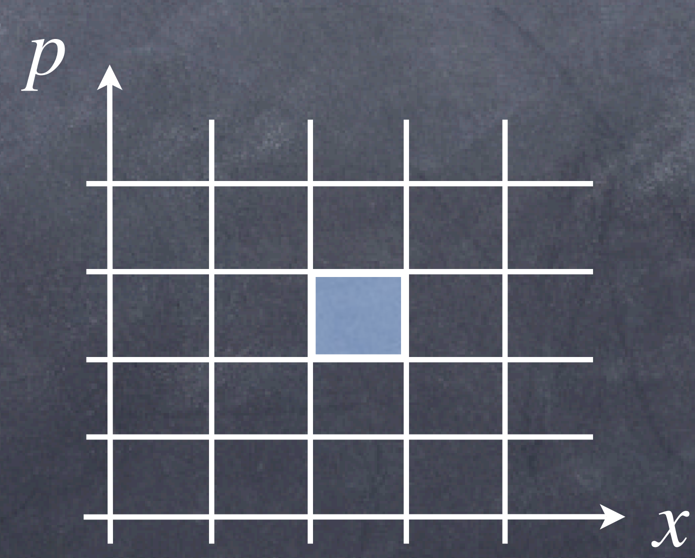
- **bosons** (spin=0, 1, 2, ...), and
- **fermions** (spin=1/2, 3/2, ...)

Because of quantum statistics, each type occupies **phase space** in a different way.

The **number of particles** in a phase space cell is:

$$dN = f(\vec{p},\vec{x})\,\frac{d^3p\,d^3x}{(2\pi)^3}$$

$$n(\vec{x}) = \int \frac{dN}{dV} = \int \frac{d^3p}{(2\pi)^3}\, f(\vec{p},\vec{x})$$



chemical potential

$$f_{BE} = \frac{1}{e^{\frac{E-\mu}{k_B T}} - 1}$$

$$f_{FD} = \frac{1}{e^{\frac{E-\mu}{k_B T}} + 1}$$

Let's forget about the chemical potential (μ) for a moment.

Then these distributions are:

$$f_{\pm} = \frac{1}{e^{E/T} \pm 1}$$

Units of $k_B=1$:
[Temperature] = [Energy]
1 K = 8.3 x 10^{-5} eV
1 eV = 1.1 x 10^{4} K

There is a beautiful result from Hamiltonian and Statistical mechanics, called the **Liouville Theorem** (Liouville 1838, Gibbs 1902), which states that the **phase-space distribution function** is **constant** on all trajectories.

This theorem is equivalent to the **collisionless Boltzmann equation**, which reads:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \left(\frac{\vec{p}}{m} \cdot \vec{\nabla}_x + \vec{F} \cdot \vec{\nabla}_p \right) f$$

In an homogeneous and isotropic (FLRW) Universe, there are **no preferred directions**, so the phase space function f must be conserved in time!

Since $E \sim 1/a$, we must have that $T \sim 1/a$, so that the **form** of the BE or FD distributions are preserved by the expansion of the Universe!

Energy and pressure

The energy and momentum of particles are related: $E^2 - \vec{p}^{\,2} = m^2 \quad \leftrightarrow \quad E\,dE = p\,dp$

The stress-energy density for a **system of particles** can be written in terms of their 4-momenta and 4-velocities:

|v| = |**p**|/E!

$$T^{\mu\nu}(x) = \sum_n p^\mu\, U^\nu\, \delta^{(4)}(x - x_n) = \sum_n p^\mu\, \frac{p^\nu}{E}\, \delta^{(4)}(x - x_n)$$

The simplest quantity is the particle number density: $dn = f_\pm(p,x)\,\frac{d^3p}{(2\pi)^3}$

The energy density for this ensemble of particles is: $d\rho = E\, f_\pm(p,x)\,\frac{d^3p}{(2\pi)^3}$

And if the momenta have no preferred directions, then: $dP = \frac{1}{3}\frac{p^2}{E}\, f_\pm(p,x)\,\frac{d^3p}{(2\pi)^3}$

The **ensemble averages** of the **number density**, **energy density** and **pressure** are:

$$n = \frac{1}{2\pi^2}\int dE\, E\, \frac{(E^2-m^2)^{1/2}}{e^{E/T}\pm 1} = \begin{cases} T \gg m & \text{BE} & \frac{\zeta(3)}{\pi^2}T^3 \\ T \gg m & \text{FD} & \frac{3}{4}\frac{\zeta(3)}{\pi^2}T^3 \\ T \ll m & \text{BE/FD} & \left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} \end{cases}$$

Equil.!

->1/V

$$\rho = \frac{1}{2\pi^2}\int dE\, E^2\, \frac{(E^2-m^2)^{1/2}}{e^{E/T}\pm 1} = \begin{cases} T \gg m & \text{BE} & \frac{\pi^2}{30}T^4 \\ T \gg m & \text{FD} & \frac{7}{8}\frac{\pi^2}{30}T^4 \\ T \ll m & \text{BE/FD} & m\,n + \frac{3}{2}\,n\,T \end{cases}$$

$$P = \frac{1}{6\pi^2}\int dE\, \frac{(E^2-m^2)^{3/2}}{e^{E/T}\pm 1} = \begin{cases} T \gg m & \text{BE} & \frac{1}{3}\rho \\ T \gg m & \text{FD} & \frac{1}{3}\rho \\ T \ll m & \text{BE/FD} & n\,T \ll \rho \end{cases}$$

Energy density of all relativistic matter

Let's collect **all the types of particles** which are **relativistic** - i.e., whose **masses** are much smaller than their **equilibrium temperatures**.

Let's also allow for some of these particles to be **out of equilibrium** with **other** particles, so that their temperatures may be different.

The total energy density in the relativistic degrees of freedom is, then:

$$\rho_r = \sum_i g_i \, T_i^4 \frac{1}{2\pi^2} \int_{m_i/T_i}^{\infty} dx \, x^2 \, \frac{\sqrt{x^2 - m_i^2/T_i^2}}{e^x \pm 1} = T^4 \sum_i g_i \, \frac{T_i^4}{T^4} \times \begin{cases} \frac{\pi^2}{30} & BE \\ \frac{7}{8}\frac{\pi^2}{30} & FD \end{cases}$$

We can define an **effective number of relativistic degrees of freedom** as:

$$\rho_r = g_* \frac{\pi^2}{30} T^4 \qquad\qquad g_* = \sum_{i:\,BE} g_i \left(\frac{T_i}{T}\right) + \frac{7}{8} \sum_{j:\,FD} g_j \left(\frac{T_j}{T}\right)$$

Inventory of degrees of freedom

Type	Statistic	g (# of D.o.f.)	mass
photons	BE	2	-
neutrinos	FD	(3) x 2	< 1.2 eV
e^+, e^-	FD	(2) x 2	0.511 MeV
p^+, p^-	FD	(2) x 2	0.938 GeV
n	FD	2	0.940 GeV
...	...	...	...

Problem #19: Between temperatures of $1 \text{ MeV} < T < 100 \text{ MeV}$, what is the effective number of degrees of freedom? (Assume that the temperatures of all species are the same.)

Problem #20: One of the main reactions that keep photons, electrons and positrons in equilibrium is Thomson and Compton scattering, $e + \gamma \rightarrow e + \gamma$.

The reaction rate for this interaction is $\Gamma_{\gamma e} = n_e \, \sigma_T \, c$, where n_e is the number density of electrons, and $\sigma_T = 6.65 \times 10^{-25} \text{ cm}^2$ is the cross section for Thomson scattering.

(a) Is this rate "fast" or "slow"? (Compared to **what**?)

(b) What is the mean free path for photons?

(c) Knowing that $n_e \approx 10^{-7} \text{ cm}^{-3} (1+z)^3$ for $z < 10^3$, and $n_e \approx 10^{-5} \text{ cm}^{-3} (1+z)^3$ for $z > 10^3$, show that the rate of these scatterings is **always** very "fast".

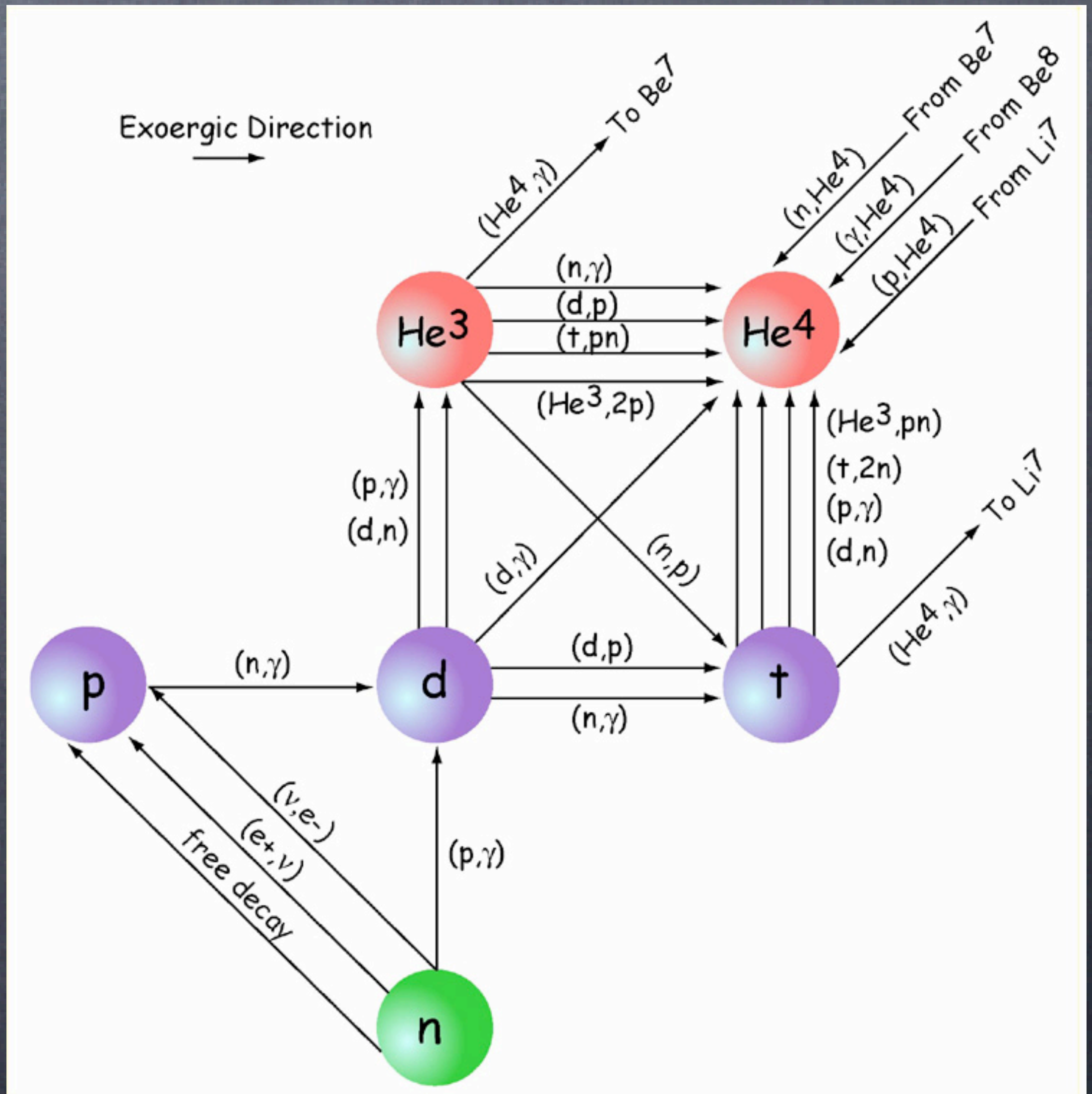
Big Bang Nucleosynthesis (BBN) - very briefly

There is a slight difference between the mass of the proton and that of the neutron: $m_n = 939.57$ MeV , $m_p = 938.27$ MeV -> $B = 1.3$ MeV ("binding energy") .

At sufficiently high temperatures ($T >> 1.3$ MeV), the two particles are in equilibrium, so there are basically the same numbers (i.e., densities) of each one.

Moreover, as the Universe expands and its **temperature falls**, protons and neutrons can start to **bind** to produce the first **nuclei**:

^{A}Z	B_A	g_A
^{2}H	2.22 MeV	3
^{3}H	6.92 MeV	2
^{3}He	7.72 MeV	2
^{4}He	28.3 MeV	1
...	...	...



So, as the **temperature** of the bath **drops** below $\sim 10\ \mathrm{MeV}$ (at $t \sim 0.1\ \mathrm{s}$ after the Big Bang), many things start to happen:

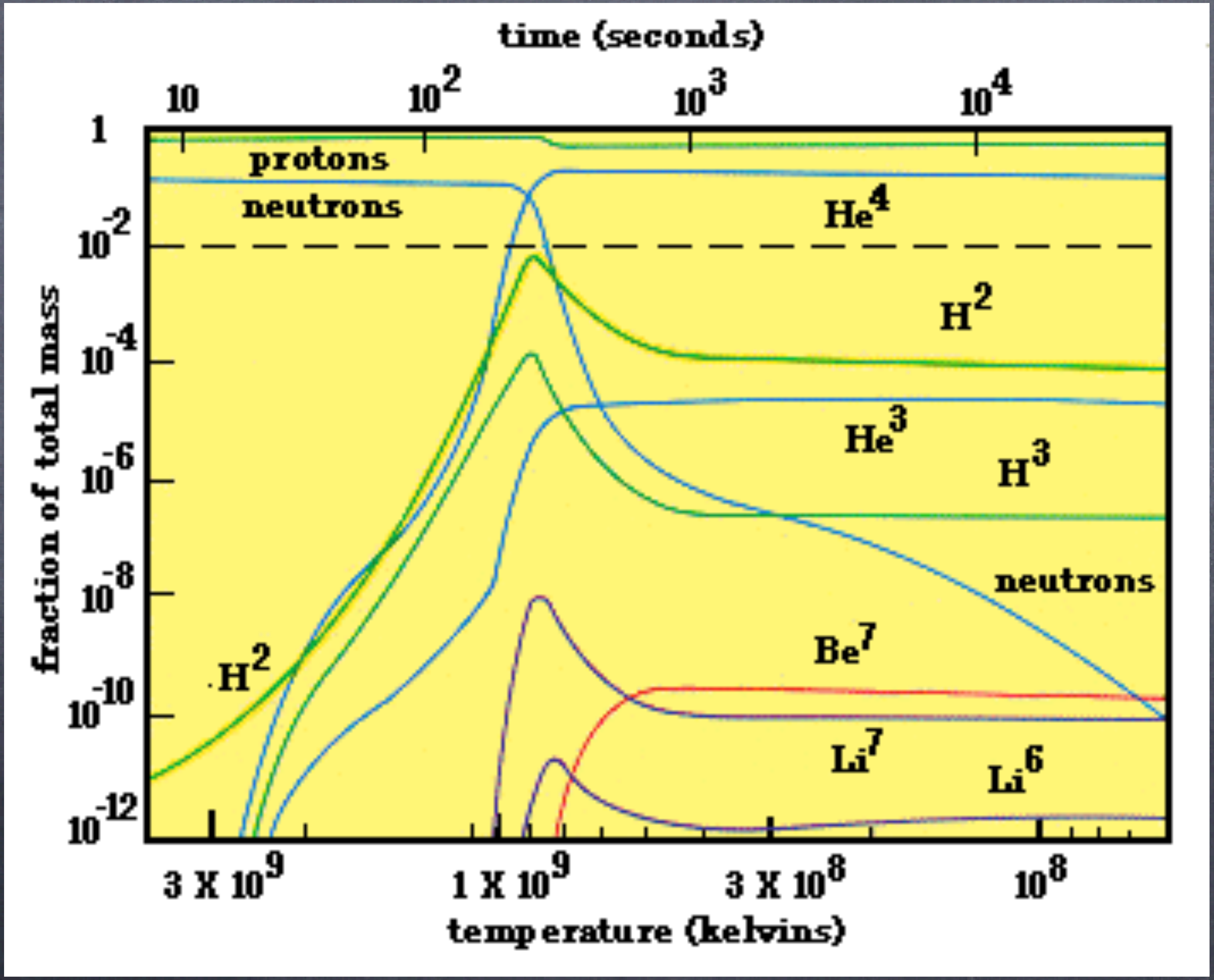
- **Light nuclei** start to **form**, "eating up" protons and neutrons;
- Free **neutrons** start to **decay** into protons;
- The more photons (radiation) there are per baryon, the easier it is to reverse this, converting neutrons back into protons, and breaking up the nuclei.
- With less neutrons around, it gets **harder** for the **nuclei to form**;
- At a temperature of about $0.05\ \mathrm{MeV}$ ($t \sim 100\ \mathrm{s}$) , all neutrons have either been captured by nuclei, or they have decayed into protons. "**Freeze-out**"!

Of course, it's **much** more complicated than that!!!

