Describe fluid perturbation by a *Lagrangian displacement* ξ^a that joins each fluid element in the unperturbed star to the corresponding fluid element in the perturbed configuration: If $c(t)$ is the fluid trajectory in the unperturbed star, then the perturbed trajectory $\bar{c}(t)$ is given in Cartesian coordinates by

$$\bar{c}^i(t) = c^i(t) + \xi^i[c(t), t].$$

Adiabatic index: $$\Gamma_1 := \left.\frac{\partial \log P}{\partial \log \rho}\right|_{s,Y_i}$$

Fractional change in volume of a fluid element is

$$\frac{\Delta V}{V} = \nabla_a \xi^a,$$

implying

$$\frac{\Delta \rho}{\rho} = \frac{\Delta P}{\Gamma_1 P} = -\nabla_a \xi^a \tag{1}$$

Here $\Delta\rho$ is the Lagrangian change in ρ, the difference in the density of the same fluid element:

$$\Delta\rho = \bar{\rho}(\mathbf{x} + \boldsymbol{\xi}) - \rho(\mathbf{x}) = \delta\rho + \pounds_{\boldsymbol{\xi}}\rho.$$

The Eulerian change $\delta\rho$ is the change at a fixed point in space, and the relation between them is

$$\Delta = \delta + \pounds_{\boldsymbol{\xi}} \tag{2}$$

Equivalently, one can consider a family of fluid flows, with the original fluid flow is mapped by the family of diffeos $\psi_\lambda(t,x)$ generated by ξ^a to the perturbed flow. If ψ_λ were time-independent, the velocity field of the new flow would be Lie-dragged by ψ_λ to the new velocity field implying Δv^a would vanish. That means Δv^a can depend only on $\partial_t\xi^a$:

$$\Delta v^a = \partial_t \xi^a. \tag{3}$$

Here's the derivation:

$$
\begin{aligned}
\bar{v}^i[\bar{c}(t)] &= \frac{d}{dt}[c^i(t) + \xi^i[t, c(t)] = v^i + v^j \partial_j \xi^i + \partial_t \xi^i \\
\bar{v}^i + \xi^j \partial_j \bar{v}^i &= v^i + v^j \partial_j \xi^i + \partial_t \xi^i \\
\delta v^i &= \partial_t \xi^i - \pounds_{\boldsymbol{\xi}} v^i \\
\Delta v^i &= \partial_t \xi^i
\end{aligned}
$$

Eqs. (1) and (3) give the perturbed fluid variables in terms of ξ^a. The perturbed Euler equation then becomes a dynamical equation for ξ^a.

Let ρ, P, v^a be a solution to the equilibrium equations

$$\begin{aligned} &P = P(\rho) \\ &\nabla_a(\rho v^a) = 0 \\ &\nabla^2\Phi = 4\pi G\rho \\ &v\cdot\nabla v_a + \frac{1}{\rho}\nabla_a P + \nabla_a\Phi = 0 \end{aligned}$$

Perturbed equation of continuity (conservation of mass) and perturbed EOS:
Already satisfied by $\Delta\rho/\rho = -\nabla_a\xi^a = \Delta P/\Gamma_1 P$.

Perturbed field equation:

$$\nabla^2 \delta\Phi = 4\pi\delta\rho = -4\pi\nabla\cdot(\rho\xi). \tag{4}$$

Perturbed Euler:

$$0 = \rho\Delta\left[(\partial_t + v\cdot\nabla)v_a + \frac{1}{\rho}\nabla_a P + \nabla_a\Phi\right] \Rightarrow$$

$$\begin{aligned} 0 &= \rho\partial_t^2\xi^a + 2\rho v^b\nabla_b\partial_t\xi^a + \rho(v^b\nabla_b)^2\xi^a - \nabla^a(\Gamma_1 P\nabla_b\xi^b) \\ &\quad -\nabla^b P\nabla_a\xi^b + \rho\xi^b\nabla_b\nabla_a\Phi + \rho\nabla_a\delta\Phi \end{aligned}$$

$$= A^a_b\ \partial_t^2\xi^a + B^a_b\ \partial_t\xi^b + C^a_b\ \xi^b \equiv L^a_b\xi^b.$$

The operators A^a_b, B^a_b, C^a_b are self-adjoint, anti-self-adjoint, and self-adjoint, respectively, for $\delta\Phi$ satisfying (4).

