Lecture 4

Gravitational Wave Production and Detection

Jolien Creighton

University of Wisconsin–Milwaukee

July 17, 2012

Properties of gravitational waves

- Polarizations of a gravitational wave
- Energy carried by a gravitational wave
- Beyond Newtonian motion and quadrupole radiation

Gravitational wave sources

- Rotating triaxial ellipsoid
- Orbiting binary system

Gravitational wave detectors

- Interferometers
- Pulsar timing arrays

Polarizations of a gravitational wave

Recall plane wave solution in TT gauge:

$$\mathbf{h}_{\mathrm{TT}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $h_+ = h_+(t - z/c)$ and $h_\times = h_\times(t - z/c)$
Independent components of Riemann tensor are

$$R_{0101} = -R_{0202} = -\frac{1}{2}\ddot{h}_+ \quad \text{and} \quad R_{0102} = R_{0201} = -\frac{1}{2}\ddot{h}_\times$$

Remaining components from Riemann symmetries and the identity $R_{3\alpha\beta\gamma} + cR_{0\alpha\beta\gamma} = 0$

Polarizations of a gravitational wave

Geodesic deviation equation: let

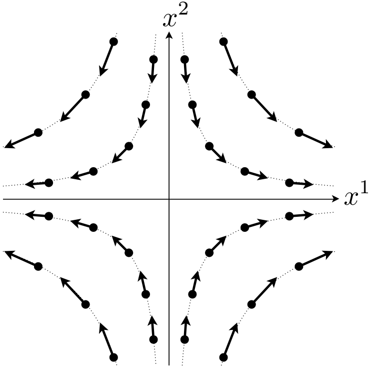
$$\boldsymbol{\zeta} = [\zeta \sin\theta\cos\phi,\ \zeta\sin\theta\sin\phi,\ \zeta\cos\theta]$$

Then,

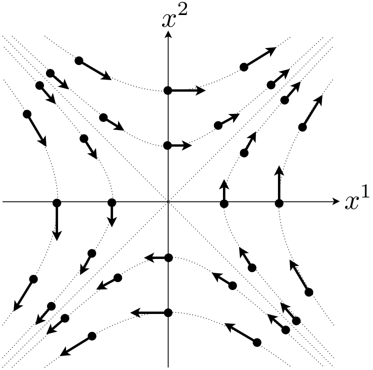
$$\begin{aligned}
\frac{d^2\zeta_1}{dt^2} &= -R_{010i}\zeta^i = \frac{1}{2}\ddot{h}_+\zeta\sin\theta\cos\phi + \frac{1}{2}\ddot{h}_\times\zeta\sin\theta\sin\phi \\
\frac{d^2\zeta_2}{dt^2} &= -R_{020i}\zeta^i = -\frac{1}{2}\ddot{h}_+\zeta\sin\theta\sin\phi + \frac{1}{2}\ddot{h}_\times\zeta\sin\theta\cos\phi \\
\frac{d^2\zeta_3}{dt^2} &= 0
\end{aligned}$$

Tidal force is transverse

Polarizations of a gravitational wave

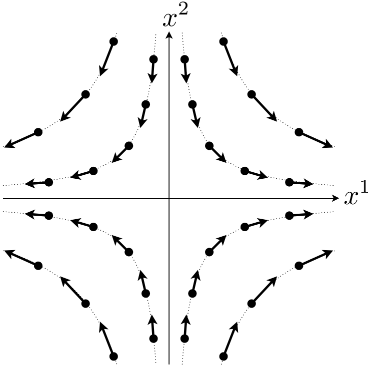


plus polarization

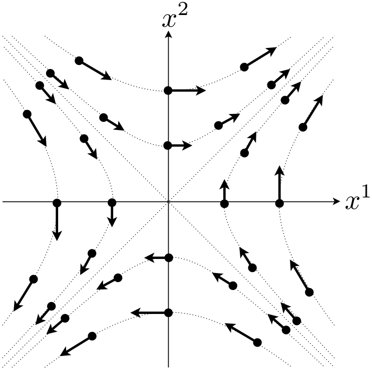


cross polarization

Polarizations of a gravitational wave

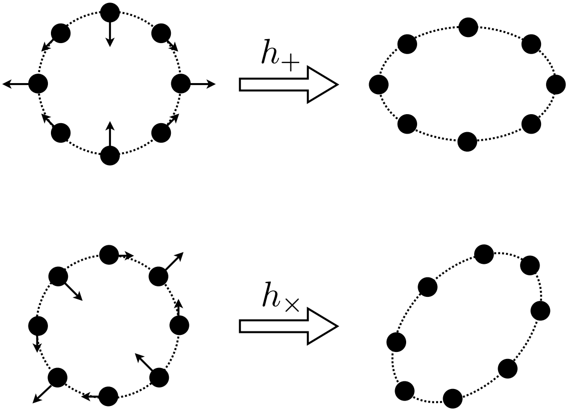


plus polarization



cross polarization

Polarizations of a gravitational wave



Energy carried by a gravitational wave

Vacuum field equations:

$$0 = G_{\mu\nu} = \underbrace{\overset{0}{G}_{\mu\nu}\,[\eta_{\alpha\beta}]}_{=0} + \underbrace{\overset{1}{G}_{\mu\nu}\,[h_{\alpha\beta}]}_{O(h)} + \underbrace{\overset{2}{G}_{\mu\nu}\,[h_{\alpha\beta}]}_{O(h^2)} + \cdots$$

