LECTURE #3 – DYNAMICS OF FLRW WITH GENERAL RELATIVITY!

In conformal-Cartesian coordinates, the FLRW metric is:

$$ds^2 = -dt^2 + a^2(t)\,\gamma_{ij}\,dx^i dx^j$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & a^2\gamma_{ij} \end{pmatrix}$$

$$\gamma_{ij} = \left(1 + k\vec{x}^2\right)^{-2} \delta_{ij}$$

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & a^{-2}\gamma^{-1}_{ij} \end{pmatrix}$$

k : UNITS OF $[\text{LENGTH}]^{-2}$

$k>0$, $k<0$, $k=0$?...

The connections for this metric were given previously:

$$\Gamma^0_{ij} = \dot{a}a\,\gamma_{ij} \qquad \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j$$

$$\Gamma^i_{jk} = \frac{2k}{1+kx^2}\left(\delta_{ij}x^k - \delta_{jk}x^i - \delta_{ki}x^j\right)$$

$$\Gamma^0_{00} = \Gamma^i_{00} = \Gamma^0_{i0} = 0$$

We can compute the Ricci tensor directly,

$$R^{\mu}{}_{\nu\mu\beta} = R_{\nu\beta}$$

$$\Rightarrow R_{00} = 3\frac{\ddot{a}}{a}$$

$$R_{0i} = 0$$

$$R_{ij} = -\left(a\ddot{a} + 2\dot{a}^2 + 2k\right)\gamma_{ij}$$

$$R = g^{\mu\nu} R_{\mu\nu} = -3\frac{\ddot{a}}{a} - 3a^{-2}\left(a\ddot{a} + 2\dot{a}^2 + 2k\right)$$

$$\Rightarrow \boxed{R = -6\frac{\ddot{a}}{a} - 6\left(\frac{\dot{a}}{a}\right)^2 - \frac{6k}{a^2}}$$

The Einstein tensor is, therefore,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

OR:

$$\begin{cases} G_{00} = -3\left(\frac{\dot{a}}{a}\right)^2 - \frac{3k}{a^2} \\ G_{ij} = \left(2a\ddot{a} + \dot{a}^2 + k\right)\gamma_{ij} \end{cases} \qquad G_{0i} = 0$$

Einstein's equations: My conventions!

$$G_{\mu\nu} = -8\pi G\, T_{\mu\nu}$$

What is this "matter"?

We saw earlier that, for a "fluid" in Minkowski spacetime, energy, momentum and the 4-velocity mix together to form the stress-energy tensor:

$$T^{(M)}_{\mu\nu} = (\rho + p) U_\mu U_\nu + p \eta_{\mu\nu}$$

CURVED SPACETIME

$$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p\, g_{\mu\nu}$$

The only "fluids" that can satisfy the Einstein field Equations are those for which

$$T_{0i} = 0 \Rightarrow U_i = 0 \Rightarrow U_0 = -1$$

$$T_{00} = \rho = \rho(t)$$

$$T_{ij} = p\, a^2 \gamma_{ij} = p(t)\, a^2 \gamma_{ij}$$

In principle, $\rho(t)$ an $p(t)$ are completely free functions of cosmic time; however they must be related by the conservation of the stress-energy tensor,

$$T^{\mu\nu}{}_{;\mu} = 0$$

We have:

$$T^{\mu\nu}{}_{;\mu} = T^{\mu\nu}{}_{,\mu} + \Gamma^{\mu}_{\mu\sigma} T^{\sigma\nu} + \Gamma^{\nu}_{\mu\sigma} T^{\mu\sigma}$$

* For $\nu = 0$,

$$T^{\mu 0}{}_{;\mu} = T^{00}{}_{,0} + \Gamma^{\mu}_{\mu 0} T^{00} + \cancel{\Gamma^{0}_{00}}^{\,0}\, T^{00} + \Gamma^{0}_{ij} T^{ij}$$

$$= \dot\rho + 3\frac{\dot a}{a}\rho + \dot a a \underbrace{\gamma_{ij} \times p\, a^{-2} \gamma^{ij}}_{=3} \; ; \; \gamma_{ij}\gamma^{jk} = \delta_i^k$$

$$\Rightarrow \quad \boxed{\dot\rho + 3\frac{\dot a}{a}(\rho + p) = 0}$$

PROBLEM # 14 – Show that the equation $T^{\mu i}{}_{;\mu} = 0$ in fact vanishes identically

Notice that the equation above, which is called the <u>continuity equation</u>, is equivalent to the statement that

$$dE = -p\, dV$$

$p\,dV$ is the work done by the system

$$\frac{d(\rho V)}{dt} = \dot\rho V + \rho \dot V = -p \dot V$$

$$\Rightarrow \dot\rho + (\rho + p)\frac{\dot V}{V} = 0$$

Since the 3D physical volume $V \sim a^3$, we have that

$$\frac{\dot V}{V} = 3H \qquad \boxed{\frac{\dot a}{a} \equiv H}$$

$$\Rightarrow \boxed{\dot\rho + 3H(\rho + p) = 0}$$

CHECK!