See, e.g., Mukhanov's book, or **G. Steigman, 0712.1100**

A good review by Tytler et al. (2000) can also be found online at:

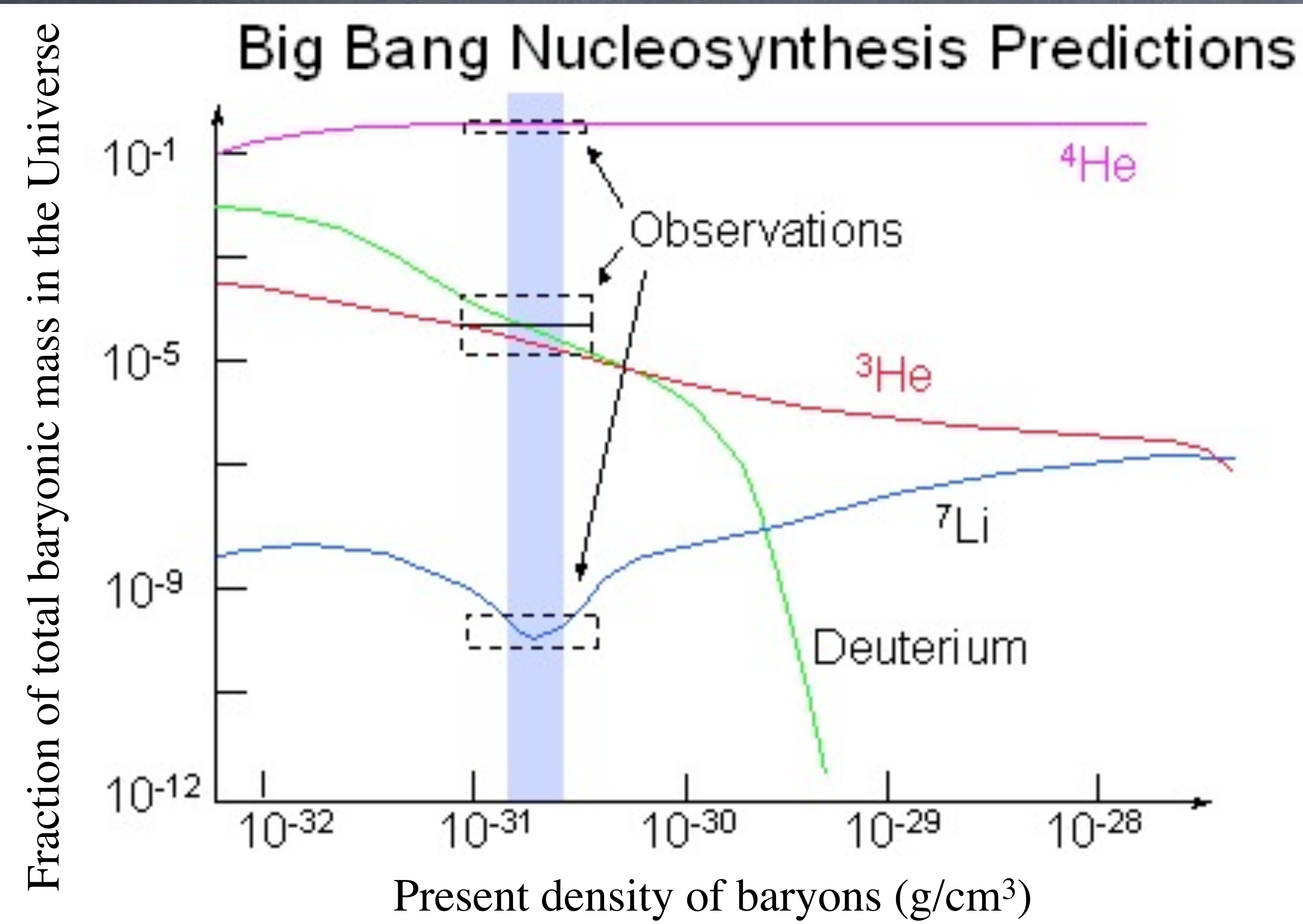
http://ned.ipac.caltech.edu/level5/Tytler2/Tytler_contents.html

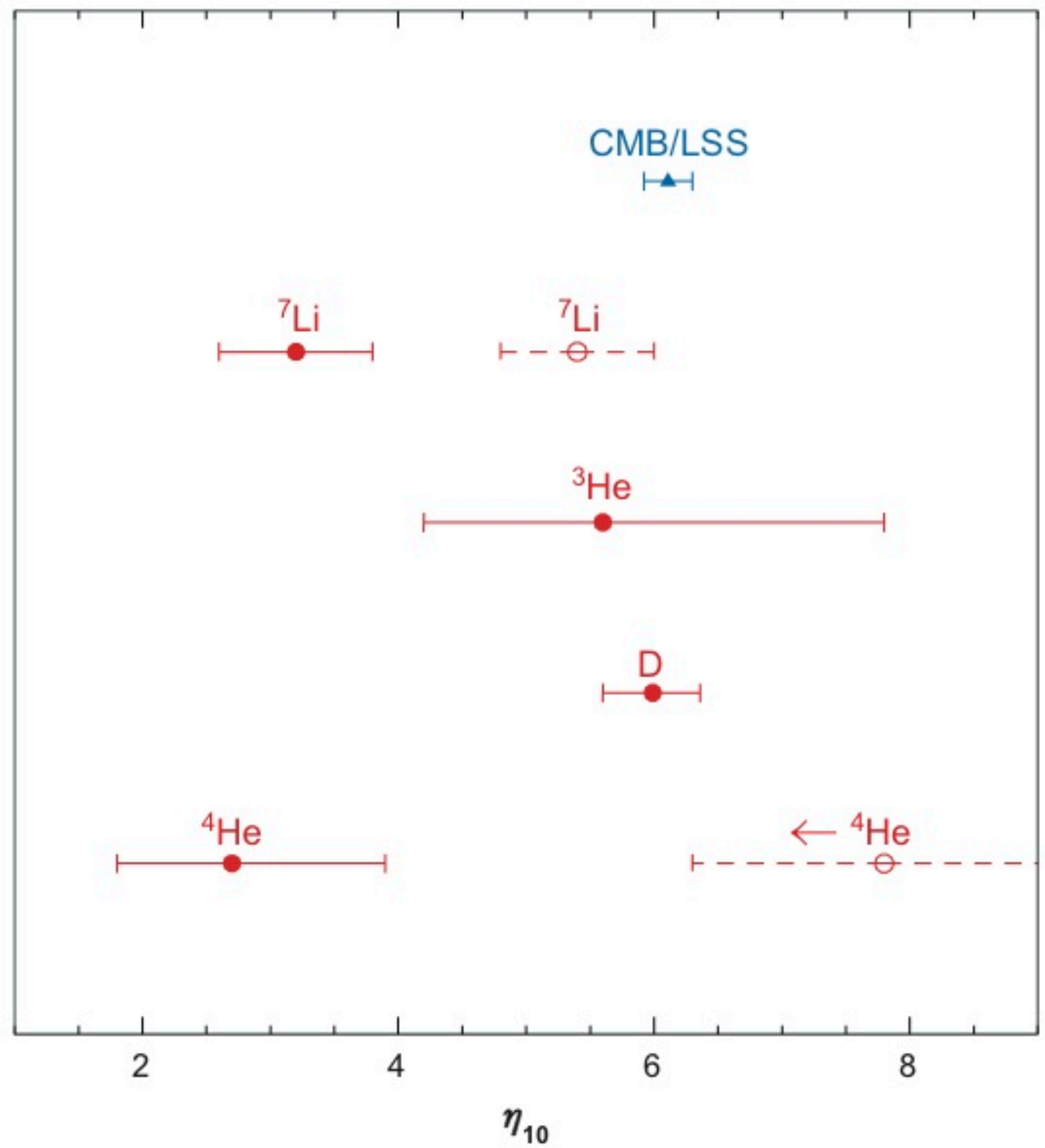


10^9 K =~ 0.1 MeV

Present observations and constraints

(Steigman 2007)





$10^{10}\,(n_p/n_\gamma)$

Decoupling ("recombination")

Peebles 1968 (2-level atom; "Peebles recombination")
Seager et al. 2000 ("RECFAST")
Switzer & Hirata 2007; Wong et al. 2007
Nice review in Hu 2008, arXiv: 0802.3688

The transition between an **ionized & opaque** Universe to a **neutral** & **transparent** one is governed by the process:

$$p + e^- \to H + \gamma \quad , \quad B = 13.6\,\mathrm{eV} \qquad\qquad \mu_p + \mu_e = \mu_H + \mu_\gamma \approx \mu_H$$

Maxwell-Boltzmann distribution f/ phase space: $f_i^{\mathrm{MB}} = e^{-(m_i-\mu_i)/T}\, e^{-p_i^2/(2m_iT)}$

$$\Rightarrow \quad n_i = g_i \int \frac{d^3p_i}{(2\pi)^3} f_i = g_i \left(\frac{m_iT}{2\pi}\right)^{3/2} \times e^{(\mu_i - m_i)/T}$$

This means that the **relative abundance** of p, e and H is:

$$\frac{n_p n_e}{n_H} \approx \left(\frac{m_e T}{2\pi}\right)^{3/2} \times e^{(\mu_p+\mu_e-\mu_H)/T} \times e^{-B/T} \approx \left(\frac{m_e T}{2\pi}\right)^{3/2} \times e^{-B/T}$$

Introducing the ionized fraction, $n_e = X_e n_b$, we get the **Saha equation**:

We live in a **low-baryon** Universe!

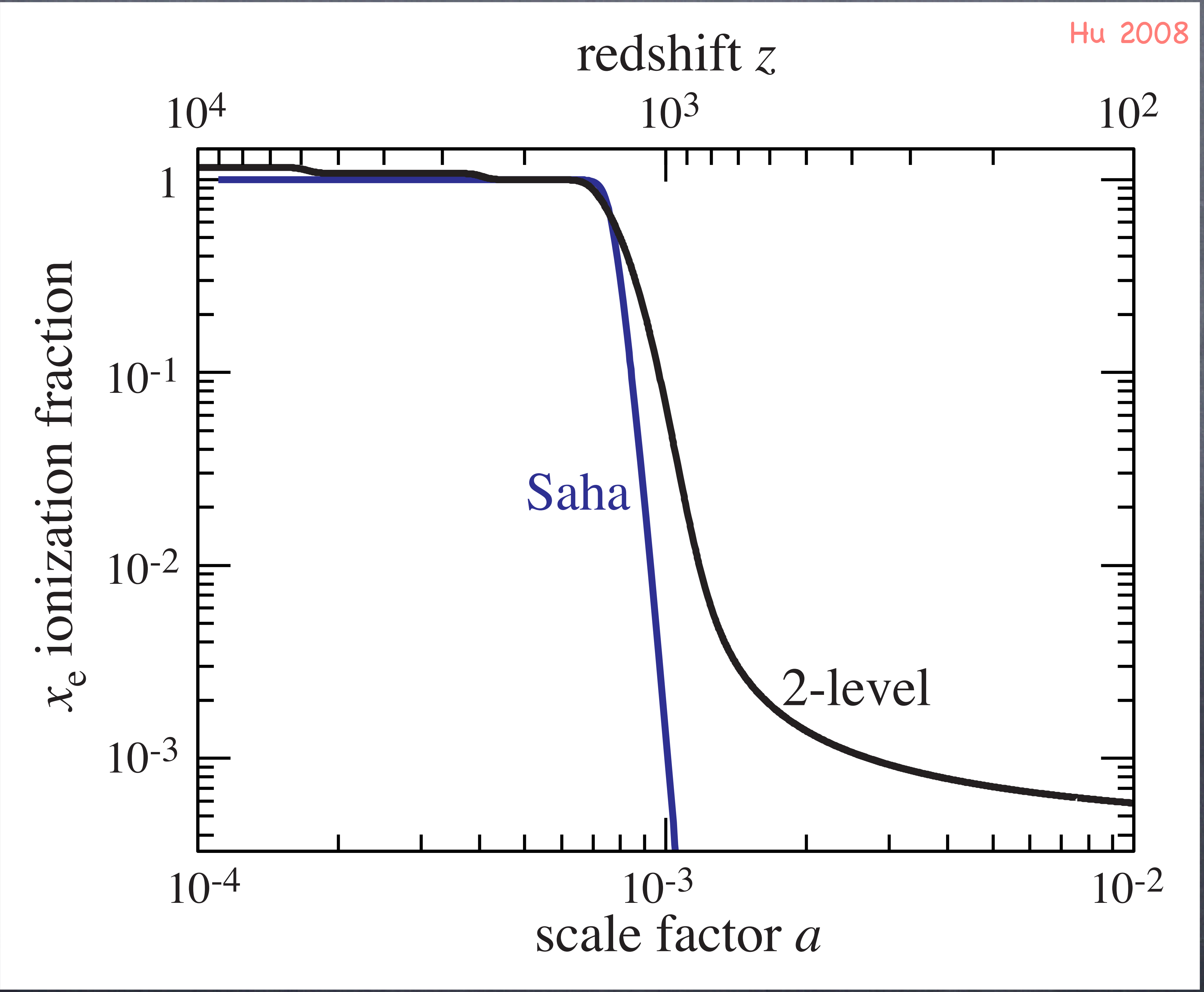
$$\frac{n_p n_e}{n_H n_b} = \frac{X_e^2}{1-X_e} = \frac{1}{n_b}\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-B/T} = \frac{\pi^{1/2}}{2^{5/2}\eta_{b\gamma}\zeta(3)}\left(\frac{m_e c^2}{k_B T}\right)^{3/2} e^{-B/k_B T}$$

~10^{-9} !

Ionized fraction

Correcting for He: $n_e = (1-Y)X_e n_b = x_e \times 1.12 \times 10^{-5}\Omega_b h^2 (1+z)^3 \text{cm}^{-3}$

The number of free **e⁻** (sources of Thomson/Compton scattering) **plunges** at temperatures of around $\mathrm{T} \sim 3\,\mathrm{eV}$ and redshifts $z \sim 1100$:



But how do the free e^- impact the CMB photons during recombination?

Essentially **all scatterings** of photons at these low energies involve the **free electrons**. The **cross-section** for Thomson/Compton scattering is:

$$\sigma_T = \frac{8\pi\alpha^2}{3m_e^2} = 6.65\times 10^{-25}\,\mathrm{cm}^2$$

The **mean free path** is given by:

$$\lambda^{mfp}_{phys} = \frac{1}{n_e\sigma_T} = \frac{1}{X_e n_b \sigma_T} \quad \Rightarrow \lambda^{mfp}_c = \frac{1}{X_e n_b \sigma_T a}$$

Then the **normalized probability** that a photon **will** be scattered between some time t_s and $t_s + dt_s$, but **not later**, in the interval $[\,t_i\,, t_f\,]$, is :

$$dP = \frac{dt_s}{\lambda(t_s)}\left(1-\frac{dt_1}{\lambda(t_1)}\right)\left(1-\frac{dt_2}{\lambda(t_2)}\right)\ldots\left(1-\frac{dt_N}{\lambda(t_N)}\right)$$

$$N\to\infty \ \downarrow$$

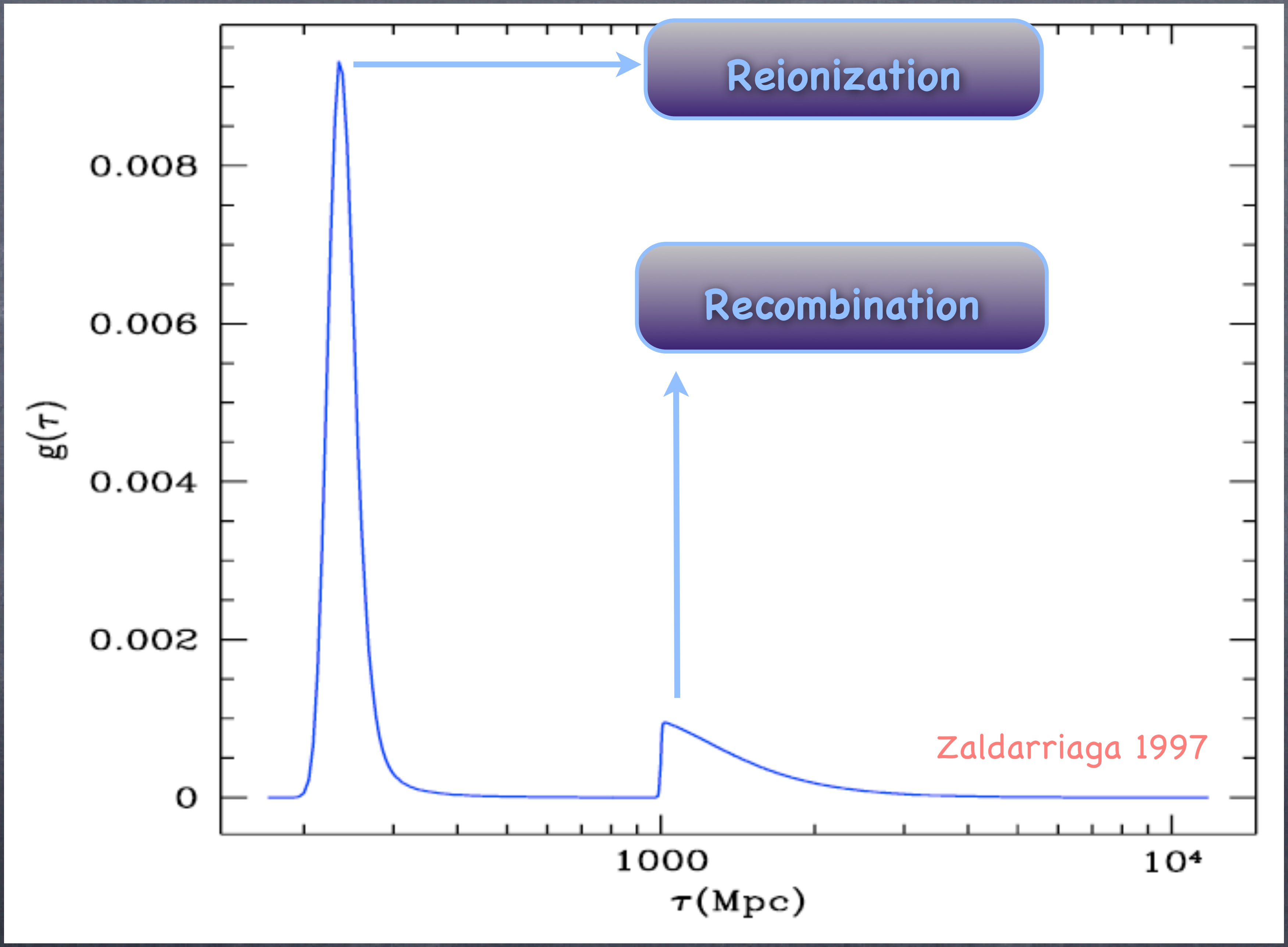
$$\Rightarrow dP = \mu'(\eta_s)d\eta_s \times e^{-\mu(\eta_s)} \quad , \quad \mu(\eta) \equiv \int_\eta^{\eta_f} d\eta'\, X_e(\eta')n_b(\eta')\sigma_T$$

$$P = \int_0^\infty d\mu\, e^{-\mu} = 1$$

Optical depth to Thomson scatt.

