1. Check that the components of the stress tensor in the ZAMO frame (the orthonormal frame of zero-angular-momentum observers) are given by the expressions in part II of the notes.

2. A Newtonian star rotates slowly with uniform angular velocity Ω.

   (a) Use a symmetry argument to argue that the change in R, ρ and Φ due to the slow rotation cannot be proportional to Ω.

   (b) Show that the assumption that the change in the potential and density of the star is $\mathrm{O}(\Omega^2)$ is consistent with Laplace's equation and the first integral of the equation of hydrostatic equilibrium.

   (c) Again argue for a slowly rotating relativistic star that the diagonal components of the metric cannot change to order Ω and check that the assumption that they are order Ω^2 is consistent with the change in the diagonal components of the stress tensor. The metric of a slowly rotating star then has to order Ω the form

$$ds^2 = -e^{2\Phi}dt^2 - 2\omega r^2 \sin^2\theta \, d\phi dt + e^{2\lambda}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2,$$

   where Φ and λ are the potentials of the spherical star.

3. (a) Show that the mass of a stationary spacetime is given by $M = -\dfrac{1}{4\pi}\displaystyle\int_\infty \nabla^\alpha t^\beta dS_{\alpha\beta}$, by explicitly evaluating this integral on a sphere at spatial infinity, for a metric that agrees asymptotically with the Schwarzschild metric to $O(1/r)$.

   (b) Similarly, by evaluating

$$J = \frac{1}{8\pi}\int_\infty \nabla^\alpha \phi^\beta dS_{\alpha\beta},$$

   check the angular momentum is the constant J that appears in the asymptotic form $\omega = 2J/r^3$. Use for the asymptotic metric the form $ds^2 = -\left(1 - \dfrac{2M}{r^2}\right)dt^2 + \left(1 - \dfrac{2M}{r^2}\right)^{-1} dr^2 - 2\omega r^2 \sin^2\theta d\phi dt + r^2 d\Omega^2$. In the integrals, note that only the dominant term in the asymptotic form of each metric component contributes to the integral, with $g^{tt} = -1 + O(1/r), g^{t\phi} = -2J/r^3 + O(1/r^4), g^{rr} = 1 + O(1/r)$.