When one includes radiation reaction, the equation has the form

$$A^a_b\ \partial_t^2 \xi^a + B^a_b\ \partial_t \xi^b + C^a_b\ \xi^b = -F^a.$$

The antisymmetry of B_{ab} implies the inner product with $\dot{\xi}^a$ is

$$\int dV [A_{ab} \dot{\xi}^a \ddot{\xi}^b + C_{ab} \dot{\xi}^a \xi^b] = -\int dV \dot{\xi}_a F^a \leq 0.$$

or

$$\frac{d}{dt} E_c = -\int dV \dot{\xi}_a F^a \leq 0,$$

where

$$E_c = \int dV \frac{1}{2} [A_{ab} \dot{\xi}^a \dot{\xi}^b + C_{ab} \xi^a \xi^b]$$

The star is then unstable (or marginally stable) when there is initial data for which $\langle \xi | C \xi \rangle < 0$.

The symmetry of the operators also means that L^a_b is symmetric in a 4-dimensional sense,

$$\hat{\xi}^a L_{ab} \xi^b = \xi^a L_{ab} \hat{\xi}^b + \nabla_\alpha \Theta^\alpha(\xi).$$

and this implies that the action for the perturbed Euler equations has the form

$$\begin{aligned} I &= \int dt dV \mathcal{L}, \\ \mathcal{L} &:= \frac{1}{2}\left[A_{ab}\dot{\xi}^a\dot{\xi}^b - B_{ab}\xi^a\dot{\xi}^b - C_{ab}\xi^a\xi^b\right] \end{aligned}$$

The momentum density conjugate to ξ^a is

$$\Pi_a = \frac{\partial \mathcal{L}}{\partial}\dot{\xi} = A_{ab}\dot{\xi}^b + \frac{1}{2}B_{ab}\xi^b.$$

The symplectic product,

$$W(\hat{\xi}, \xi) = \int dV (\hat{\Pi}_a \xi^a - \Pi_a \hat{\xi}^a)$$

is conserved and gives simple forms for the energy and angular momentum of a perturbation and a relation between them:

$$E_c = \frac{1}{2} W(\dot{\xi}, \xi), \quad J_c = -\frac{1}{2} W(\pounds_\phi \xi, \xi).$$

Then for a mode with time-dependence $e^{im\phi - \omega t}$, and ω real, we have

$$\frac{E_c}{J_c} = \omega/m, \qquad \frac{\dot{E}_c}{\dot{J}_c} = \omega/m,$$

ω/m the *pattern speed* of the mode. For $J_c < 0$ (a mode that moves backward relative to the star), the energy E_c will be less than zero when $\omega > 0$, when the star rotates fast enough that the mode is dragged forward relative to the inertial frame.

Relativistic perturbation theory

Variations of the metric and fluid

Here's roughly the same introduction a second time, this time in the context of spacetime and the exact theory. First order departures from an initial configuration can be described in two ways. The Eulerian perturbations in the quantities $Q(\lambda)$ are defined by

$$\delta Q = \frac{d}{d\lambda} Q(\lambda)|_{\lambda=0} \tag{5}$$

and compare values of Q at the same point of the spacetime.

In the region occupied by the original fluid, one can also introduce the Lagrangian perturbations

$$\Delta Q = \frac{d}{d\lambda}[\chi_{-\lambda} Q(\lambda)]|_{\lambda=0} \quad (6)$$
$$= (\delta + \pounds_\xi) Q, \quad (7)$$

where ξ^α generates the family of diffeomorphisms χ_λ. That is, the curve $\lambda \to \chi_\lambda(P)$ has tangent $\xi^\alpha(P)$ at the point P. The field ξ^α is termed a Lagrangian displacement and may be regarded as the connecting vector joining fluid elements in the unperturbed configuration to the corresponding elements in the perturbed spacetime.

The first order changes in the variables Q can be expressed in terms of the displacement ξ^α and the Eulerian change in the metric

$$h_{\alpha\beta} = \delta g_{\alpha\beta}. \quad (8)$$

In fact, we will see that perturbations of the fluid variables can all be written in terms of $\Delta g_{\alpha\beta}$,

$$\Delta g_{\alpha\beta} = h_{\alpha\beta} + \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha. \quad (9)$$

We begin with the change in the four-velocity u^α. Let $t \to c(t)$ be the initial path of a fluid element, $c_\lambda = \chi_\lambda \circ c$ the new path. Because χ_λ drags c to c_λ, the Lagrangian change in c and in its tangent vector vanishes. That is, if w^α is tangent to c, then $w^\alpha_\lambda = \chi_\lambda w$ is tangent to c_λ. Thus $\chi_{-\lambda} w^\alpha_\lambda = w^\alpha$, independent of λ, implying

$$\Delta w^\alpha = \partial_\lambda(\chi_{-\lambda} w^\alpha_\lambda) = \partial_\lambda w^\alpha = 0. \tag{10}$$