Write

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} = \eta_{\alpha\beta} + \lambda\,\overset{1}{h}_{\alpha\beta} + \lambda^2\,\overset{2}{h}_{\alpha\beta} + \cdots$$

where λ is an order parameter
Then

$$0 = G_{\mu\nu} = \lambda\,\overset{1}{G}_{\mu\nu}\,[\overset{1}{h}_{\alpha\beta}] + \lambda^2\left(\overset{1}{G}_{\mu\nu}\,[\overset{2}{h}_{\alpha\beta}] + \overset{2}{G}_{\mu\nu}\,[\overset{1}{h}_{\alpha\beta}]\right) + O(\lambda^3)$$

Energy carried by a gravitational wave

The first order equation

$$\overset{1}{G}_{\mu\nu}\,[\overset{1}{h}_{\alpha\beta}] = 0$$

gives our gravitational wave solution

The correction to first order solution is due to the back-reaction of the first order solution:

$$\overset{1}{G}_{\mu\nu}\,[\overset{2}{h}_{\alpha\beta}] = \underbrace{-\,\overset{2}{G}_{\mu\nu}\,[\overset{1}{h}_{\alpha\beta}]}_{\frac{8\pi G}{c^4} T^{\mathrm{GW}}_{\mu\nu}}$$

Here

$$T^{\mathrm{GW}}_{\mu\nu} = -\frac{c^4}{8\pi G}\,\overset{2}{G}_{\mu\nu}\,[\overset{1}{h}_{\alpha\beta}]$$

is interpreted as the stress energy tensor of the gravitational wave

Energy carried by a gravitational wave

Volume average over several wavelengths:

$$T_{\mu\nu}^{\mathrm{GW}} = -\frac{c^4}{8\pi G}\left\langle \overset{2}{R}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\,\overset{2}{R} \right\rangle$$

Result is

$$\boxed{T_{\mu\nu}^{\mathrm{GW}} = \frac{c^4}{32\pi G}\left\langle \frac{\partial h_{\mathrm{TT}}^{ij}}{\partial x^{\mu}}\frac{\partial h_{ij}^{\mathrm{TT}}}{\partial x^{\nu}} \right\rangle}$$

Energy carried by a gravitational wave

For a plane wave

$$h_{11}^{\mathrm{TT}} = -h_{22}^{\mathrm{TT}} = h_+(t - z/c) \quad \text{and} \quad h_{12}^{\mathrm{TT}} = h_{21}^{\mathrm{TT}} = h_\times(t - z/c)$$

Energy density/flux:

$$T_{00}^{\mathrm{GW}} = -cT_{03}^{\mathrm{GW}} = -cT_{30}^{\mathrm{GW}} = c^2 T_{33}^{\mathrm{GW}} = \frac{c^4}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle$$

For a onochromatic wave of frequency ω,

$$h_+ = A_+ \cos[\omega(t - z/c)] \quad \text{and} \quad h_\times = A_\times \sin[\omega(t - z/c)]$$

$$T_{00}^{\mathrm{GW}} = -cT_{03}^{\mathrm{GW}} = -cT_{30}^{\mathrm{GW}} = c^2 T_{33}^{\mathrm{GW}} = \frac{c^4}{32\pi G}\omega^2(A_+^2 + A_\times^2)$$

Production of gravitational waves

In harmonic coordinates, $\partial \bar{h}^{\mu\alpha}/\partial x^{\mu} = 0$, the field equations are

$$\begin{aligned}\Box \bar{h}^{\alpha\beta} &= -\frac{16\pi G}{c^4} T^{\alpha\beta} + O(h^2) \\ &= -\frac{16\pi G}{c^4} \tau^{\alpha\beta}\end{aligned}$$

where $\tau^{\alpha\beta}$ is the effective stress energy tensor that includes all $O(h^2)$ terms.