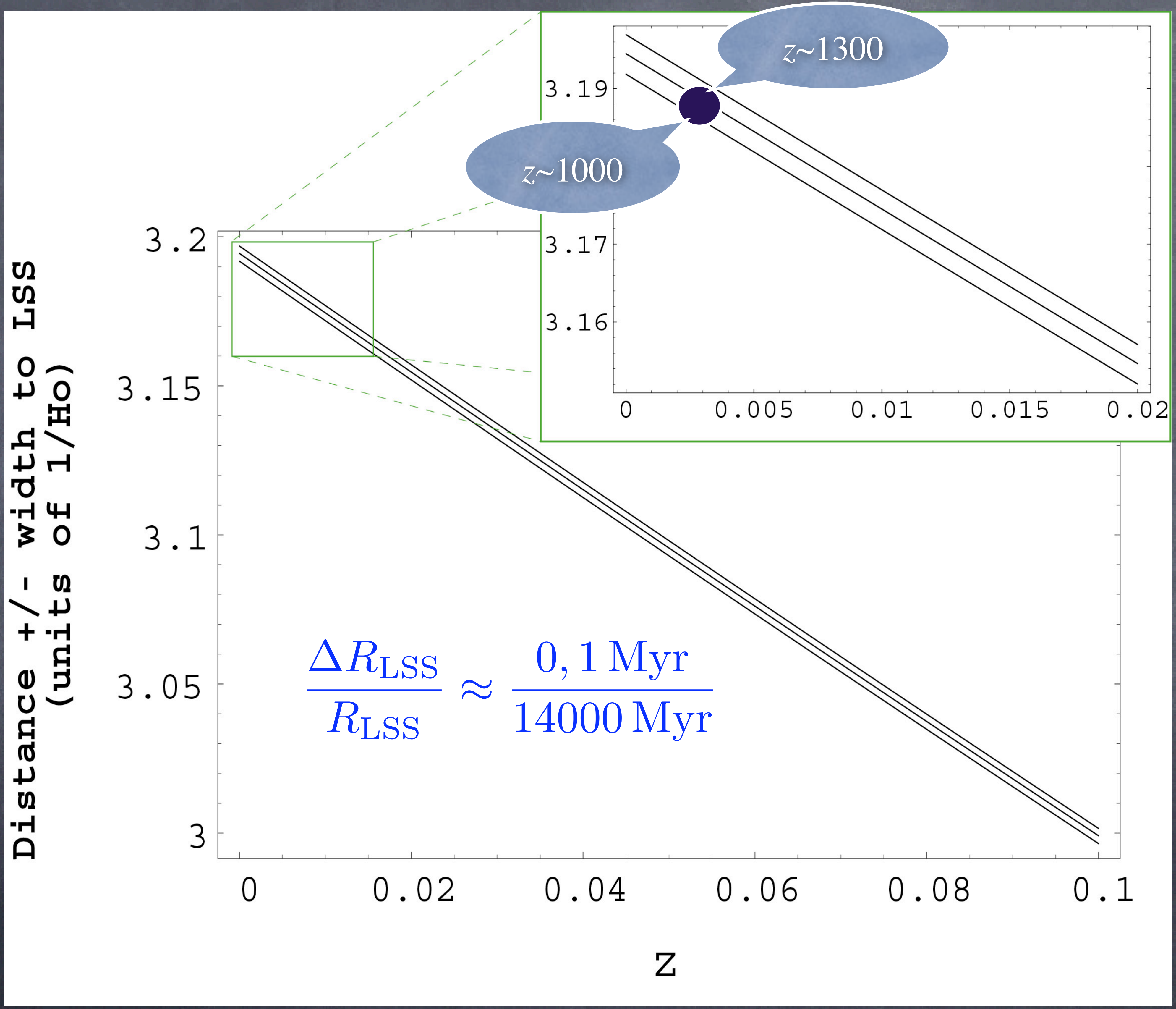
This **probability per unit length** that a photon **will be scattered** at some time **t**, **but not later**, is called the **visibility function**.

$$g(\eta) = \mu'(\eta) e^{-\mu(\eta)} \;=\; \sigma_T X_e(\eta) n_b(\eta) a(\eta) \times \exp\left[\int_0^\eta d\eta' \, \sigma_T X_e(\eta') n_b(\eta') a(\eta')\right]$$



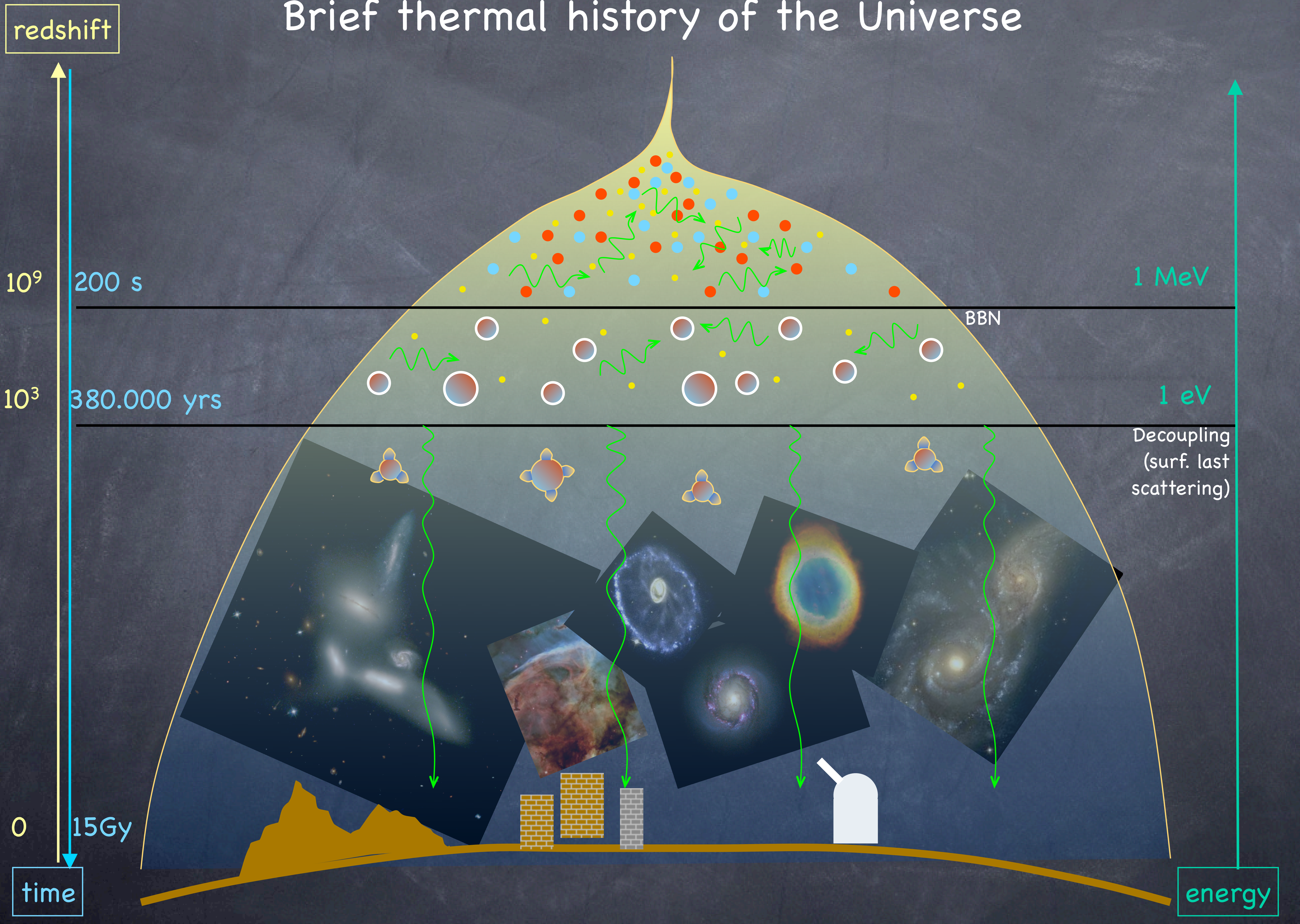
Maximal probability: $$z_* \approx 1089 \left(\frac{\Omega_m h^2}{0.14}\right)^{0.0105} \left(\frac{\Omega_b h^2}{0.024}\right)^{-0.028}$$ Hu 2005

Thickness of the last scattering surface



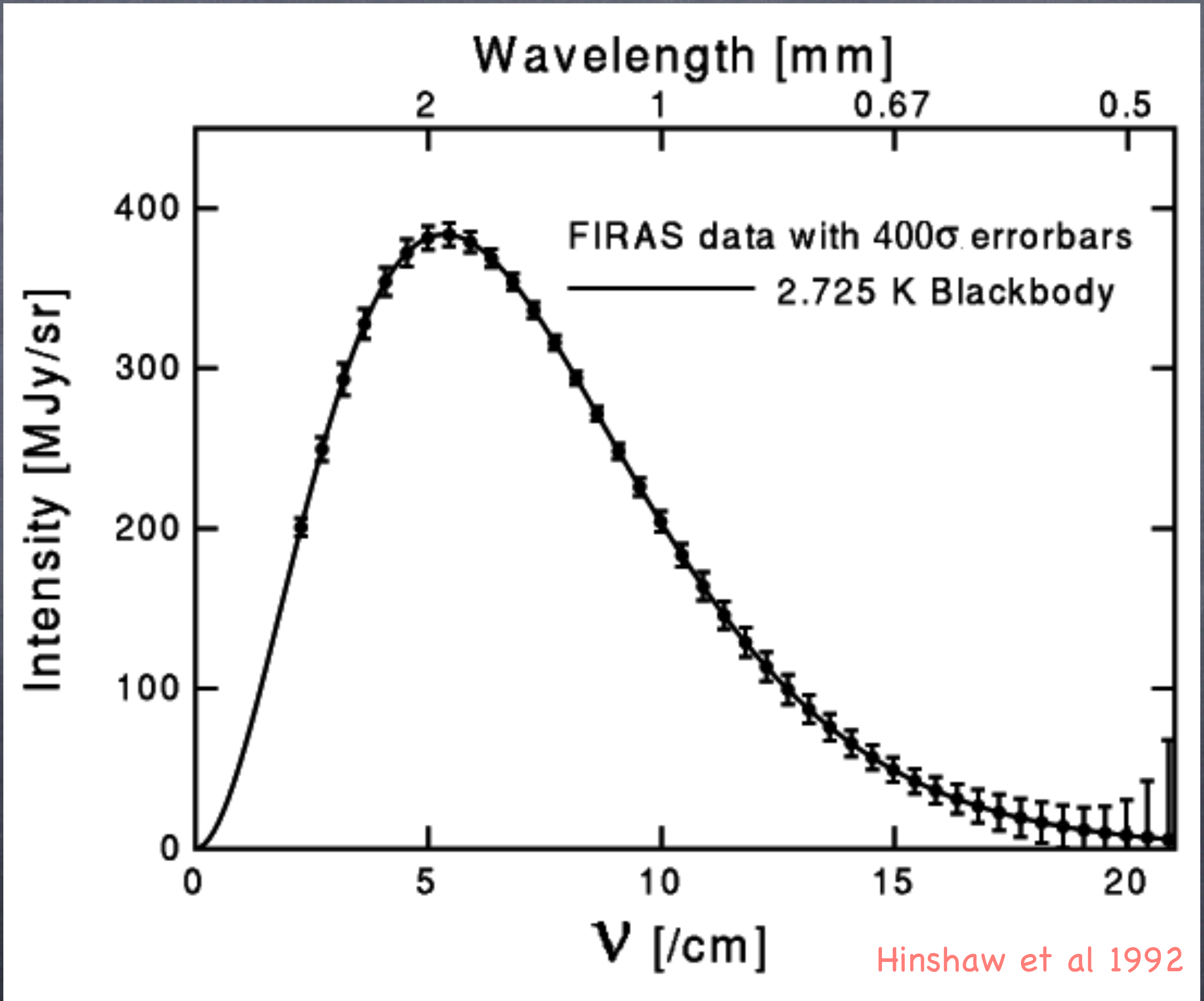
- The CMB (up to reionization) **really** probes a very **narrow spatial region** ("shell" of R_{LSS}) and a **very precise epoch** (z~1000-1300) !

Brief thermal history of the Universe



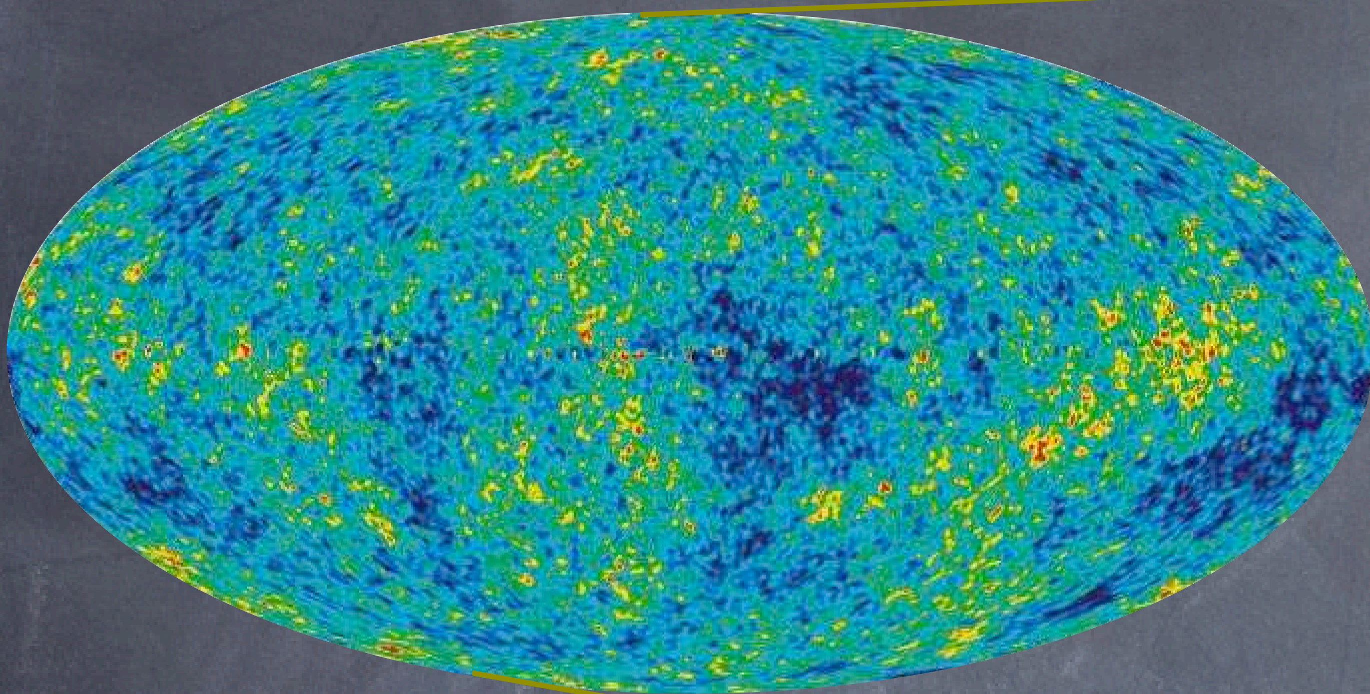
The CMB was created at a time of thermal equilibrium... or was it?

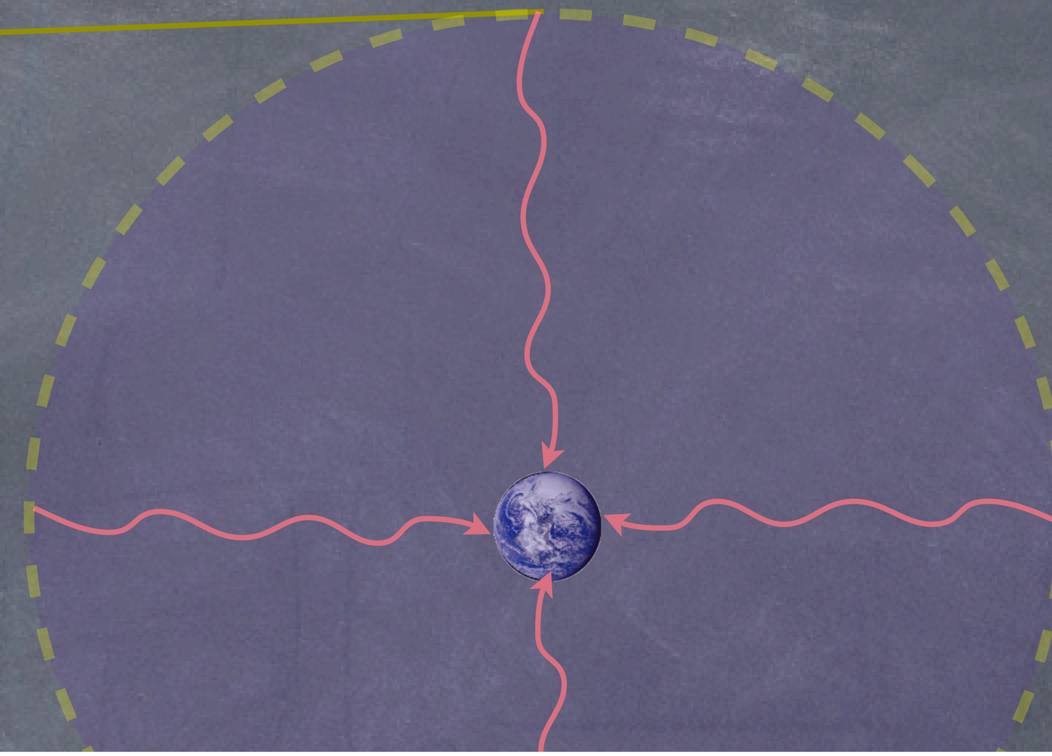
In 1992-1993, an instrument on the COBE satellite called FIRAS measured the energy distribution of the CMB photons:

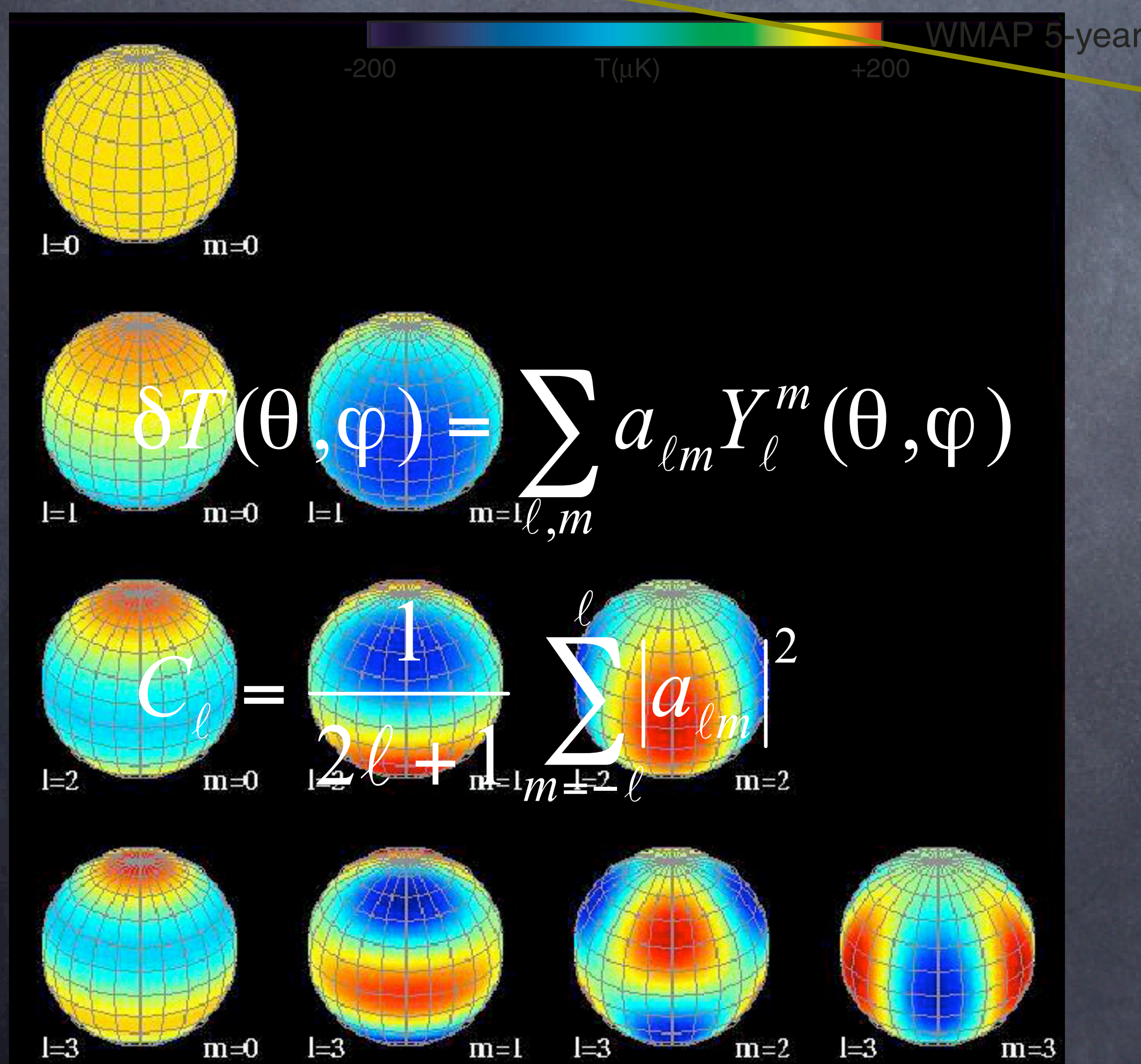


Hinshaw et al 1992

- **The CMB!**

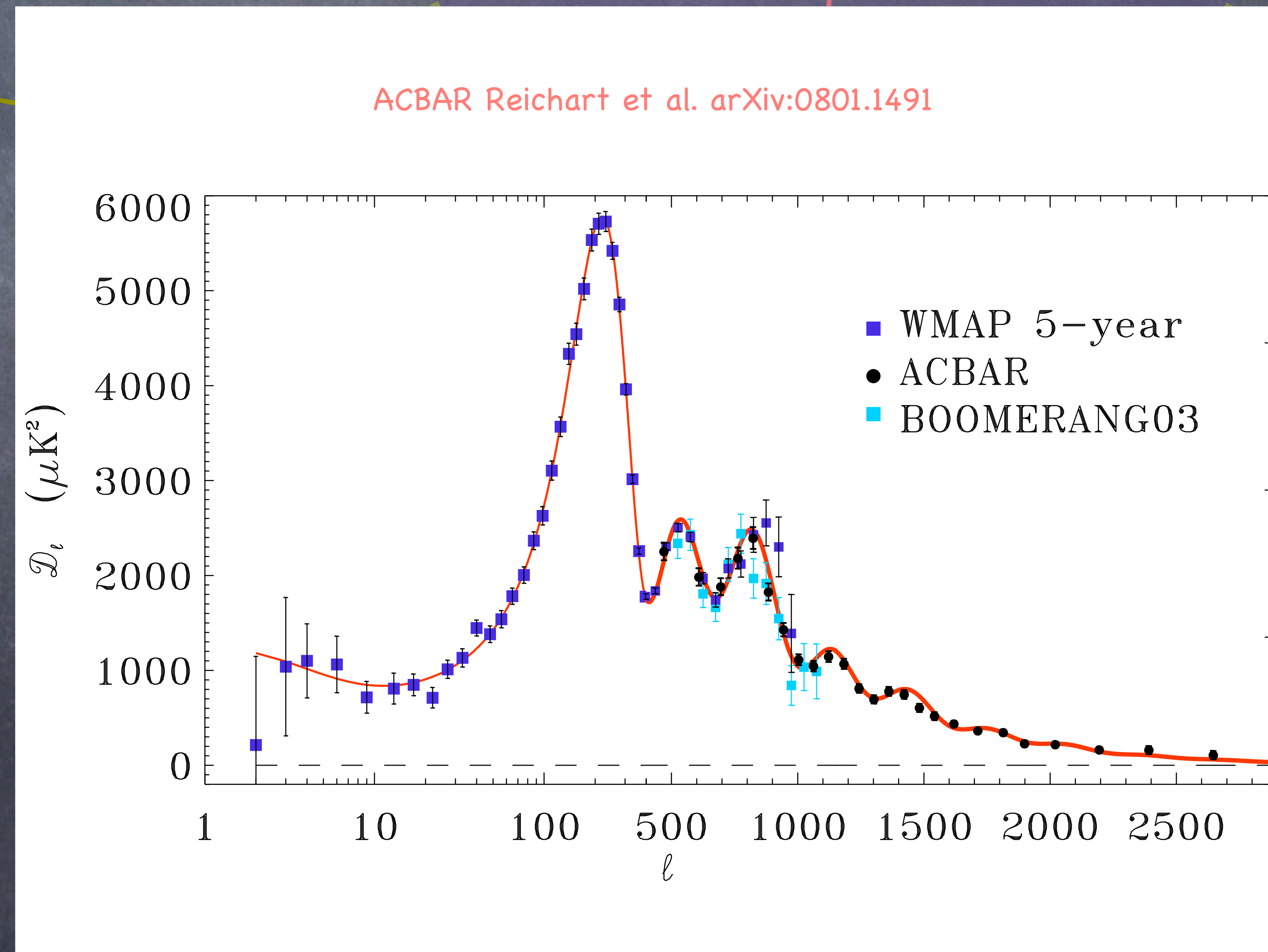




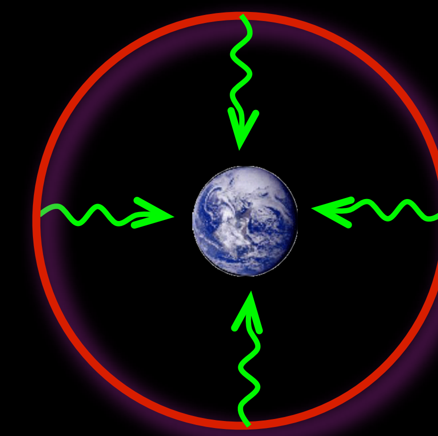


$$\delta T(\theta,\varphi) = \sum_{\ell,m} a_{\ell m} Y_\ell^m(\theta,\varphi)$$

$$C_\ell = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} \left| a_{\ell m} \right|^2$$

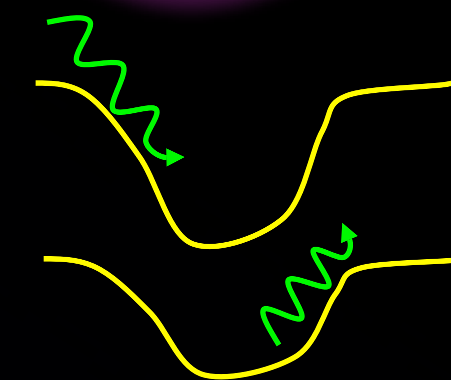


Sources of CMB anisotropies:



- Sachs-Wolfe effect

$$\frac{\Delta T(\vec{n})}{T} \approx \frac{1}{3}\psi(R_{\rm LSS}\vec{n}, \eta_{\rm LSS}) \sim \frac{\delta\rho}{\rho} \sim 10^{-5}$$

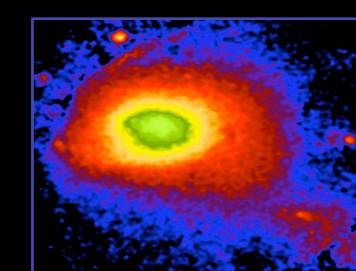


- ISW

$$\frac{\Delta T(\vec{n})}{T} = 2\int_{\vec{n}} d\eta \frac{\partial\Phi(\vec{n}\cdot\eta,\eta)}{\partial\eta}$$

- Sunyaev-Zel'dovich

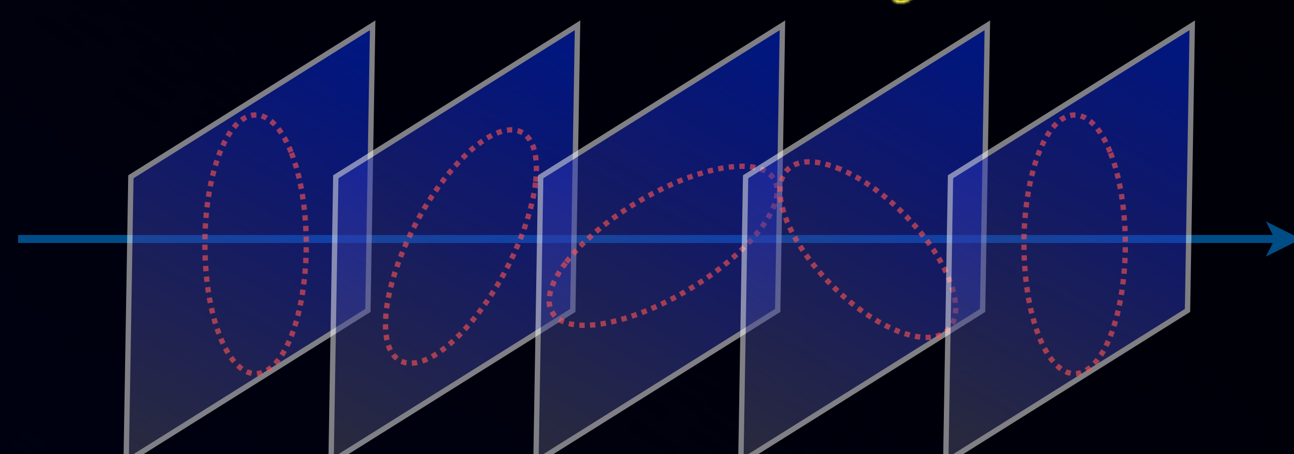
$$\frac{\Delta T(\vec{n}_c,\nu)}{T} \sim f(\nu)y(\vec{n}_c)$$



- Gravitational lensing

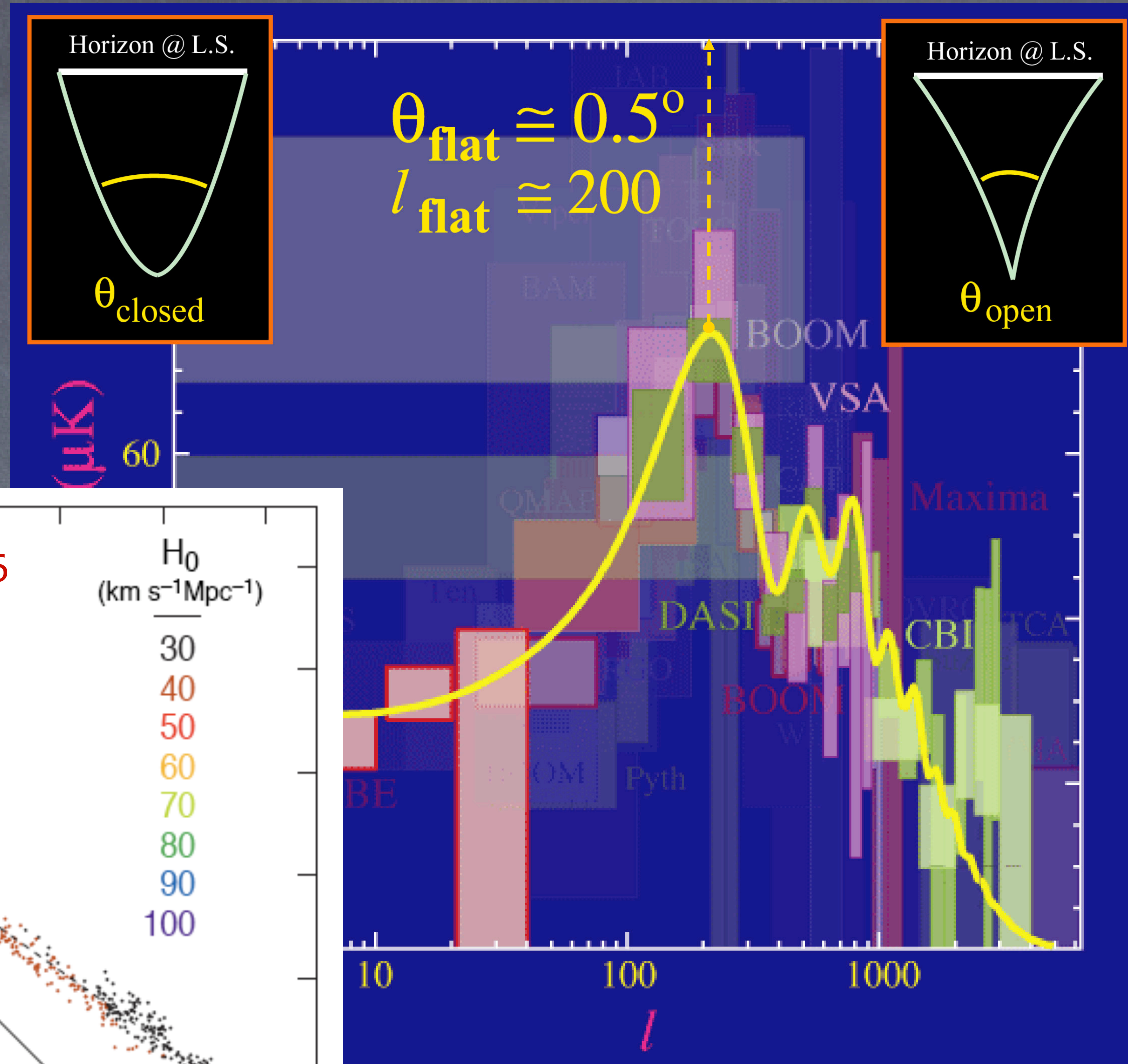
$$n^a \rightarrow \tilde{n}^a = M^a_b n^b$$

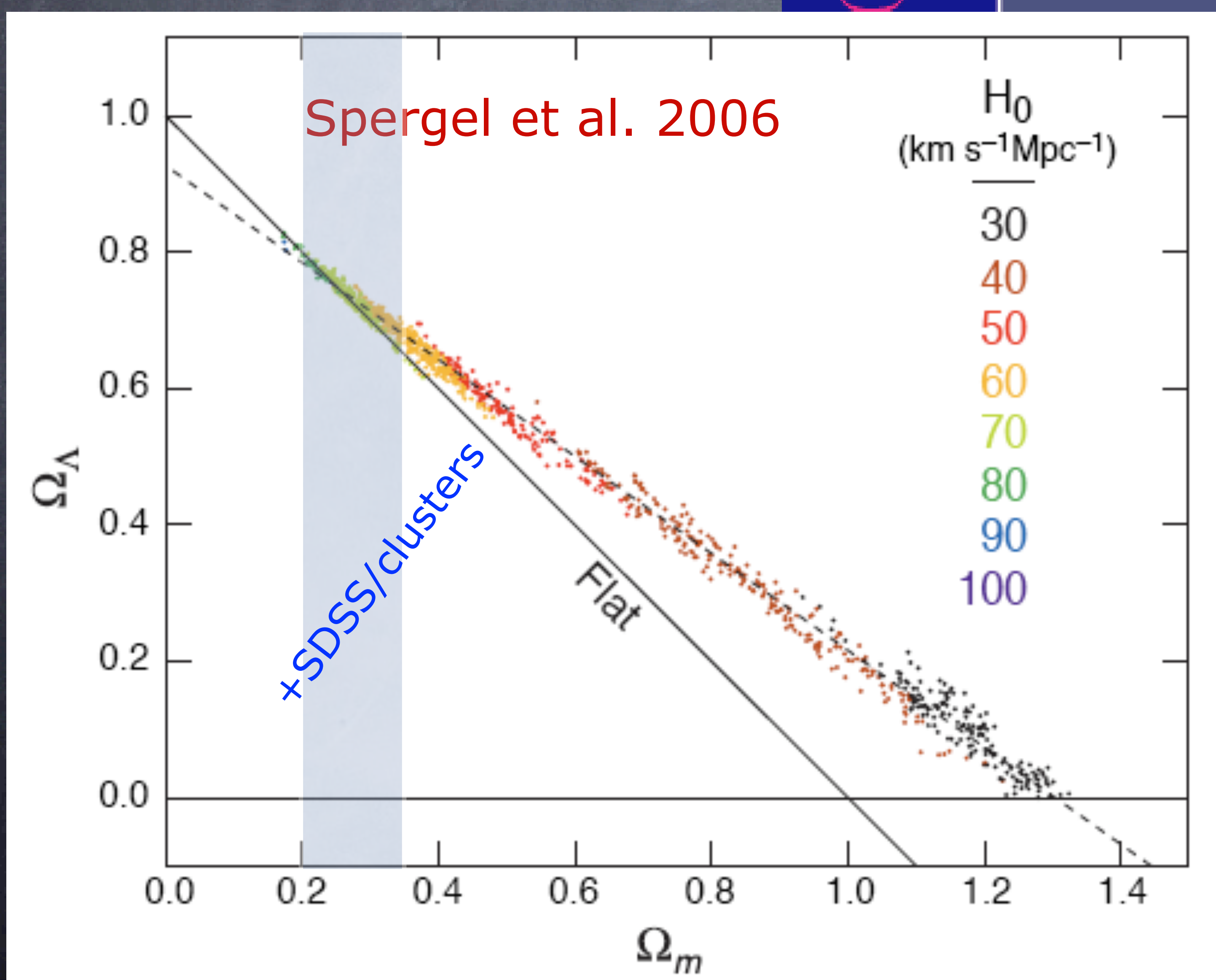
- Gravity waves



Sachs & Wolfe 1967, Silk 1968, Peebles 1968, Zel'dovich and Sunyaev 1969, Peebles & Yu 1970, Polnarev 1987
Modern theory: Bond & Efstathiou, Bond & Szalay, Hu & Sugiyama 1995, Seljak & Zaldarriaga, Hu & White ...

Observations: CMB





$$\Omega_K = 1 - \Omega_m - \Omega_\Lambda$$

2. CMB polarization

Bond & Efstathiou 1984
Polnarev 1985
Kosowski 1996
Seljak & Zaldarriaga 1997
Hu & White 1997
Cabella & Kamionkowski 2005
Li & Wandelt 2005

- Prior to decoupling (z >1100), radiation was unpolarized: $\langle I_i\rangle \neq 0$, $\langle Q_i\rangle = \langle U_i\rangle = \langle V_i\rangle = 0$

- After decoupling, **Thomson scattering** of CMB photons off free electrons generates polarization. The cross-section for an incident photon with polarization ε_i emerging with polarization ε_f is:

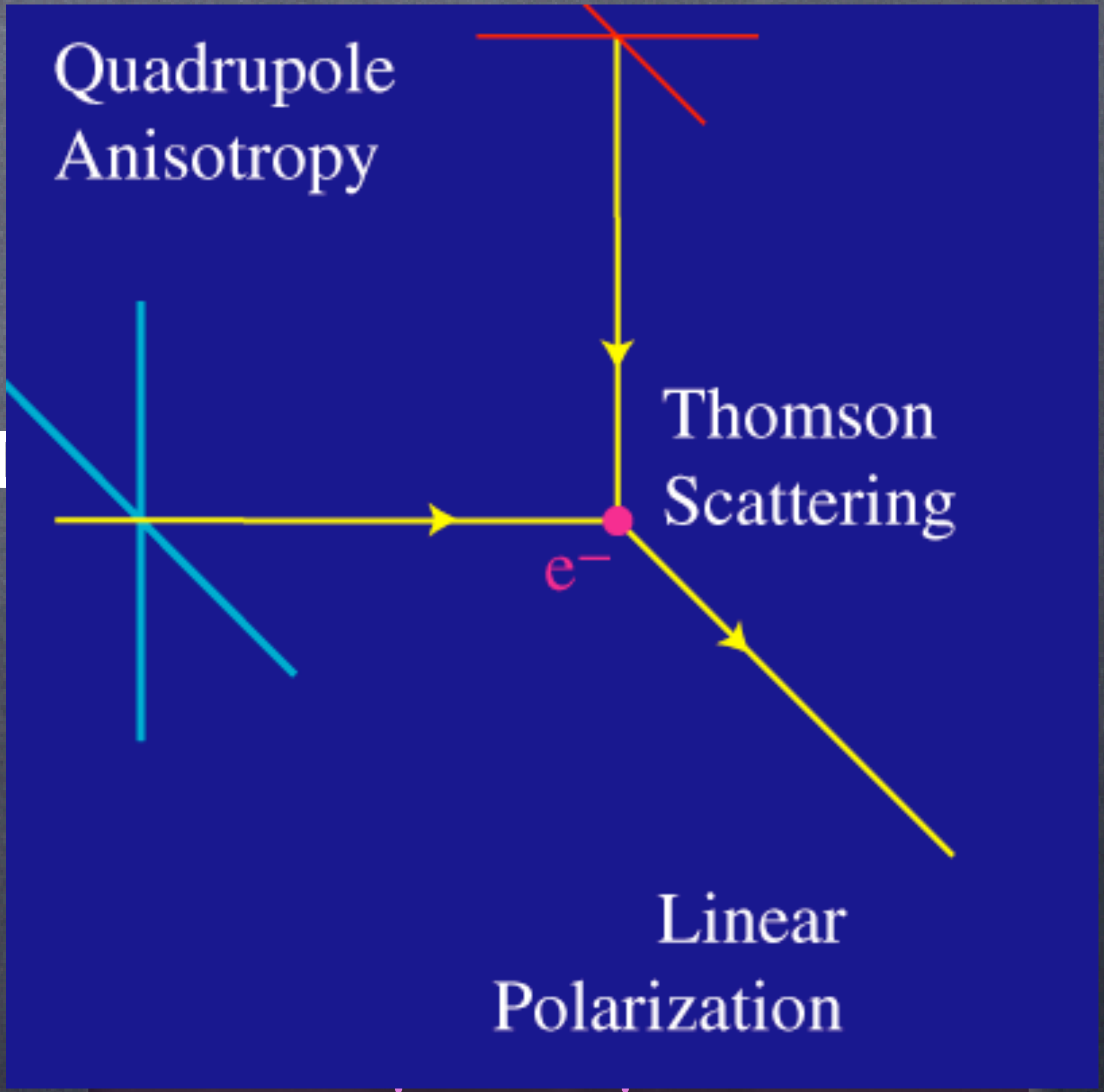
$$\frac{d\sigma}{d\Omega} = \frac{3\sigma_T}{8\pi} |\hat{\epsilon}_i \cdot \hat{\epsilon}_f|^2$$

!!!