Now w^α will not, in general have norm -1; even if we choose t to be proper time along the original path, t will not be proper time along c_λ. As a result, the Lagrangian change in the four-velocity is nonzero, depending on the change in the metric along the fluid trajectory, $\Delta g_{\alpha\beta} u^\alpha u^\beta$. We have

$$u^\alpha = \frac{w^\alpha}{(-w^\beta w_\beta)^{1/2}} = \frac{w^\alpha}{(-g_{\beta\gamma} w^\beta w^\gamma)^{1/2}};$$

$$\Delta u^\alpha = -\frac{1}{2}\frac{w^\alpha}{(-w_\delta w^\delta)^{3/2}}(-\Delta g_{\beta\gamma} w^\beta w^\gamma) = \frac{1}{2} u^\alpha u^\beta u^\gamma \Delta g_{\beta\gamma}. \tag{11}$$

Baryon density $\Delta\rho$:

Conservation of baryon mass is

$$\Delta(\rho\sqrt{q}) = 0;$$

because (as discussed in the context of the unperturbed conservation law), the volume of a fluid element perpendicular to u^α is proportional to $\sqrt{q}$, and the fractional change in its volume is

$$\frac{\Delta V}{V} = \frac{\Delta\sqrt{q}}{\sqrt{q}} = \frac{1}{2}q^{\alpha\beta}\Delta q_{\alpha\beta} = \frac{1}{2}q^{\alpha\beta}\Delta g_{\alpha\beta}.$$

Then

$$\frac{\Delta\rho}{\rho} = -\frac{1}{2}q^{\alpha\beta}\Delta g_{\alpha\beta} \tag{12}$$

The equation means that the fractional increase in ρ is equal to the fractional decrease in the volume orthogonal to the 4-velocity.

$\Delta\epsilon$ and ΔP:

We already know the relation between $\Delta\epsilon$, ΔP and $\Delta\rho$:

$$\frac{\Delta\epsilon}{\epsilon + p} = \frac{\Delta P}{\Gamma_1 P} = \frac{\Delta\rho}{\rho} = -\frac{1}{2} q^{\alpha\beta}\Delta g_{\alpha\beta}.$$

These relations imply that an action for the Einstein-Euler system is given by

$$I = \int \left[\frac{1}{16\pi} R - \epsilon\sqrt{-g}\right] :$$

Use

$$\begin{aligned} \Delta\left(\epsilon\sqrt{-g}\right) &= -\frac{1}{2} T^{\alpha\beta}\Delta g_{\alpha\beta} \\ &= -\frac{1}{2} T^{\alpha\beta} h_{\alpha\beta} + \xi_\alpha \nabla_\beta T^{\alpha\beta} - \nabla_\beta(\xi_\alpha T^{\alpha\beta}) \end{aligned}$$

$$\delta I = \int \left[-\frac{1}{16\pi}(G_{\alpha\beta} - 8\pi T^{\alpha\beta}) h_{\alpha\beta} - \xi_\alpha \nabla_\beta T^{\alpha\beta}\right] \sqrt{-g} d^4x$$

Perturbed Euler equation

$u_\beta \nabla_\alpha T^{\alpha\beta} = 0$ already satisfied by $\Delta\epsilon = -\frac{1}{2}(\epsilon + P) q^{\alpha\beta}\Delta g_{\alpha\beta}$.
For remaining part, use our expressions for all perturbed fluid quantities in terms of $\Delta g_{\alpha\beta}$ to write the perturbed Euler equation of a uniformly rotating barotropic star:

$$\begin{aligned} u^t q_\alpha{}^\beta u^\gamma \pounds_{\mathbf{k}} \Delta g_{\beta\gamma} &= -q_\alpha{}^\beta \nabla_\beta \Delta \ln \frac{h}{u^t} \\ &= -\frac{1}{2} q_\alpha{}^\beta \nabla_\beta \left(u^\gamma u^\delta \Delta g_{\gamma\delta} + \frac{\Gamma_1 p}{\epsilon + p} q^{\gamma\delta} \Delta g_{\gamma\delta} \right). \end{aligned}$$

Perturbed Einstein equation

In a Lorenz gauge (deDonder, transverse, harmonic, ...), given by

$$\nabla_\beta \widetilde{h}^{\alpha\beta} = 0, \text{ where } \widetilde{h}_{\alpha\beta} := h_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} h.$$