The *exact* equations of motion are $\dfrac{\partial}{\partial x^{\mu}} \tau^{\mu\alpha} = 0$

The solution to the field equation is

$$\bar{h}^{\alpha\beta}(t, \boldsymbol{x}) = \frac{4G}{c^4} \int \frac{\tau^{\alpha\beta}(t - \|\boldsymbol{x} - \boldsymbol{x}'\|/c, \boldsymbol{x}')}{\|\boldsymbol{x} - \boldsymbol{x}'\|} \, d^3\boldsymbol{x}'$$

Production of gravitational waves

- Far field: $\|\boldsymbol{x} - \boldsymbol{x}'\| \approx r$
- Slow motion: $t - \|\boldsymbol{x} - \boldsymbol{x}'\|/c \approx t - r/c$ over entire source

$$\bar{h}^{\alpha\beta}(t, \boldsymbol{x}) \approx \frac{4G}{c^4 r} \int \tau^{\alpha\beta}(t - r/c, \boldsymbol{x}')\, d^3\boldsymbol{x}'$$

Only need to compute $\bar{h}^{ij}$
Use the identity

$$\tau^{ij} = \frac{1}{2}\frac{\partial^2}{\partial t^2}(x^i x^j \tau^{00}) + \frac{\partial}{\partial x^k}(x^i \tau^{jk} + x^j \tau^{ik}) - \frac{1}{2}\frac{\partial^2}{\partial x^k \partial x^l}(x^i x^j \tau^{kl})$$

$$\bar{h}^{ij}(t, \boldsymbol{x}) \approx \frac{2G}{c^4 r}\frac{\partial^2}{\partial t^2} \int x'^i x'^j \tau^{00}(t - r/c, \boldsymbol{x}')\, d^3\boldsymbol{x}'$$

Production of gravitational waves

Define the **quadrupole tensor**

$$I^{ij}(t) = \int x^i x^j \tau^{00}(t, \boldsymbol{x})\, d^3\boldsymbol{x}$$

$$\boxed{h_{ij}^{\mathrm{TT}}(t) \approx \frac{2G}{c^4 r} \frac{\partial^2}{\partial t^2} I_{ij}^{\mathrm{TT}}(t - r/c)}$$

where

$$I_{ij}^{\mathrm{TT}} = P_{ik} I^{kl} P_{lj} - \frac{1}{2} P_{ij} P_{kl} I^{kl}$$

is the TT projection and the operator $P_{ij} = \delta_{ij} - \hat{n}_i \hat{n}_j$ projects into the plane transverse to the normal vector $\hat{n}^i = x^i/r$

Production of gravitational waves

Gravitational wave luminosity:

$$\boxed{-\frac{dE}{dt} = \frac{1}{5}\frac{G}{c^5}\left\langle \frac{d^3\mathfrak{I}_{ij}}{dt^3}\frac{d^3\mathfrak{I}^{ij}}{dt^3}\right\rangle}$$

quadrupole formula

Gravitational wave torque:

$$\boxed{-\frac{dJ_i}{dt} = \frac{2}{5}\frac{G}{c^5}\epsilon_{ijk}\left\langle \frac{d^2\mathfrak{I}^j{}_l}{dt^2}\frac{d^3\mathfrak{I}^{kl}}{dt^3}\right\rangle}$$

Here $\mathfrak{I}$ is the trace-free quadrupole tensor

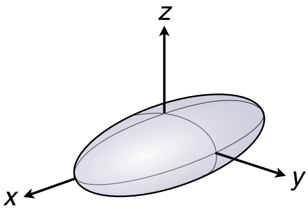
$$\mathfrak{I}^{ij}(t) = \int (x^i x^j - \tfrac{1}{3} r \delta^{ij})\tau^{00}(t, \boldsymbol{x})\, d^3\boldsymbol{x}$$

Beyond Newtonian motion and quadrupole radiation

- Expand equations of motion in powers of $1/c$:
  - $O(1/c^2)$ terms are "first post-Newtonian order"
  - $O(1/c^4)$ terms are "second post-Newtonian order"
- Include higher multipole terms in radiation

$$
\begin{aligned}
h^{ij}_{\mathrm{TT}} &= \frac{4G}{c^4 r}\int \tau^{ij}_{\mathrm{TT}}(t - r/c - \hat{\boldsymbol{n}}\cdot\boldsymbol{x}'/c, \boldsymbol{x}')\, d^3\boldsymbol{x}' \\
&= \frac{4G}{c^4 r}\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m}{\partial t^m}\int \tau^{ij}_{\mathrm{TT}}(t - r/c, \boldsymbol{x}')(\hat{\boldsymbol{n}}\cdot\boldsymbol{x}'/c)^m\, d^3\boldsymbol{x}' \\
&= \frac{2G}{c^4 r}\frac{\partial^2}{\partial t^2}\sum_{m=0}^{\infty}\Big[\underbrace{I^{ij}}_{\text{quarupole}} + \underbrace{I^{ijk}\hat{n}_k}_{\text{octupole}} + \underbrace{I^{ijkl}\hat{n}_k\hat{n}_l}_{\text{hexadecapole}} + \cdots\Big]_{\mathrm{TT}}
\end{aligned}
$$