- Integrating over incident radiation field I
final state:

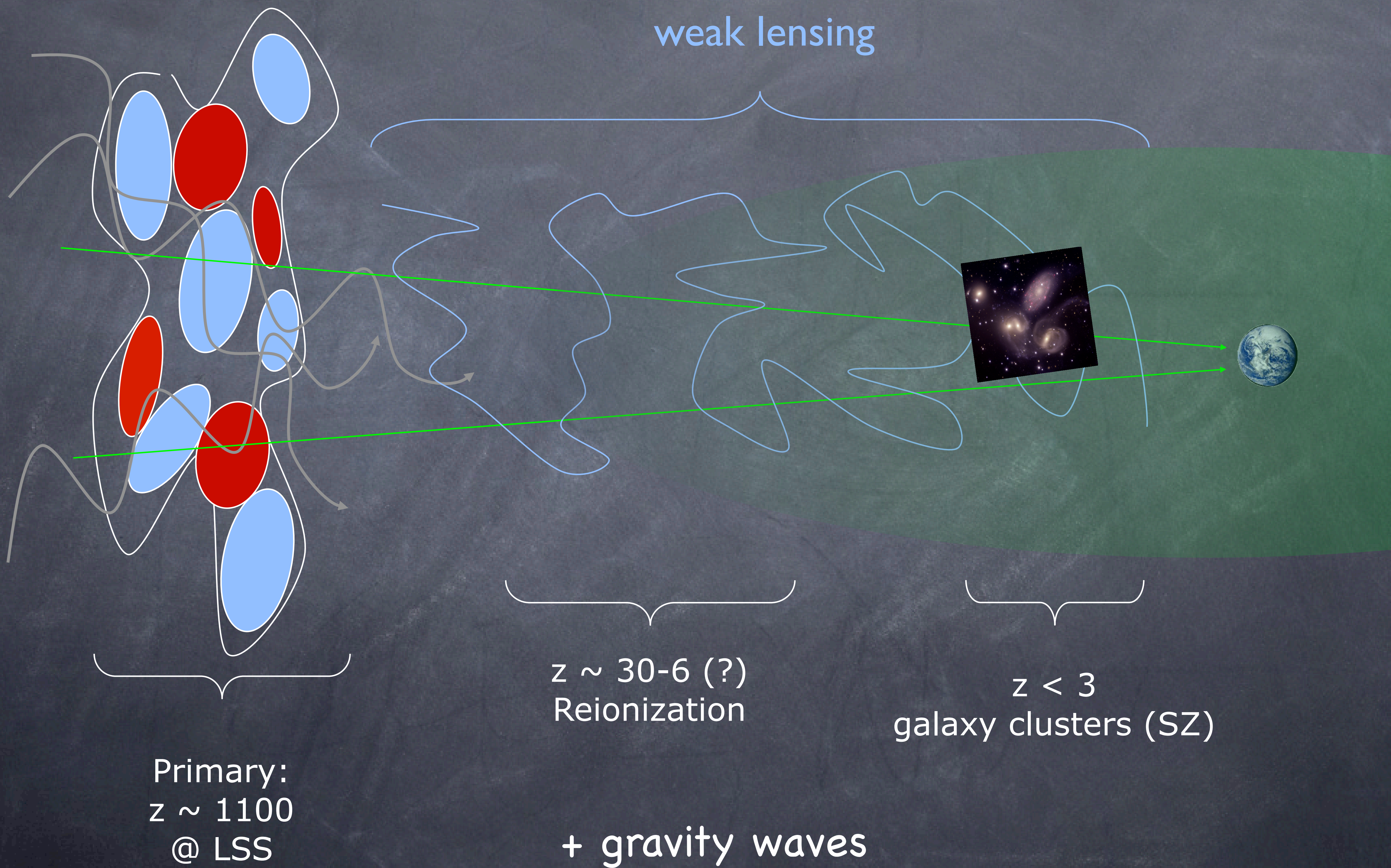
$$Q_f = \frac{3\sigma_T}{16\pi} \int d\Omega \sin^2\theta \cos 2\varphi I_i(\theta,\varphi)$$

$$U_f = -\frac{3\sigma_T}{16\pi} \int d\Omega \sin^2\theta \sin 2\varphi I_i(\theta,\varphi)$$



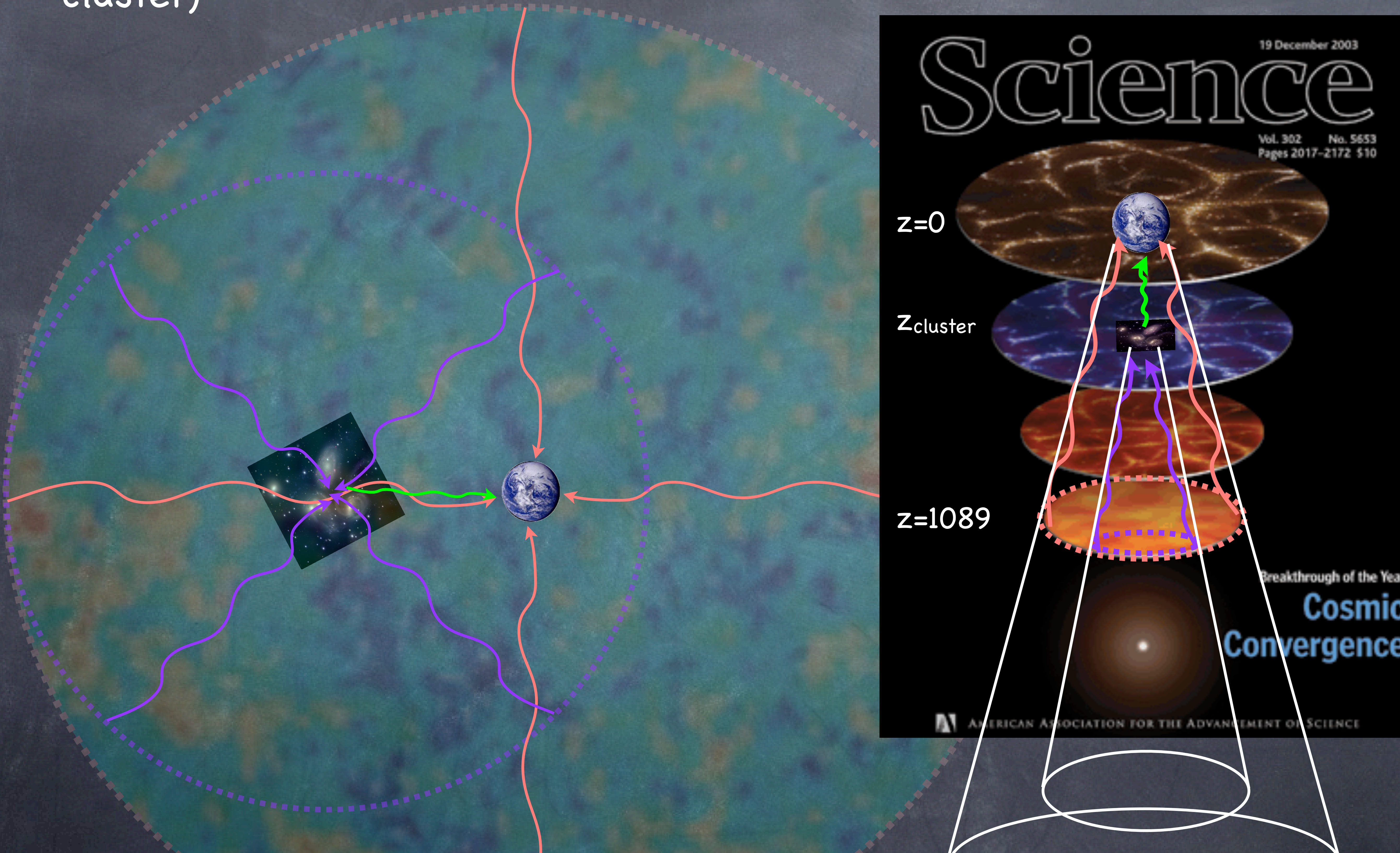
of incident radiation

Main sources of primary/secondary CMB polarization:
free electrons, with optical depth to Thomson scattering: $d\tau = \sigma_T \, n_e \, d\eta$



+ gravity waves

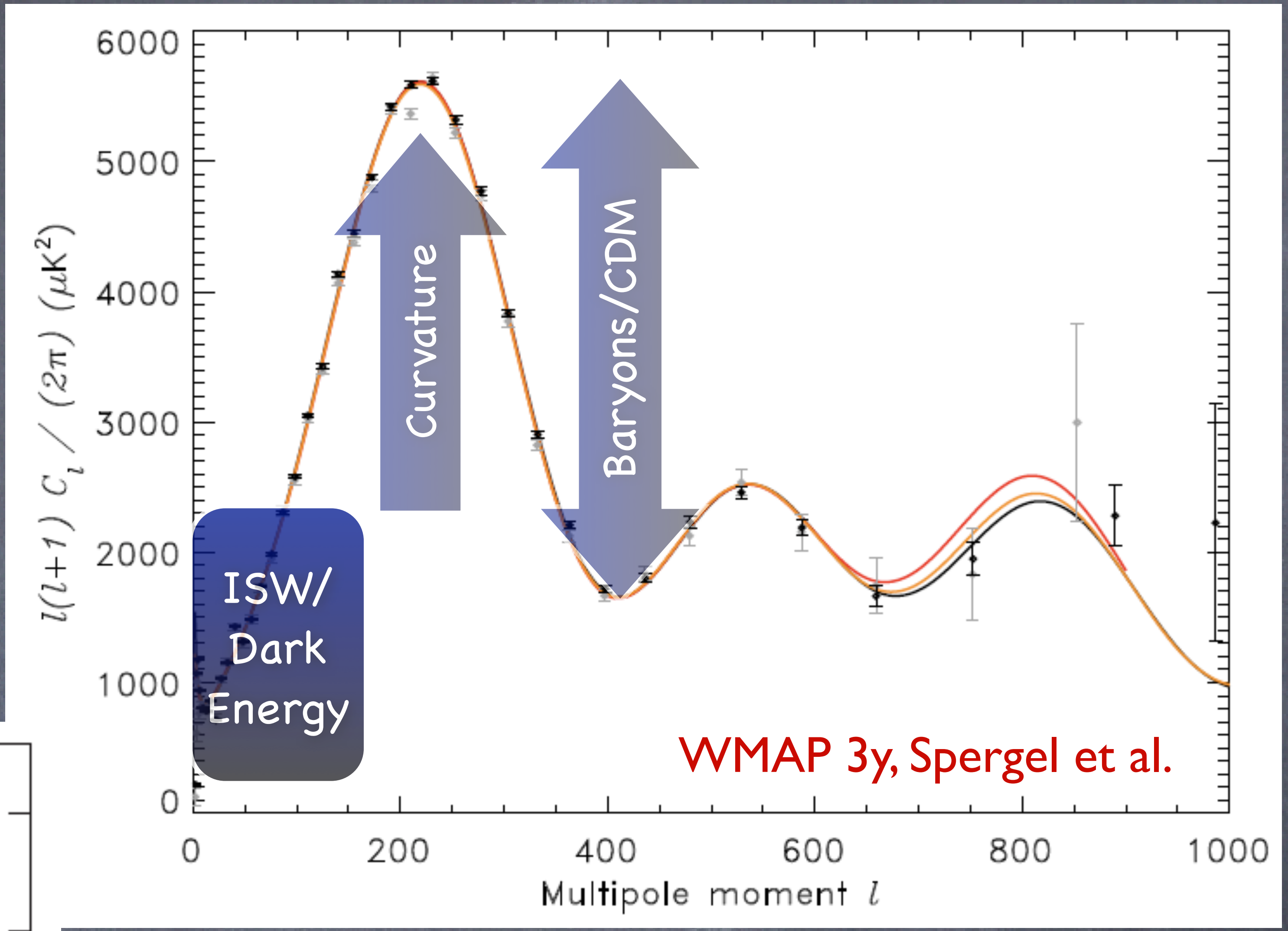
CMB polarization from a cloud of free electrons (e.g., a galaxy cluster)

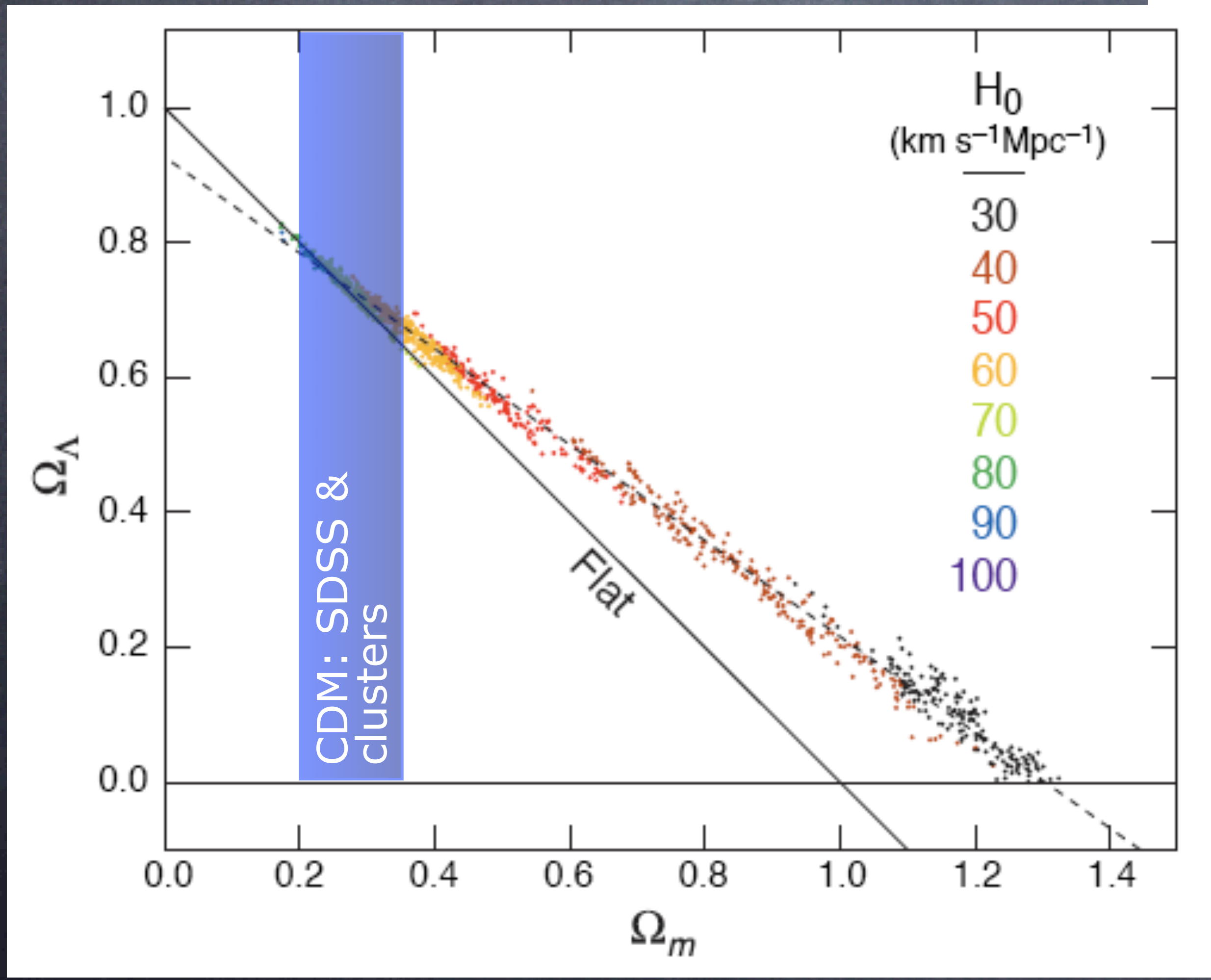


Description in position (real) space:
R.A. & H. Xavier PRD 2007
R. A., P. Reimberg & H. Xavier, PRD 2010

CMB Temperature: Precision cosmology

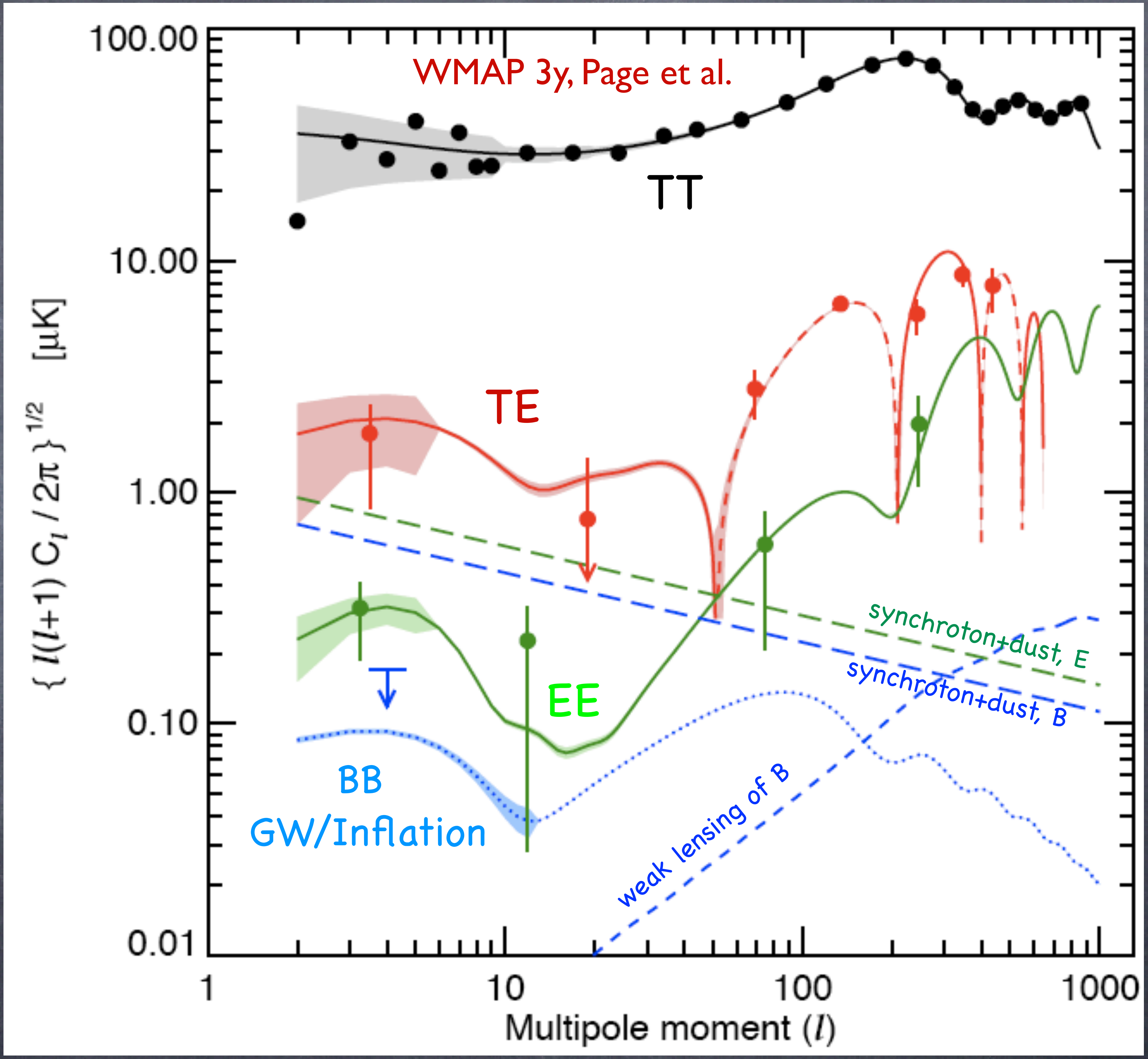
H_0,
Ω_{cdm}, Ω_b, Ω_Λ, Ω_K,
n_s, σ_8,
etc.