The perturbed vacuum field equation has the form

$$\boldsymbol{\delta} G^{\alpha\beta} = -\frac{1}{2} \nabla_\gamma \nabla^\gamma \tilde{h}^{\alpha\beta} - 2 R^{\alpha\gamma\beta\delta} \tilde{h}_{\gamma\delta} = 0,$$

with a more elaborate expression

$$\boldsymbol{\delta} G^{\alpha\beta} = -\frac{1}{2} \nabla_\gamma \nabla^\gamma \tilde{h}^{\alpha\beta} - G^{\alpha\gamma\beta\delta} \tilde{h}_{\gamma\delta}$$

in the star.

Because the field equations come from an action, the perturbed field equations satisfy a symmetry relation: For any pairs $(\xi^\alpha, h_{\alpha\beta})$ and $(\widehat{\xi}^\alpha, \widehat{h}_{\alpha\beta})$, the symmetry relation has the form

$$\widehat{\xi}_\beta \delta(\nabla_\gamma T^{\beta\gamma}\sqrt{|g|}) \quad + \quad \frac{1}{16\pi}\widehat{h}_{\beta\gamma}\delta\left[(G^{\beta\gamma} - 8\pi T^{\beta\gamma})\sqrt{|g|}\right] = -2\mathcal{L}(\widehat{\xi}, \widehat{h}; \xi, h) + \nabla_\beta \Theta^\beta,$$

where $\mathcal{L}$ is symmetric under interchange of (ξ, h) and $(\widehat{\xi}, \hat{h})$, implying an action of the form

$$I = \int d^4x \mathcal{L} = \frac{1}{2}\mathcal{L}(\xi, h; \xi, h).$$

Again we define a symplectic product and conserved energy and angular momentum of the perturbation by

$$W(\widehat{\xi}, \widehat{h}; \xi, h) := \int_\Sigma (\widehat{\Pi}_\alpha \xi^\alpha + \widehat{\pi}^{\alpha\beta} h_{\alpha\beta} - \Pi_\alpha \widehat{\xi}^a - \pi^{\alpha\beta}\widehat{h}_{\alpha\beta}) d^3x.$$

$$
\begin{aligned}
E_c &= \frac{1}{2} W(\pounds_{\boldsymbol{t}} \xi^\alpha, \pounds_{\boldsymbol{t}} h_{\alpha\beta}, \xi^\alpha, h_{\alpha\beta}) \\
J_c &= -\frac{1}{2} W(\pounds_{\boldsymbol{\phi}} \xi^\alpha, \pounds_{\boldsymbol{\phi}} h_{\alpha\beta}, \xi^\alpha, h_{\alpha\beta})
\end{aligned}
$$

The stability discussion is now identical to that in the Newtonian approximation, with longer expressions for the operators.

Why are the perturbed field equations self-adjoint in this 4-dimensional sense (symmetric up to a divergence)?

This follows quite generally from the existence of an unconstrained action. Let $\{\phi^I\}$ be a set of fields whose field equations $E_I(\phi) = 0$ are given by a variation of the action $I[\phi] = \int \mathcal{L}(\phi) d^4x$. That is, $E_I = \dfrac{\delta I}{\delta \phi^I}$, with $\delta I = \displaystyle\int \frac{\delta I}{\delta \phi^I(x)} \delta\phi^I(x) d^4x$, for any choice of perturbed fields $\delta\phi^I$ vanishing sufficiently rapidly at the boundary of the region of integration that no surface terms arise. Then the perturbed field equations, $\delta E_I = 0$, governing a perturbation $\delta\phi^I$, have the form

$$\delta E_I(x) = \int \frac{\delta^2 I}{\delta\phi^I(x)\delta\phi^J(x')} \delta\phi^J(x') d^4x',$$

and the symmetry of the system corresponds to an exchange of the order of the two derivatives of the action:

$$\begin{aligned}\int \hat{\delta}\phi^I(x)\delta E_I(x)d^4x &= \int \hat{\delta}\phi^I(x)\frac{\delta^2 I}{\delta\phi^I(x)\delta\phi^J(x')}\delta\phi^J(x')d^4x'd^4x \\ &= \int \delta\phi^J(x')\frac{\delta^2 I}{\delta\phi^J(x')\delta\phi^I(x)}\hat{\delta}\phi^I(x)d^4xd^4x' \\ &= \int \delta\phi^J(x')\hat{\delta}E_J(x')d^4x'.\end{aligned}$$