Rotating triaxial ellipsoid



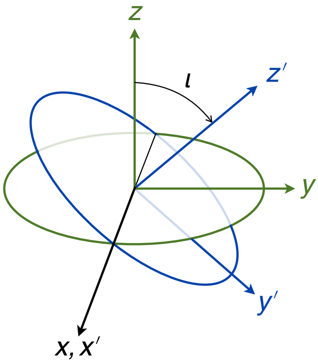
$$\mathbf{I}_0 = \begin{bmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{bmatrix}$$

$$\mathbf{I}(t) = \mathbf{R}_3^{-1}(\omega t) \cdot \mathbf{I}_0 \cdot \mathbf{R}_3(\omega t)$$

$$\frac{d^2}{dt^2}\mathbf{I}(t) = 2\varepsilon I_3 \omega^2 \begin{bmatrix} -\cos 2\omega t & -\sin 2\omega t & 0 \\ -\sin 2\omega t & \cos 2\omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where $\varepsilon = (I_1 - I_2)/I_3$ is the **ellipticity**

Rotating triaxial ellipsoid



Observer on z axis: $\mathbf{h}_{\mathrm{TT}} = \frac{2G}{c^2 r}\frac{d^2}{dt^2}\mathbf{I}_{\mathrm{TT}}$

$$h_+ = -\frac{4G\varepsilon I_3\omega^2}{c^4 r}\cos 2\omega t$$

$$h_\times = -\frac{4G\varepsilon I_3\omega^2}{c^4 r}\sin 2\omega t$$

Observer on z' axis:

$$\mathbf{I}' = \mathbf{R}_1(\iota)\cdot\mathbf{I}\cdot\mathbf{R}_1^{-1}(\iota)$$

$$h_+ = -\frac{4G\varepsilon I_3\omega^2}{c^4 r}\frac{1+\cos^2\iota}{2}\cos 2\omega t$$

$$h_\times = -\frac{4G\varepsilon I_3\omega^2}{c^4 r}\cos\iota\sin 2\omega t$$

Rotating triaxial ellipsoid

$$\frac{d^3}{dt^3}\mathbf{I}(t) = 4\varepsilon I_3 \omega^3 \begin{bmatrix} \sin 2\omega t & -\cos 2\omega t & 0 \\ -\cos 2\omega t & -\sin 2\omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Luminosity: $-\frac{dE}{dt} = \frac{32}{5}\frac{G}{c^5}\varepsilon^2 I_3^2 \omega^6$
- Torque: $-\frac{dJ_3}{dt} = \frac{32}{5}\frac{G}{c^5}\varepsilon^2 I_3^2 \omega^3$

$$\left.\right\} \quad \frac{dE}{dt} = \omega \frac{dJ_3}{dt}$$

Crab pulsar

rotation period	P	$= 0.0333\,\mathrm{s}$
spin-down rate	$\dot{P}$	$= 4.21 \times 10^{-13}$
mass	M	$= 1.4\,M_\odot$
radius	R	$= 10\,\mathrm{km}$
quadrupole	I_3	$\approx \frac{2}{5} MR^2 = 1.1 \times 10^{38}\mathrm{m}^2\,\mathrm{kg}$
inclination	ι	$= 62^\circ$
distance	r	$= 2.5\,\mathrm{kpc}$

If the spin-down is explained *entirely* by gravitational radiation then

$$\dot{P} = -\frac{P^2}{2\pi}\,\dot{\omega} = -\frac{P^2}{2\pi I_3}\frac{dJ_3}{dt} = \frac{512\pi^4}{5}\frac{G}{c^5}\varepsilon^2 I_3 P^{-3}$$

Required ellipticity is $\varepsilon = \sqrt{\frac{5}{512\pi^4}\frac{c^5}{G}\frac{\dot{P}P^3}{I_3}} \approx 7.2 \times 10^{-4}$

Crab pulsar

Sinusoidal gravitational waves produced at $\sim 60\,\mathrm{Hz}$ with amplitude

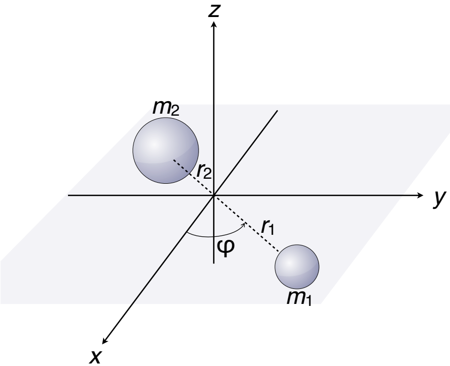
$$A_+ = \frac{4G\varepsilon I_3\omega^2}{c^4 r}\frac{1+\cos^2\iota}{2} = 7.3\times 10^{-25}$$
$$A_\times = \frac{4G\varepsilon I_3\omega^2}{c^4 r}\cos\iota = 5.6\times 10^{-25}$$