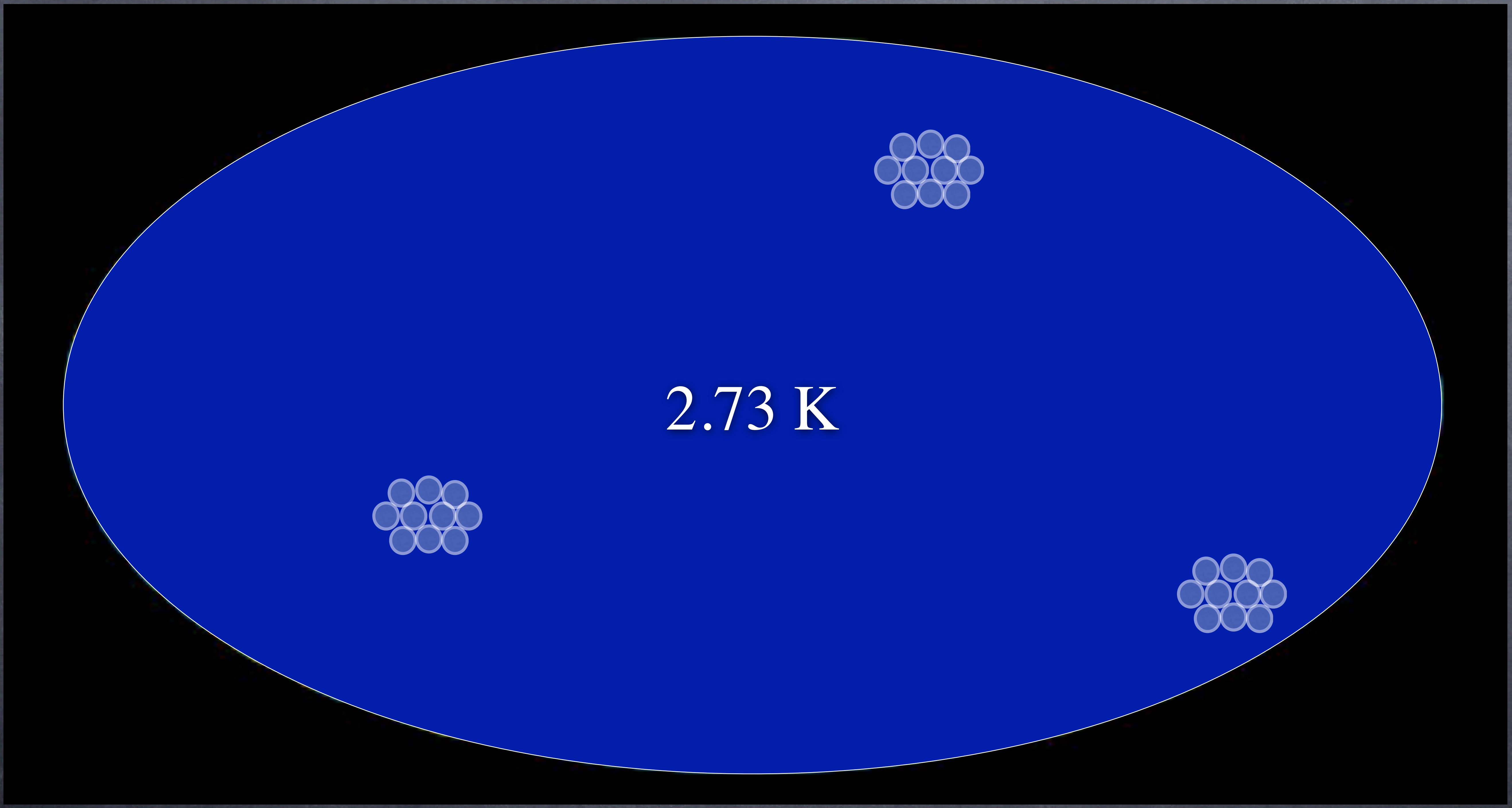


CMB Polarization:

all that
+
infl. gravity waves
inflation v. defects
reionization
magnetic fields
adiabatic/
isocurvature
etc. etc.



CMB and causality



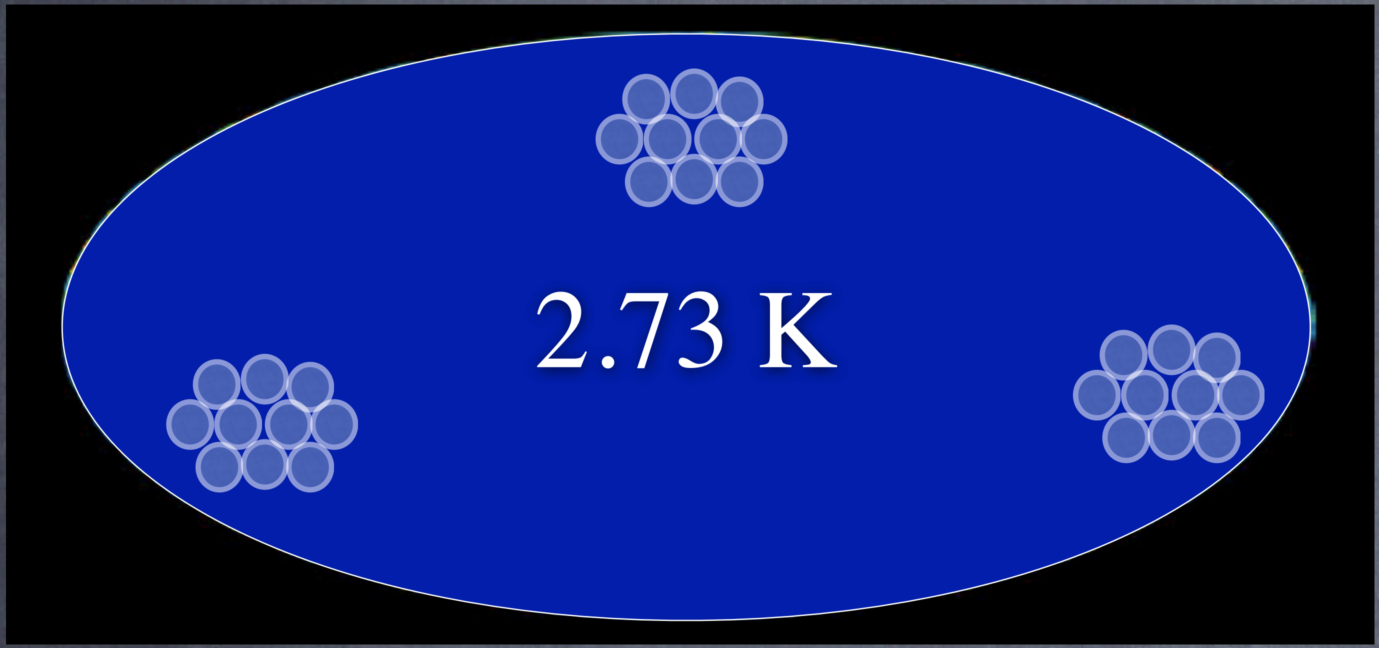
- How come these regions have all the **same** temperature, and the (tiny!) **fluctuations** also share the **same statistical nature** (Gaussian random) ???...

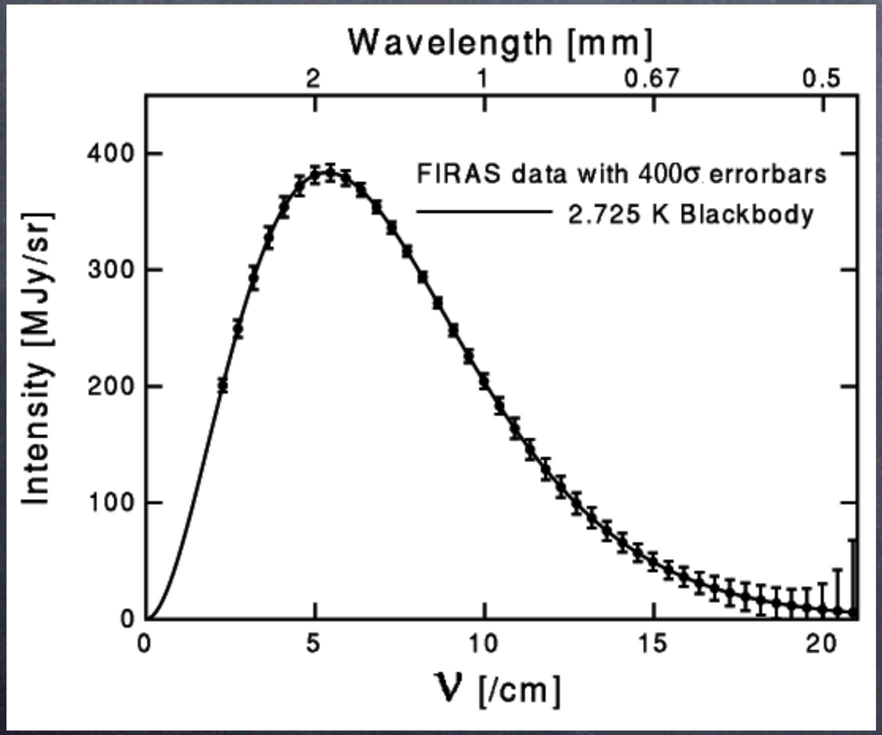
Part 6: Big Bang, horizons, inflation, and all that

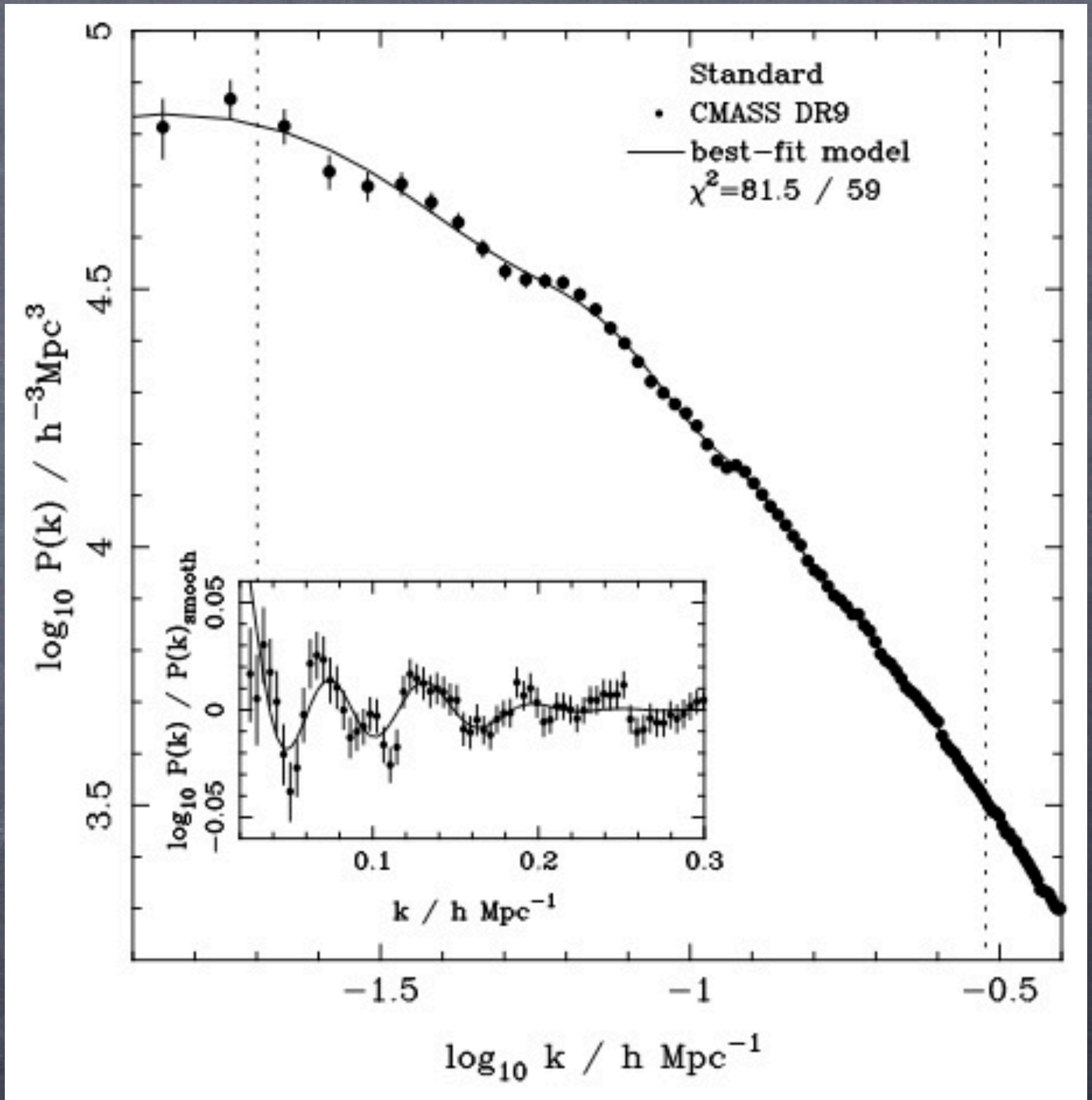
Some crucial observations to understand our Universe:

- **Cosmic microwave background, or CMB** (COBE, Boomerang, Dasi, QUAD, …, WMAP… PLANCK…)

- Matter distribution over large scales – "**Matter Power Spectrum**"







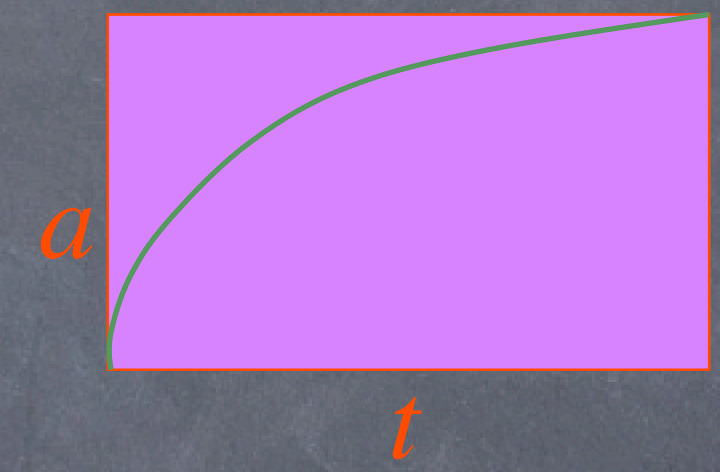
- The physics of the CMB involves **propagation** and **scattering** of photons

- The CMB is also the **most distant direct observation** we have of the universe in its infancy, hence it is a key observable to **test physical processes, as well as correlations and causality,** over the **largest observable scales**

- In an FLRW spacetime, proper distances for light-speed signals can be **finite** even when the travel time extends arbitrarily into the past or into the future.

- For instance, let's take a **decelerating** FLRW:

$$a(t) = a_0 \left(\frac{t}{t_0} \right)^p \quad , \quad 0 < p < 1$$

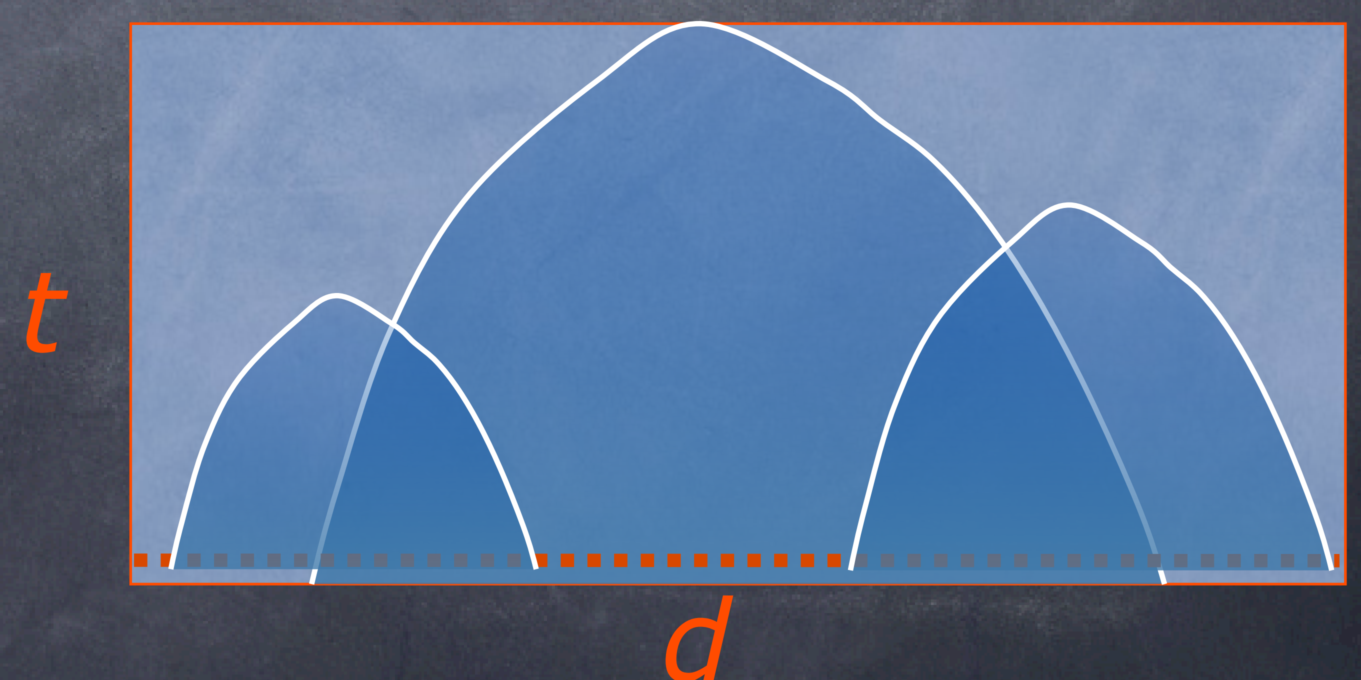


This spacetime can be continued to the past only down to t=0 (when **a=0**). Then:

$$d_{pH}(t) = a(t) \int \frac{dt'}{a(t')} = \left(\frac{t}{t_0} \right)^p \int_0^t dt' \left(\frac{t'}{t_0} \right)^{-p}$$

$$= \frac{1}{1-p} t = \frac{p}{1-p} H^{-1}(t)$$

d_{Hp} is the maximum physical distance a light ray can cover if it was emmitted at the earliest possible time in the past. This means that the **past light cone** in this scenario is **bounded**, and cannot be extended beyond that limit.