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The Friedman Equations

$$G_{\mu\nu} = -8\pi G\, T_{\mu\nu} \Rightarrow \begin{cases} \boxed{3\left(H^2 + \frac{k}{a^2}\right) = 8\pi G \rho} & (*) \\ -H^2 - 2\frac{\ddot a}{a} - \frac{k}{a^2} = 8\pi G p & (**) \end{cases}$$

It is also useful to write (and easier to remember):

$$(*) + 3*(**) \Rightarrow \boxed{\frac{\ddot a}{a} = -4\pi G(\rho + 3p)}$$

"Acceleration" "Force"

THESE EQUATIONS ARE FUNCTIONS OF TIME ONLY!
(~~NOTICE THAT~~ $R_{\mu\nu}$, $G_{\mu\nu}$ and $g_{\mu\nu}$ are <u>not</u> functions of time only!)

These equations are particular to Einstein's General Relativity. In theories of gravity where, for instance,

$$S_{EH} = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\, R \longrightarrow \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,\tilde{f}(R)$$

the Friedman equations are modified.

We can write: $\tilde{f}(R) = R + f(R)$

$\nwarrow$ f = const. recovers General Relativity w/ Λ
$f(R) \to +2\Lambda$

Modified Einstein Equations:

$$G_{\mu\nu} + \frac{df}{dR} R_{\mu\nu} + \left(\Box \frac{df}{dR} - \frac{f}{2}\right) g_{\mu\nu} - \nabla_\mu \nabla_\nu \frac{df}{dR} = 8\pi G T_{\mu\nu}$$

$f_R \equiv \frac{df}{dR}$ is like a scalar field!

For $k=0$, the 0-0 EFE becomes:

$$H^2 - f_R(H^2 + HH') - \frac{1}{6}f + H^2 f_{RR} R' = \frac{8\pi G}{3}\rho$$

$$H' = \frac{dH}{d\log a} = \frac{1}{H}\frac{\ddot{a}}{a} + H$$

~~Exercise~~ Problem # 15

Consider the metric: $ds^2 = -N^2(t)\,dt^2 + a^2(t)\,\gamma_{ij}\,dx^i dx^j$

$$\gamma_{ij} = \left(1 + k x^2\right)^{-2} \delta_{ij}$$

(a) Compute the connections, Ricci tensor and Ricci scalar

(b) Find the modified Friedman equation 0-0 for the theory with

$$S_{MOD} = \int d^4x \sqrt{g}\, \frac{R + f(R)}{16\pi G}$$

By varying this action with respect to N, and then taking $N \to 1$.

(c) By imposing the continuity equation for matter (the RHS of the modified Friedman equations), find the modified i-i Friedman equation.

Matter

Usually we assume that, almost always, the number density of particles is conserved, for each individual species (protons, e, H, He, ν...).

$$n(t) = \frac{N}{V} \propto a^{-3}$$

The 4-momentum of the particles is

$$p^\mu = (E, \vec{p}) \quad , \quad p^\mu p_\mu = -m^2$$

$\Rightarrow$ (Flat spacetime) $E^2 = \vec{p}^{\,2} + m^2$

The energy density of a given species is:

$$\rho_s = n_s \cdot \bar{E}_s$$ average energy of S particles

The pressure (density) is:

$$p_s^i = n_s \bar{p}_s^i \bar{v}_s^i$$ (no summation over i!)

If we assume that the distribution of this species is homogeneous and isotropic, then

$$\sum_i p_s^i = n_s \sum_i \bar{p}_s^i \bar{v}_s^i = 3 p_s$$

$$\Rightarrow \boxed{p_s = \frac{1}{3} n_s \vec{p}_s \cdot \vec{v}_s}$$

Interesting limits:

(a) $m \gg |\vec{p}|$, $E \simeq m$

$|\vec{v}| \ll 1 (c)$

$$\Rightarrow \begin{cases} \rho_m \cong n_m \cdot m \\ P_m \cong \frac{1}{3} n_m \cdot m \frac{\vec{p}^2}{m^2} \ll \rho_m \end{cases}$$

In this case, $\rho_m \sim n_m \sim a^{-3}$ "DUST" "Pressureless matter"