Upper bounds since most spin-down is from electromagnetic breaking

For $\tau = 1\,\mathrm{year}$ of observation, "root-sum-square" amplitude is

$$h_{\mathrm{rss}} \sim \sqrt{\tau}\,h \sim 10^{-21}\,\mathrm{Hz}^{-1/2}$$

Orbiting binary system



Center-of-mass coords:

- Orbital separation
  $a = r_1 + r_2$
- Total mass
  $M = m_1 + m_2$
- Reduced mass
  $\mu = m_1 m_2 / M$

Quadrupole tensor:

$$
\begin{aligned}
I_{11} &= \mu a^2 \cos^2 \varphi \\
I_{12} &= \mu a^2 \sin \varphi \cos \varphi \\
I_{22} &= \mu a^2 \sin^2 \varphi
\end{aligned}
$$

where $\varphi = \omega t$

Orbiting binary system

Kepler's 3rd law: $a^3\omega^2 = GM$ Let $v = a\omega = (GM\omega)^{1/3}$

Waveform is:

$$
\begin{aligned}
h_+ &= -\frac{4G\mu}{c^2 r}\left(\frac{v}{c}\right)^2 \frac{1+\cos^2\iota}{2}\cos 2\varphi \\
h_\times &= -\frac{4G\mu}{c^2 r}\left(\frac{v}{c}\right)^2 \cos\iota \sin 2\varphi
\end{aligned}
$$

Gravitational wave frequency is $2\pi f = 2\omega$ so

$$v = (\pi G M f)^{1/3}$$

also

$$v = \left(\frac{2\pi GM}{P}\right)^{1/3} \qquad v = \sqrt{\frac{GM}{a}}$$

Orbiting binary system

Gravitational radiation causes orbit to decay

- Orbital energy:

$$E = \frac{1}{2}\mu v^2 - \frac{GM\mu}{a} = -\frac{1}{2}\mu v^2$$

- Gravitational wave luminosity:

$$-\frac{dE}{dt} = \frac{32}{5}\frac{c^5}{G}\left(\frac{\mu}{M}\right)^2\left(\frac{v}{c}\right)^{10}$$

- Orbital evolution:

$$\frac{d}{dt}\frac{v}{c} = \frac{32\eta}{5}\frac{c^3}{GM}\left(\frac{v}{c}\right)^9$$

  where $\eta = \mu/M$

Orbiting binary system

Generalize: define an *energy function* and a *flux function*

$$\mathcal{E}(v) = \frac{E(v) - Mc^2}{Mc^2} \quad \text{and} \quad \mathcal{F}(v) = -\frac{G}{c^5}\frac{dE}{dt}(v)$$
$$-\frac{GM}{c^3}\frac{d\mathcal{E}}{dv}\frac{dv}{dt}$$

$$\frac{dt}{dv} = -\frac{GM}{c^3}\frac{1}{\mathcal{F}}\frac{d\mathcal{E}}{dv}$$

- Time until coalescence:

$$\boxed{\tau = t_c - t(v) = -\frac{GM}{c^3}\int_v^{v_c} \frac{1}{\mathcal{F}}\frac{d\mathcal{E}}{dv}\, dv}$$

where v_c is the value of v at coalescence, $t = t_c$

Orbiting binary system

- Phase evolution:

$$\begin{aligned}\frac{d\varphi}{dv} &= \frac{d\varphi}{dt}\frac{dt}{dv} \\ &= -\omega\frac{GM}{c^3}\frac{1}{\mathcal{F}}\frac{d\mathcal{E}}{dv} \\ &= -\left(\frac{v}{c}\right)^3\frac{1}{\mathcal{F}}\frac{d\mathcal{E}}{dv}\end{aligned}$$

$$\boxed{\varphi(v) = \varphi_c + \int_v^{v_c}\left(\frac{v}{c}\right)^3\frac{1}{\mathcal{F}}\frac{d\mathcal{E}}{dv}\,dv}$$

Orbiting binary system

Binary inspiral "chirp" waveform:

$$h_+\big(t(v)\big) = -\frac{4G\mu}{c^2 r}\left(\frac{v}{c}\right)^2 \frac{1+\cos^2\iota}{2}\cos 2\varphi(v)$$

$$h_\times\big(t(v)\big) = -\frac{4G\mu}{c^2 r}\left(\frac{v}{c}\right)^2 \cos\iota\sin 2\varphi(v)$$

$$t(v) = t_c + \frac{GM}{c^3}\int_v^{v_c}\frac{1}{\mathcal{F}}\frac{d\mathcal{E}}{dv}\,dv \qquad \varphi(v) = \varphi_c + \int_v^{v_c}\left(\frac{v}{c}\right)^3\frac{1}{\mathcal{F}}\frac{d\mathcal{E}}{dv}\,dv$$