• This **maximal distance** is called a **horizon**. Since in this case ($p<1$) the horizon refers to a truncation of the PLC, it is a past-like horizon, a.k.a. a **particle horizon**. This horizon is usually approximately equal to the **curvature radius** of the FLRW, $r \sim 1/R^{-1/2} \sim H^{-1}$ – i.e., the Hubble radius!

• The particle horizon separates observers which **never had causal contact prior to the time t**. Therefore, when there is a particle horizon, the Universe can be separated into regions which are (up to that time) **causally disconnected**

• Since the Universe has been, for most of its history, dominated by either radiation (p=1/2) or matter (p=2/3), if that were true down to t=0, then our particle horizon today would be:

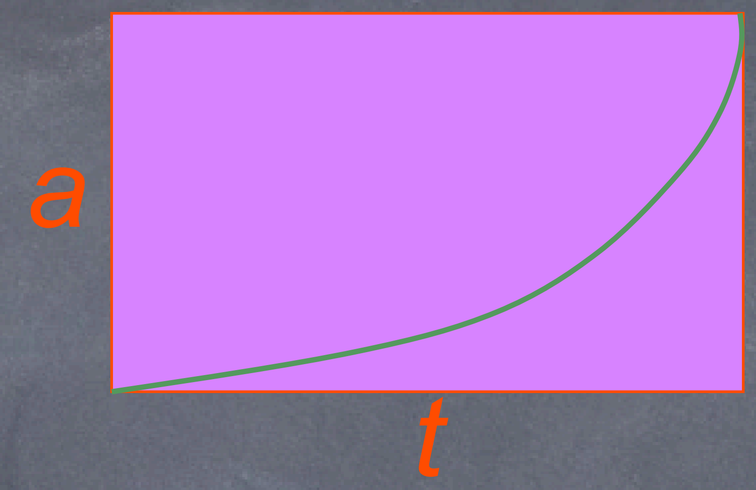
$$d_H(0) \approx c \times t_0 \approx 4600 \text{ Mpc}$$

¿
How can the CMB be so homogeneous
over the whole sky
?

Problem #20: Compute the particle horizon at the time of decoupling (t~380.000 y, z~1100), assuming that p=1/2. Answer: ~200 Kpc

- Now take an **accelerating** scale factor:

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^p \quad , \quad p > 1$$



We still have some initial time t=0. **However**:

$$d(t) = \left(\frac{t}{t_0}\right)^p \int_{t_i}^{t} dt' \left(\frac{t'}{t_0}\right)^{-p}$$

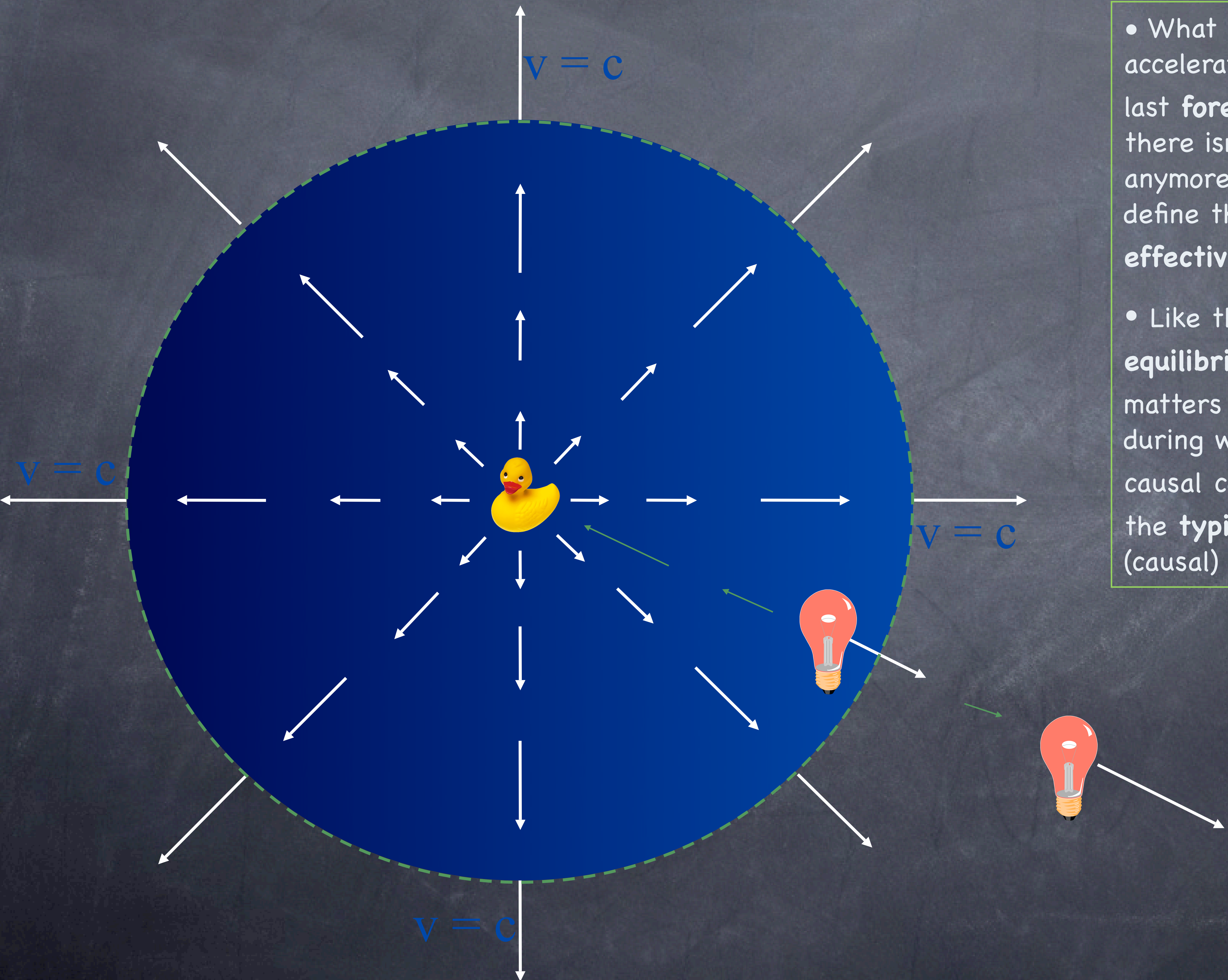
is now an **arbitrarily large** distance as we take the lower limit $t_i \to 0$, and hence **there is no particle horizon in this case!**

However, consider, instead, what happens if the **upper limit** is take to be $t_f \to \infty$, and take the **lower limit** to be t.

This distance would then correspond to the **maximal length** that separates two objects such that they could **ever** exchange a light-speed **signal emmitted at time** t. If that maximal distance is **not infinity**, then there an **event horizon**:

$$d_{eH}(t) = \left(\frac{t}{t_0}\right)^p \int_{t}^{\infty} dt' \left(\frac{t'}{t_0}\right)^{-p} = \frac{1}{p-1} t = \frac{p}{p-1} H^{-1}(t)$$

- The physical meaning of an **event horizon** is that it marks the **boundary** beyond which observers lose the possibility of **causal connection** in the **future**.



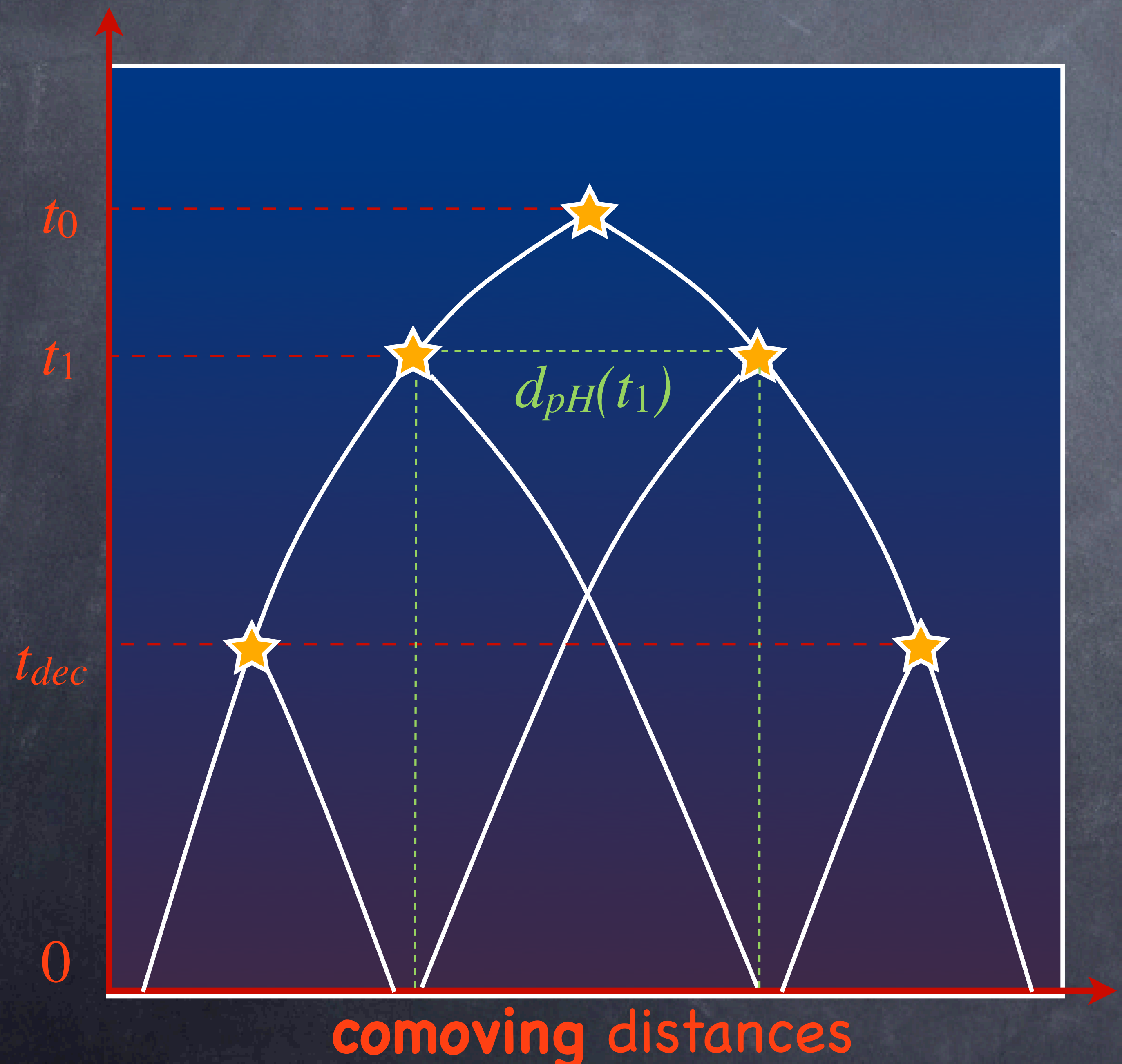
- What happens when the accelerated expansion doesn't last **forever**? Mathematically, there isn't an event horizon anymore - but still we can define the notion of an **effective** horizon:

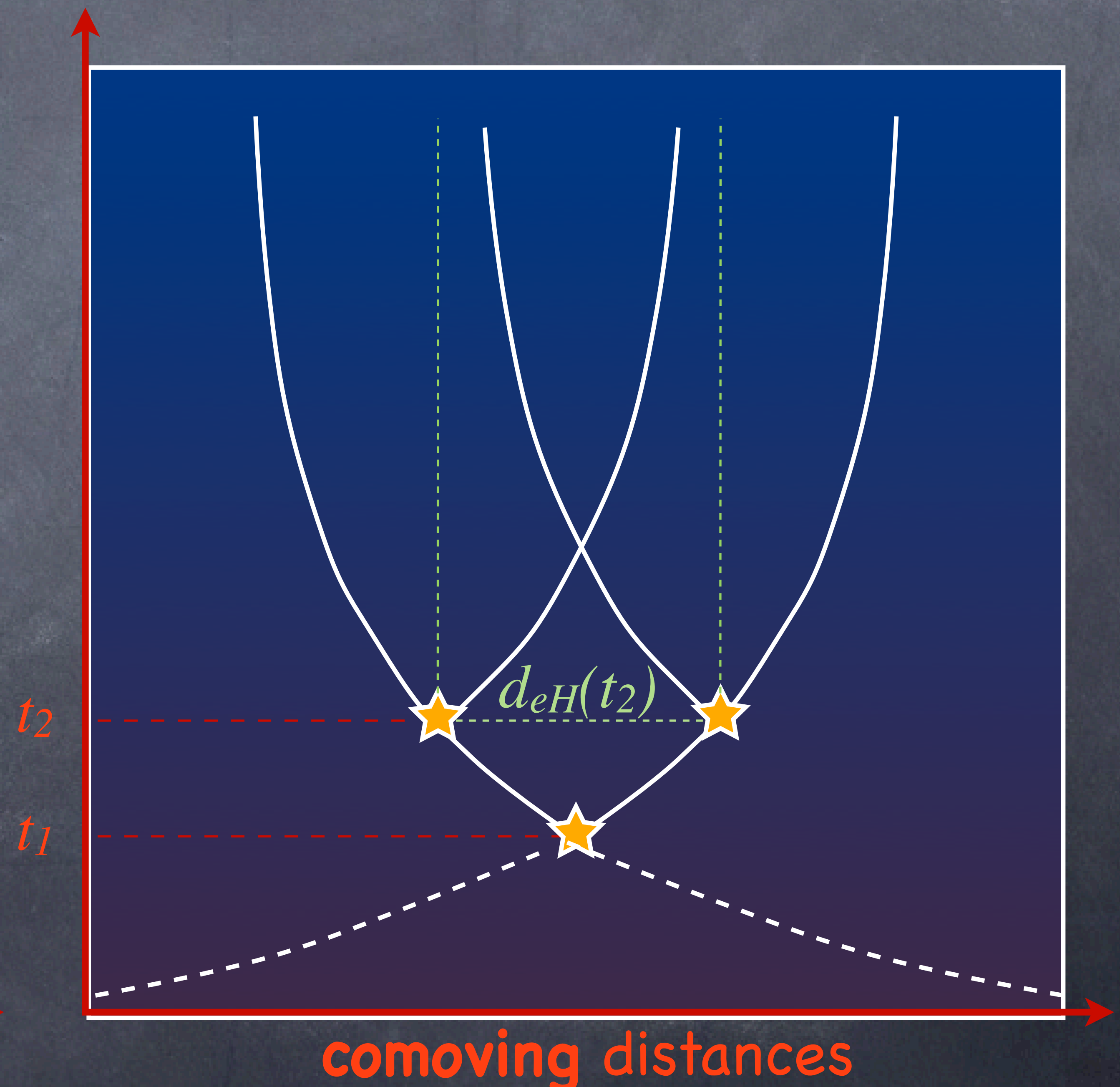
- Like the notion of **thermal equilibrium**, what really matters is the **time interval** during which there is no causal contact, **compared** to the **typical times** for other (causal) physical processes

- The physical significance of a **horizon** is profound, as it clearly marks **causality boundaries**:

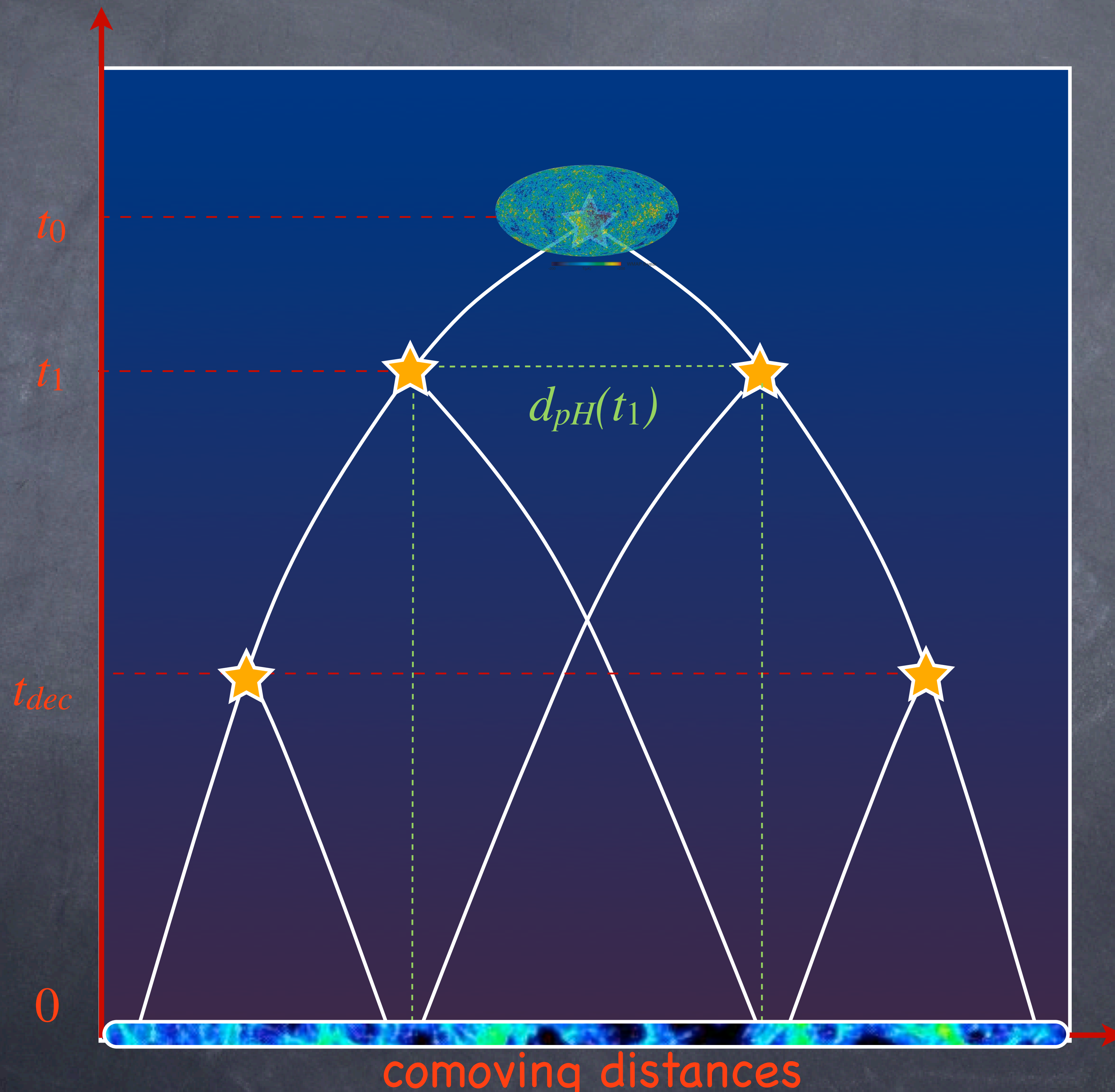
➡ A **particle horizon** sets a limit to the **past light-cone** of observers at time t: pairs of inertial observers separated by a distance larger than d_{pH} at time t **have never been** in causal contact **before** t.