(b) Relativistic limit ("ultra-relativistic")

$m \to 0$, $E \cong |\vec{p}|$, $\vec{p} \cdot \vec{v} \cong |\vec{p}|$

$\Rightarrow \rho_r \cong n_r E_r$

$P_r \cong \frac{1}{3} n_r \vec{p} \cdot \vec{v} \cong \frac{1}{3} n |\vec{p}| = \frac{1}{3} n_r E_r$

Note that, in an expanding universe, $E_r \sim \frac{1}{a}$

$\Rightarrow$ "radiation": $\begin{cases} \rho_r \sim a^{-4} \\ P_r = \frac{1}{3} \rho_r \end{cases}$

PROBLEM # 16

Verify that the expressions for the energy density and pressure of "dust" and "radiation" are consistent with the continuity equation

(c) Cosmological constant (Λ)

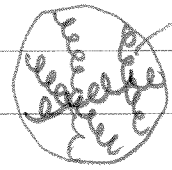
Consider a type of matter/energy for which

$$\rho_\Lambda = \text{constant} = \frac{\Lambda}{8\pi G}$$

By the continuity equation, $\dot{\rho}_\Lambda + 3H(\rho_\Lambda + p_\Lambda) = 0$,

$\Rightarrow \; p_\Lambda = -\rho_\Lambda$ Negative pressure

Example of negative pressure:
Bag model of QCD



gluons, $F \sim x^2$

$\Rightarrow p \simeq -\frac{1}{3}\rho$

(d) General expression

$p = w\rho$ w: EQUATION OF STATE

constant, or function of time.

$w_m = 0$ (dust)
$w_r = \frac{1}{3}$ (radiation)

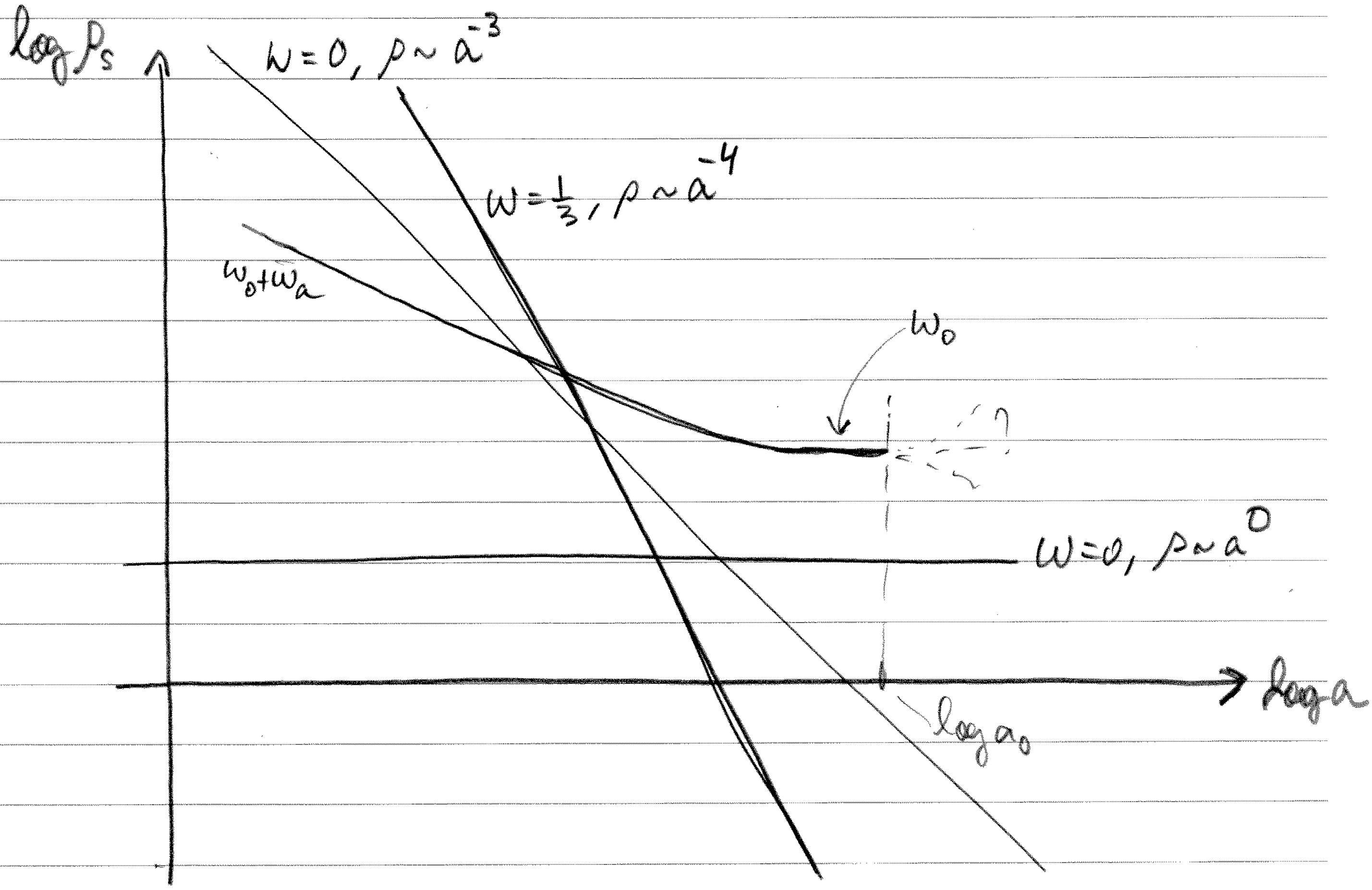
$w_\Lambda = -1$ (Λ)