Orbiting binary system

Newtonian chirp: $\mathcal{E} = -\frac{1}{2}\left(\frac{v}{c}\right)^2$ and $\mathcal{F} = -\frac{32}{5}\eta\left(\frac{v}{c}\right)^{10}$

$$t(v) = t_c - \frac{5}{256\eta}\frac{GM}{c^3}\left(\frac{v}{c}\right)^{-8} \quad \text{and} \quad \varphi(v) = \varphi_c - \frac{1}{32\eta}\left(\frac{v}{c}\right)^{-5}$$

(Note: as $v \to \infty$, $t \to t_c$, $\varphi \to \varphi_c$)

- Gravitational wave frequency evolution: use $f = v^3/\pi GM$

$$\frac{df}{dt} = \frac{df}{dv}\frac{dv}{dt} = \frac{96}{5}\pi^{8/3}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3} f^{11/3}$$

where

$$\boxed{\mathcal{M} = \eta^{3/5}M = \mu^{3/5}M^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}}$$

chirp mass

Orbiting binary system

$$h_+(t) = -\frac{1+\cos^2\iota}{2}\frac{G\mathcal{M}}{c^2 r}\left(\frac{t_c - t}{5G\mathcal{M}/c^3}\right)^{-1/4}\cos\left[\varphi_c - \left(\frac{t_c - t}{5G\mathcal{M}/c^3}\right)^{5/8}\right]$$

$$h_\times(t) = -\cos\iota\frac{G\mathcal{M}}{c^2 r}\left(\frac{t_c - t}{5G\mathcal{M}/c^3}\right)^{-1/4}\sin\left[\varphi_c - \left(\frac{t_c - t}{5G\mathcal{M}/c^3}\right)^{5/8}\right]$$

Hulse-Taylor binary pulsar

pulsar mass	m_1	$= 1.4414\, M_\odot$
companion mass	m_2	$= 1.3867\, M_\odot$
orbital period	P	$= 0.322\,997\,448\,930\,\mathrm{days}$
orbital decay	$\dot{P}$	$= -75.9\,\mathrm{\mu s\,yr^{-1}} = -2.405 \times 10^{-12}$
orbital eccentricity	e	$= 0.617\,133\,8$

If the orbital decay is explained *entirely* by gravitational radiation,

$$\dot{P} = -\frac{192\pi}{5}\left(\frac{2\pi G\mathcal{M}}{c^3 P}\right)^{5/3}\left[\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}}\right]$$

$$\approx -2.402 \times 10^{-12}$$

Binary neutron star coalescence

Characteristic amplitude:

$$h \sim \frac{2G\mu}{c^2 r}\left(\frac{v}{c}\right)^2$$

Characteristic chirp timescale for $\Delta f \sim f$:

$$\tau = \frac{1}{d\ln f/dt} = \frac{1}{3}\left(\frac{1}{v/c}\frac{d(v/c)}{dt}\right)^{-1} \sim \frac{5}{96\eta}\frac{GM}{c^3}\left(\frac{v}{c}\right)^{-8}$$

Root-sum-squared amplitude:

$$h_{\mathrm{rss}} \sim \sqrt{\tau}h \sim \sqrt{\frac{5}{96}\frac{G\mu}{c^3}}\frac{2G\mu}{c^2 r}\left(\frac{v}{c}\right)^{-2} \sim \sqrt{\frac{5}{24}\frac{G\mathcal{M}}{c^3}}\frac{G\mathcal{M}}{c^2 r}\left(\frac{\pi G\mathcal{M}f}{c^3}\right)^{-2/3}$$

Binary neutron star coalescence

neutron star masses	$m_1 = m_2 = 1.4\, M_\odot$
chirp mass	$\mathcal{M} = 1.22\, M_\odot$
gravitational wave frequency	$f = 100, \mathrm{Hz}$
distance	$r = 100\,\mathrm{Mpc}$

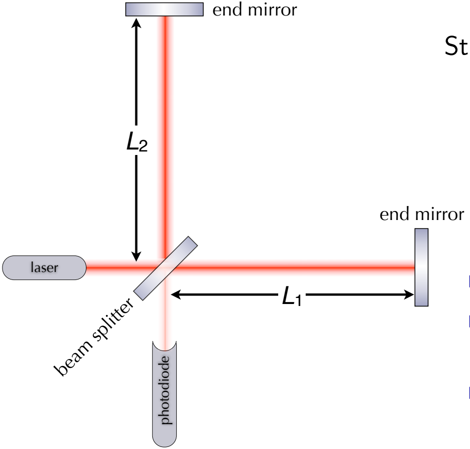
Chracteristic amplitude:

$$h_{\text{rss}} \sim 4 \times 10^{-23}$$

Interferometers

LIGO	Hanford, WA, USA	2nd-Generation (construction)
	Livingstong, LA, USA	2nd-Generation (construction)
	India	2nd-Generation (planning)
Virgo	Pisa, Italy	2nd-Generation (construction)
KAGRA	Kamioka, Japan	2nd-Generation (construction)
Einstein Telescope	Europe	3rd-Generation (planning)

Interferometers



Strain sensitivity:

$$h = \frac{\Delta L_1 - \Delta L_2}{L} \sim \eta \frac{\lambda_*}{L}$$

- Laser wavelength: λ_*
- Resolvable fraction of a fringe: η
- Effective arm-length: L

Interferometers

- Resolvable fraction of a fringe is determined by **shot noise**:

  $$\eta = N_\gamma^{-1/2}$$

  where N_γ is the number of photons collected in time τ:

  $$N_\gamma = \frac{P_*}{2\pi\hbar c/\lambda_*}\,\tau$$

  and P_* is the laser power

- Effective arm-length can be larger than actual arm length, e.g., by having multiple bounces

  To remain in long-wavelength limit: $L < \dfrac{c}{f_0}$

Interferometers

Strain **power spectral density** is $S_h \sim \tau h^2$

Detectable signal has $h_{\rm rss} \gtrsim (\text{a few}) \times S_h^{1/2}$ where

$$S_h^{1/2} \sim \tau^{1/2}\, h \sim \tau^{1/2}\, \eta \frac{\lambda_\ast}{L} \sim \tau^{1/2} \sqrt{\frac{2\pi\hbar c}{P_\ast} \frac{1}{\tau \lambda_\ast}}\, \frac{\lambda_\ast}{c/f_0} = \sqrt{\frac{2\pi\hbar\lambda_\ast}{P_\ast c}}\, f_0$$

Initial LIGO

$$\left.\begin{array}{lr} \text{laser wavelength} & \lambda_\ast = 1\,\mu\text{m} \\ \text{laser power} & P_\ast = 100\,\text{W} \\ \text{gravitational wave frequency} & f_0 = 100\,\text{Hz} \end{array}\right\} \quad S_h^{1/2} \sim 10^{-23}\,\text{Hz}^{-1/2}$$

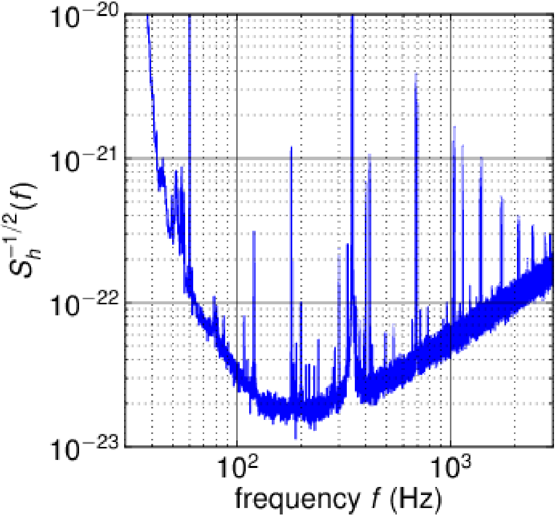
Interferometers

Other noise sources:

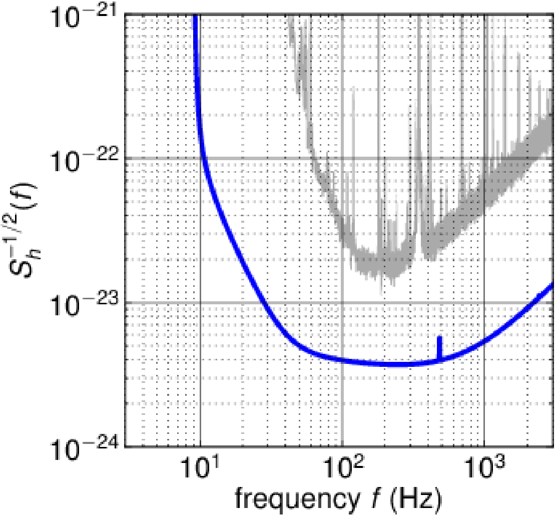
- Thermal noise
- Radiation pressure noise
- Seismic noise
- Gravity gradient noise

Interferometers — LIGO

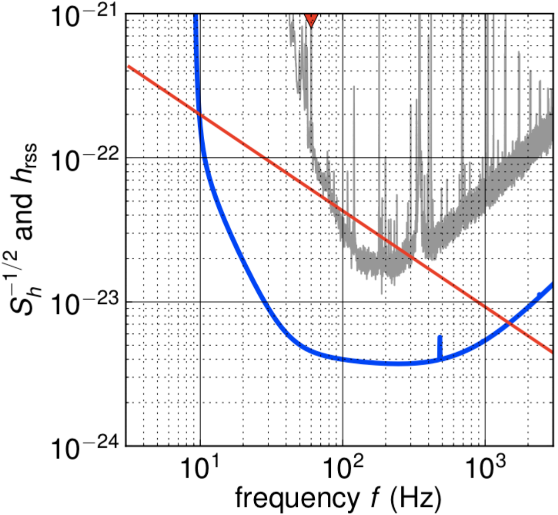
Interferometers — Initial LIGO



Interferometers — Advanced LIGO

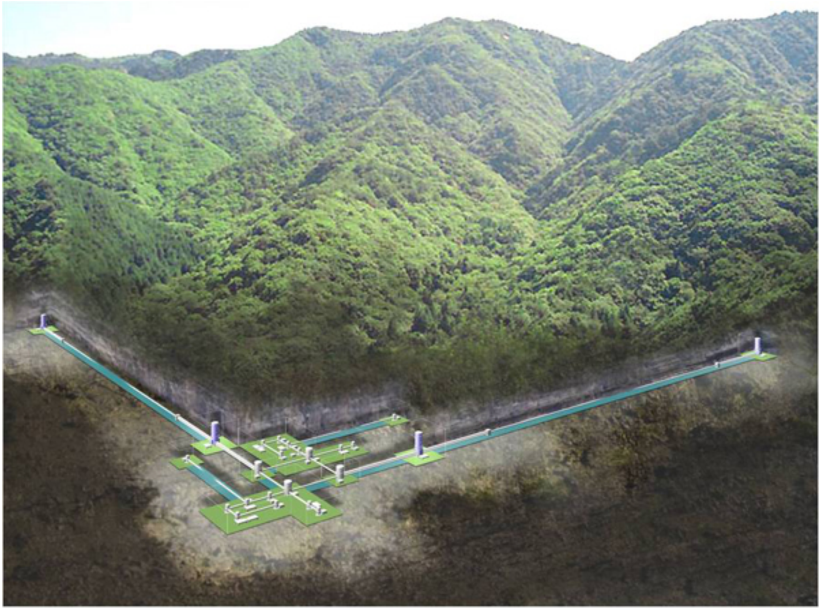


Interferometers — Advanced LIGO

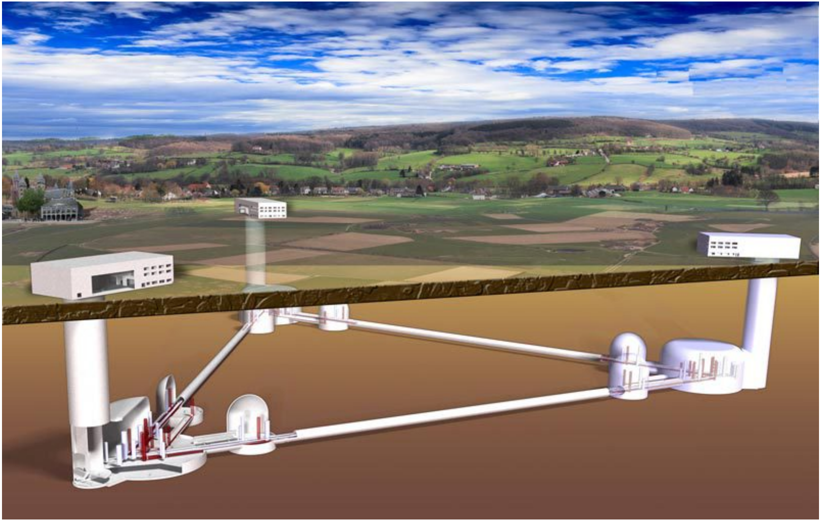


Interferometers — Virgo

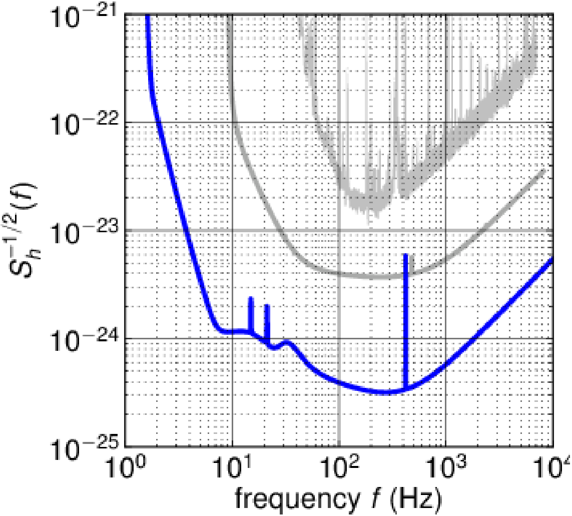
Interferometers — KAGRA



Interferometers — Einstein Telescope



Interferometers — Einstein Telescope



Pulsar timing arrays

- Pulsars are very stable “clocks”
- Timing residuals: difference between when a pulse arrives and when it was expected
- Gravitational waves affect clocks; produce timing residuals
- Decade-long obeservation $\implies$ sensitive to nano-Hertz frequencies