➡ An **event horizon** sets a limit to the **future light-cone** of inertial observers at time t: pairs of observers separated by a distance larger than d_{eH} at time t **will never again** be in causal contact **after** t.





- But for the "standard" FLRW hot Big Bang model, this is precisely what happens: our (decelerating) Universe would **have** a particle horizon - and a pretty small one!



Problems of the dust + radiation Friedmann models

- **Horizon problem**: if the expansion rate has always been decelerating, then the visible Universe today is a much "larger" region than it was at the time of decoupling. If that were true, then the CMB shows many causally disconnected regions which are incredibly similar (same temperature, density, pressure, etc.) So, why does the CMB shows such an isotropic and homogeneous Universe, on such immense scales?

- **Curvature problem**: the density of the Universe today is very close to the critical one, $\Omega \cong 1$. However, if curvature wasn't set to zero at the Big Bang, with an incredible accuracy, then it would have become dominant a long time ago! How can this be?...

- **Problem of the origin of the primordial inhomogeneities (Robert's talk!)**: the standard Big Bang model does not give any clue as to why or how there were only very small fluctuations in the density field. It does not say anything about their amplitude (which is near scale-invariant), or about their statistical nature (nearly Gaussian).

The solution to all of these problems: accelerated expansion (inflation)

The main idea is that the Universe, in its **very early stages**, went through a phase of **accelerated expansion**:

$$\ddot{a} > 0 \quad , \quad q = -\frac{a\ddot{a}}{\dot{a}^2} < 0$$

E.g., the scale factor could have behaved as:

$$a = a_1 \left(\frac{t}{t_1} \right)^p \quad , \quad p > 1 \qquad\qquad \lim p \to \infty \ : \quad a = e^{Ht}$$

$$\frac{\ddot{a}}{a} = \frac{p(p-1)}{t^2} > 0$$

This accelerated phase could be due to some type of matter with a negative equation of state, such that:

$$\rho \propto a^{-3(1+w)} \quad , \quad w = \frac{2-3p}{3p}$$

OK, but... what does that achieve?

✓ The curvature problem

The 0-0 Friedmann equation (again!) is:

$$3H^2 + \frac{K}{a^2} = 8\pi G \rho$$

If the matter that dominates the Universe decays less fast than curvature, then it will make curvature less dominant with time.
In our example of power-law expansions, if ***p>>1***, then the energy density stays nearly constant:

$$\rho \propto a^{-2/p}$$

In the limit ***p*** $\rightarrow$ **∞** , the density is ***constant*** and the contribution of spatial curvature decays exponencially:

$$\lim p \rightarrow \infty \quad : \qquad a = e^{Ht} \quad , \quad \rho \propto \text{const} \quad , \quad \frac{K}{a^2} \propto e^{-2Ht}$$

Hence, the result of an era of accelerated expansion is to **suppress the spatial curvature**. All models of inflation do that: the acceleration era is long enough that curvature today is completely negligible - even after the radiation and matter eras.

Therefore, inflation "predicts" that Ω=1.

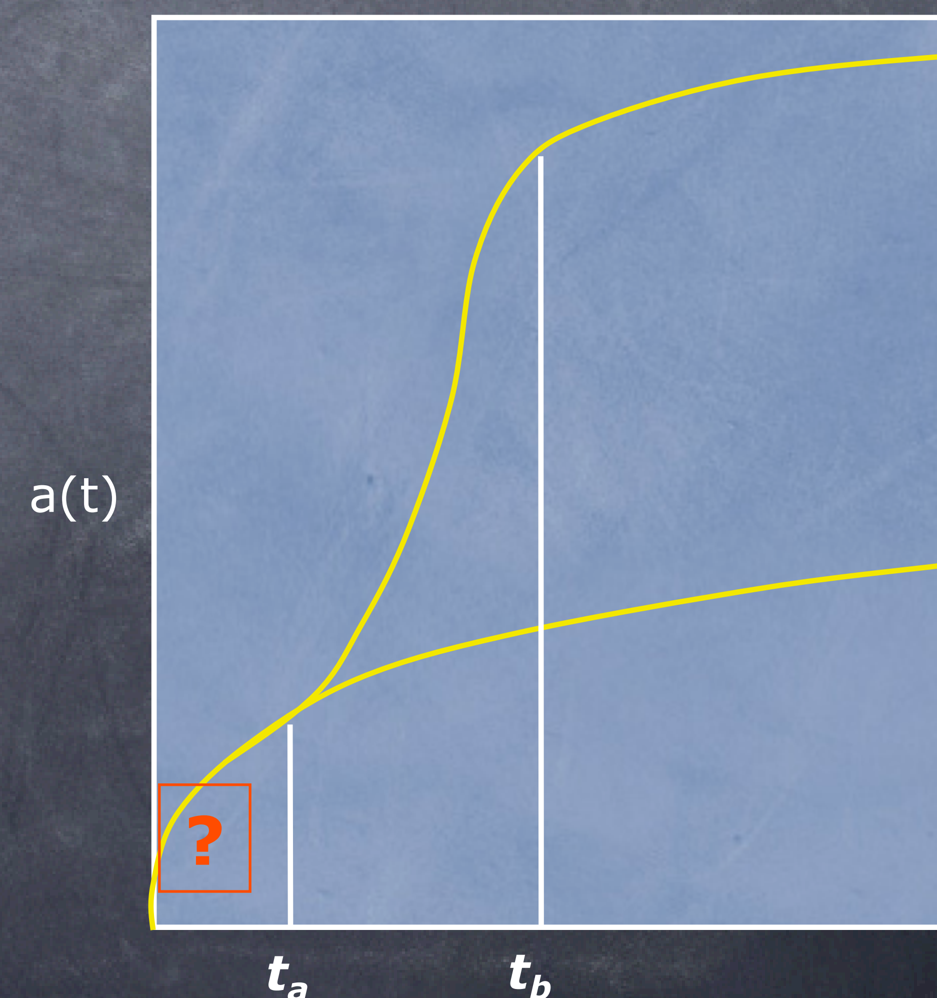
✓ The horizon problem

Consider, for simplicity, the radiation era, and let's say that spatial curvature is gone. Without inflation we would have:

$$\left.\begin{array}{l} \rho \propto a^{-4} \\ 3H^2 = 8\pi G\rho \end{array}\right\} a = t^{1/2} \qquad \Rightarrow d_{Hp} = a(t)\int_0^t \frac{dt'}{a(t')} = 2t$$

Now, let's say that between t_a e t_b, the Universe's evolution was dominated by some type of matter that caused an accelerated expansion.
The scale factor would then look something like this:

$$a(t) = \begin{cases} t^{1/2} & t \le t_a \\ t_a^{1/2}\left(\dfrac{t}{t_a}\right)^p & t_a \le t \le t_b \\ t_a^{1/2}\left(\dfrac{t_b}{t_a}\right)^p \left(\dfrac{t}{t_b}\right)^{1/2} & t \ge t_b \end{cases}$$

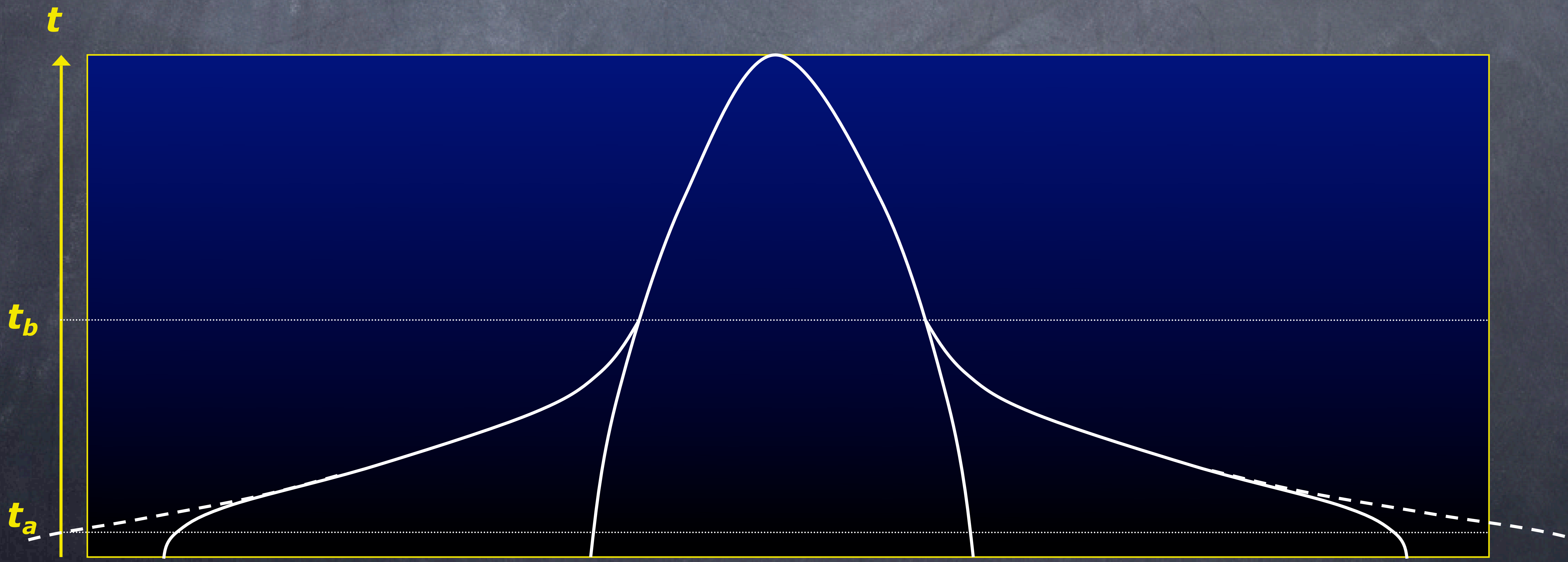


The particle horizon would then become **much larger:**

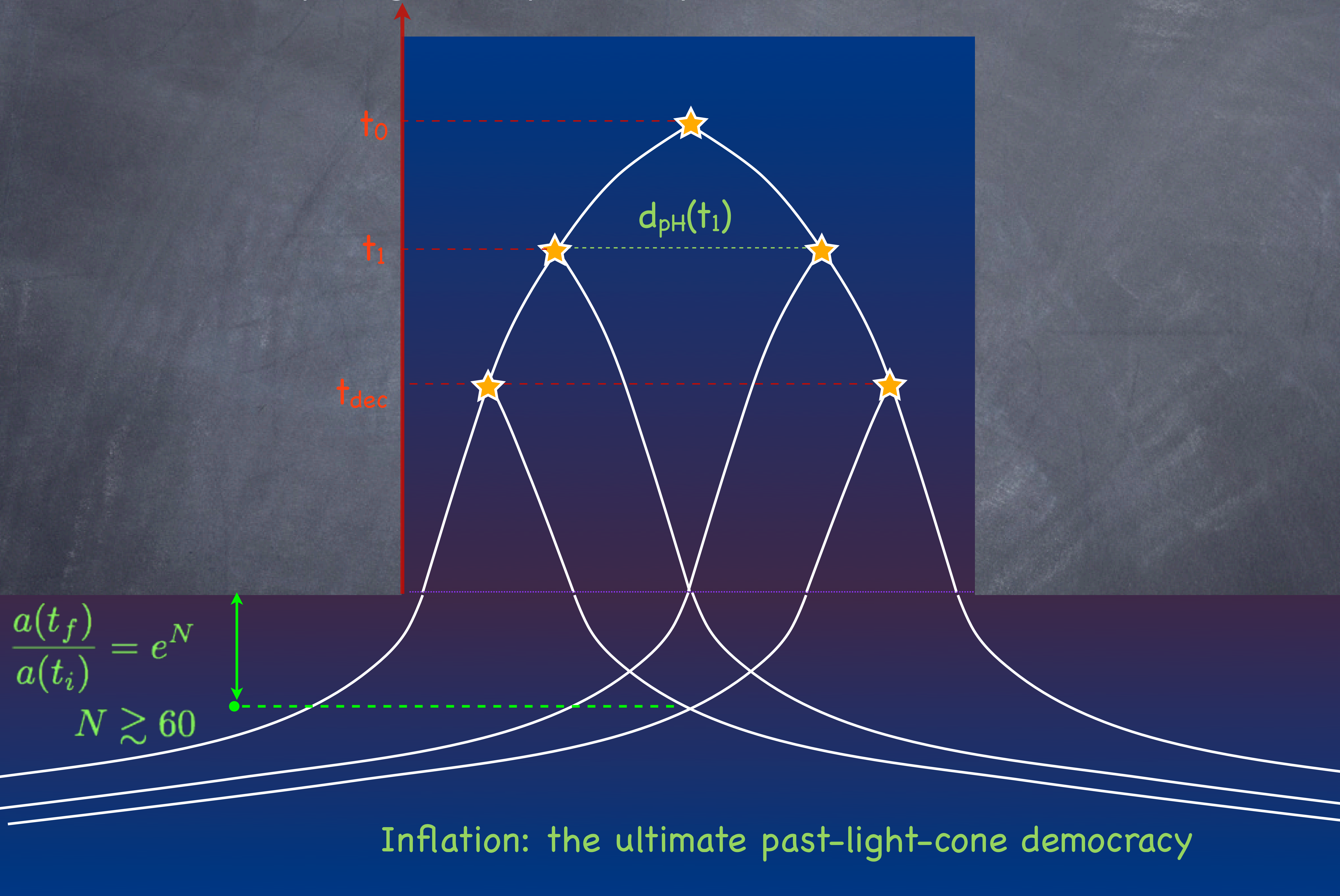
$$d_{Hp} = 2t + \frac{t_a}{p-1}\left(\frac{t_b}{t_a}\right)^p$$

☛ **Problem #22: check this!**

So, if $t_b >> t_a$ (that is, if the inflationary phase lasts long enough), and if $p>>1$, then the second term is much larger than the first (which is simply the horizon in a radiation-dominated Universe).
The effect of inflation upon the past light cone is this:



- Therefore, this is how inflation solves the puzzle of the large-scale homogeneity of the Universe: past light-cone promiscuity!



Acceleration with **scalar fields!**

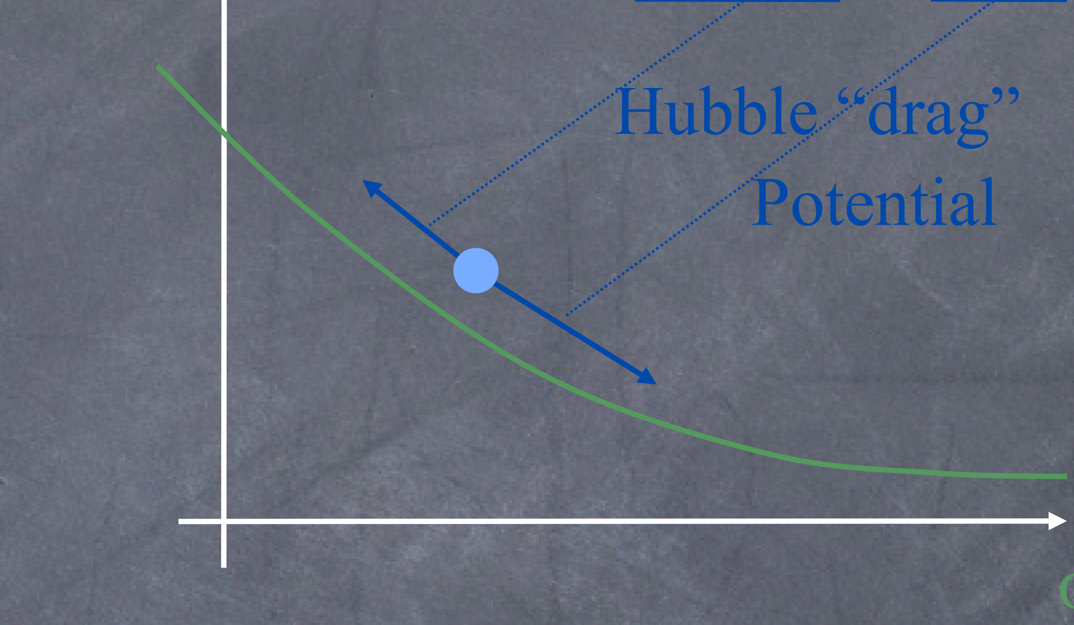
Eq. motion (Klein-Gordon)

$$L_\phi = -\tfrac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - V(\phi)$$

small

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0$$

Background, $\phi(t)$:



$$\rho_\phi = \tfrac{1}{2}(\dot{\phi})^2 + V(\phi)$$

$$p_\phi = \tfrac{1}{2}(\dot{\phi})^2 - V(\phi)$$

If the kinetic energy is << than the potential energy => "**slow roll**" :

$$\rho_\phi \approx V(\phi) \approx \text{constante}$$

$$p_\phi \approx -V(\phi)$$

With **slow-roll**, V(ϕ) works like a time-varying Λ

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}(\dot{\phi})^2 - V(\phi)}{\frac{1}{2}(\dot{\phi})^2 + V(\phi)} \Rightarrow w_\phi \cong -1$$

- Problem #33: the "power-law" model of inflation. Take an exponential potential:

$$V = M^4 e^{-\phi/s} \quad , \quad V_{,\phi} = -\frac{1}{s} V$$

The Friedmann and Klein-Gordon take this form:

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0$$

$$3H^2 = 8\pi G\left(\tfrac{1}{2}\dot{\phi}^2 + V\right)$$

Let's now try a solution of the type: $a(t) = a_0 \left(\frac{t}{t_0}\right)^p \quad , \quad H = \frac{p}{t}$

By inspection of the equations above,
we are led to consider a scalar field: $\phi(t) = \phi_0 \ln \frac{t}{t_0}$

Substituting these expressions in the equations above, we have:

$$\left.\begin{aligned} -\frac{\phi_0}{t^2} + \frac{3p}{t}\frac{\phi_0}{t} - \frac{M^4}{s} e^{-(\phi_0/s)\ln t/t_0} = 0 \\ 3\frac{p^2}{t^2} = 8\pi G\left(\frac{1}{2}\frac{\phi_0^2}{t^2} + M^4 e^{-(\phi_0/s)\ln t/t_0}\right) \end{aligned}\right\} \Rightarrow \begin{aligned} \frac{\phi_0}{s} = 2 \\ 2(3p-1) = \frac{M^4 t_0^2}{s^2} \\ 3p^2 = 8\pi G s^2\left(1 + \frac{M^4 t_0^2}{s^2}\right) \end{aligned}$$

Solving for the power ***p***, we obtain:

$$s = \sqrt{\frac{p}{16\pi G}}$$

$$\phi_0 = 2s$$

$$t_0 = \frac{p}{M^2\sqrt{8\pi G}}\sqrt{p(3p-1)}$$

Therefore, we have determined that there is an **analytic solution** in the case of an exponential potential, given by:

$$\phi(t) = \frac{\sqrt{p}}{\sqrt{4\pi G}}\ln\frac{t}{t_0}$$

$$a(t) = a_0\left(\frac{t}{t_0}\right)^p$$