PROBLEM # 17

In the CPL parameterization, $w = w(a)$, with:

$$w_{CPL} = w_0 + w_a (a_0 - a) \qquad [a_0 \to 1 \text{ w/ CPL}]$$

Find the functional form of $\rho(a)$ given this equation of state



The density parameter

The Friedmann equation reads:

$$\frac{3}{8\pi G}\left(H^2 + \frac{k}{a^2}\right) = \rho = \rho_1 + \rho_2 + \ldots$$

or,

$$\frac{3H^2}{8\pi G} = \rho - \frac{3k}{8\pi G a^2}$$

Let's define $\rho_c(t) \equiv \frac{3H^2}{8\pi G} \longrightarrow$ TODAY, $\rho_c^0 \approx 10^{-29}\, g/cm^3$, or ~ 6 protons/m^3, if all matter/energy was in the form of baryonic matter

or neutrons

$$\Rightarrow \quad 1 = \frac{\rho}{\rho_c} + \frac{\rho_k}{\rho_c} \qquad \rho_k \equiv -\frac{3k}{8\pi G a^2}$$

$$\Rightarrow \quad 1 = \frac{\rho_1}{\rho_c} + \frac{\rho_2}{\rho_c} + \ldots + \frac{\rho_k}{\rho_c}$$

Density parameter: $\boxed{\Omega_s = \frac{\rho_s}{\rho_c}}$ $\qquad \Omega_s^0 = \frac{\rho_s(a=a_0)}{\rho_c(a=a_0)}$

$\Rightarrow$ Friedman equation: $\boxed{\sum_s \Omega_s = 1 - \Omega_k}$

$$\begin{cases} \Omega_m + \Omega_r + \Omega_\Lambda + \Omega_w + \Omega_k = 1 \\ H^2 = H_0^2\left(\Omega_m^0 a^{-3} + \Omega_r^0 a^{-4} + \Omega_\Lambda^0 + \Omega_k^0 a^{-2} + \ldots\right) \end{cases}$$

Spatial curvature revisited

$$\rho_K = \frac{-3K}{8\pi G a^2}, \quad \Omega_K = \frac{\rho_K}{\rho_c}$$

$\Rightarrow$ Flat space $\Longrightarrow$ $\Omega_K = 0$, $\sum_s \Omega_s = 1$

"Closed" space $\Longrightarrow$ $\Omega_K < 0$ $\quad \sum_s \Omega_s > 1$

"Open" space $\Longrightarrow$ $\Omega_K > 0$ $\quad \sum_s \Omega_s < 1$

The CMB indicates that (*WITH BAO + H_0!)

$$\Omega_0 = \sum_s \Omega_s^0 \cong 1.01 \pm 0.01$$

$$\Rightarrow \quad -0.02 \lesssim \Omega_K^0 \lesssim 0.01$$ (WMAP 7 years + BAO + H_0)

PROBLEM #18

Let's say that there is only matter (dust, $p=0$), some spatial curvature, and also some type of matter X such that its equation of state is $w_X = -\frac{2}{3}$.

(a) Compute the luminosity-distance as a function of redshift, in the limit $z \ll 1$, to second order in z (re-derive $d_L(z)$ w/ $K \neq 0$!)

(b) An astronomical observation tells you that the deceleration parameter q_0 is:

$$-0.4 \leq q_0 \leq -0.2 .$$

Another observation tells you that

$$-0.05 \leq \Omega_k^0 \leq 0.05$$

How these observations limit the possible values of Ω_m^0 and Ω_X^0?

Try to make a plot such as: