Majorana neutrinos in dense matter and strong magnetic fields*

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*This talk is based on

M.Dvornikov, Found. Phys. **42**, 1469 (2012) arXiv:1106.3303 [hep-th];
M.Dvornikov, Nucl. Phys B **855**, 760 (2012), arXiv:1108.5043 [hep-ph];
M.Dvornikov and D.M.Gitman, arXiv:1211.5367 [hep-th].

Plan of the talk

- Introduction
- Problems in the description of a massive Majorana (Weyl) field in vacuum
- Brief review of the previous approaches to the studies of massive Weyl fields
- Hamilton formalisms for Majorana particles
- New way of quantization in vacuum
- Comparison with the standard description of Weyl fermions
- Propagators of massive Weyl fields in background matter
- Classical and quantum field theory descriptions of Weyl spinors in external fields
- Self-interacting Weyl spinors
- Discussion

Introduction

- After the experimental confirmation (SNO, KamLAND, BOREXINO, etc.) of the existence of neutrino oscillations, we believe that neutrinos are massive particles.
- The mass of the neutrinos is small. From the probes of the large-scale structure and of the cosmic microwave background anisotropies we get that $\sum \boldsymbol{m}_{\nu} < 1\ \mathrm{eV}$ (Wong, 2011).
- The smallness of the neutrino mass is explained in a most natural way by suggestion that neutrino mass eigenstates are Majorana particles (Schechter & Valle, 1980).
- Despite great experimental efforts (CUORICINO, NEMO, COBRA, etc.) to reveal the nature of neutrinos, i.e. to say whether neutrinos are Dirac or Majorana, this issue is still open.

Majorana and Weyl spinors

- Majorana condition in an extended sense

$$\psi^c = \mathrm{i}\gamma^2\psi^* = \kappa\psi, \quad |\kappa| = 1$$

- We can choose a specific form of γ-matrices to make a Majorana spinor real.
- One can express a Majorana spinor in terms of the two component Weyl spinors.

$$\psi_\eta = \begin{pmatrix} \mathrm{i}\sigma_2\eta^* \\ \eta \end{pmatrix} \text{ or } \psi_\xi = \begin{pmatrix} \xi \\ -\mathrm{i}\sigma_2\xi^* \end{pmatrix}$$

- The representation via Weyl spinors is preferable since standard model neutrinos are left-handed particles.
- The wave equations for a Weyl spinors can be formally derived from the Dirac equation.

$$\dot{\eta} - (\vec{\sigma}\nabla)\eta + m\sigma_2\eta^* = 0$$
$$\dot{\xi} + (\vec{\sigma}\nabla)\xi - m\sigma_2\xi^* = 0$$

Variation methods to describe Weyl fields dynamics

- A Lagrangian for a Dirac field is valid for both first- and second-quantized spinors

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\gamma^{\mu}\partial_{\mu} - m)\psi$$

- A Lagrangian for a Weyl field can be obtained by a direct substitution of a Majorana spinor (expressed in terms of Weyl spinors) in the Dirac Lagrangian

$$\mathcal{L} = \mathrm{i}\eta^{\dagger}(\sigma^{\mu}\partial_{\mu})\eta$$

$$-\frac{\mathrm{i}}{2}m\eta^{\mathrm{T}}\sigma_2\eta + \frac{\mathrm{i}}{2}m\eta^{\dagger}\sigma_2\eta^{*}$$

- Mass term in the Weyl Lagrangian vanishes if $\boldsymbol{\eta}$ is a c-numbers spinor

$$\mathcal{L}_m = -\frac{\mathrm{i}}{2}m\eta^{\mathrm{T}}\sigma_2\eta$$

$$+\frac{\mathrm{i}}{2}m\eta^{\dagger}\sigma_2\eta^{*} = 0$$

Possible solutions of the mass term puzzle

- Case (1957) suggested that, since η is a ½ spin field, it should be expressed via anti-commuting variables. This kind of suggestion is reasonable but not logical. Case (1957) also "quantizes" this field, i.e. he expresses it via creation and annihilation operators. However, this procedure is just a re-expression of already quantized objects in terms new variables rather than a generic quantization.
- Schechter & Valle (1981) claimed that a massive Weyl field does not have a proper description in terms of first quantized c-number spinors. The only possible description of a classical Weyl field can be made using Grassmann variables (g-numbers). However it is in contradiction to the fact that a *classical* fermionic system should have equivalent c- & g-numbers descriptions.
- Ahluwalia *et al.* (2010) introduced an exotic filed, *Eigenspinoren des LadungsKonjugationsOperators* (ELKO), which allowed them construct a Lagrange formalism for a c-number Majorana spinor.

Hamilton approach for the description of massive Weyl fields (c-numbers)

$$H=\int \mathrm{d}^3\mathbf{r}\left\{\pi^{\mathrm{T}}(\vec{\sigma}\,\nabla)\eta-\eta^{\dagger}(\vec{\sigma}\,\nabla)\pi^{*}+m\left[\eta^{\dagger}\sigma_2\pi+\pi^{\dagger}\sigma_2\eta\right]\right\}$$

If the particle mass $\boldsymbol{m}$ is real, the functional $\boldsymbol{H}$ is also real, as it should be for a classical Hamiltonian.

$$\dot{\eta}=\frac{\delta H}{\delta\pi}=(\vec{\sigma}\,\nabla)\eta-m\sigma_2\eta^{*},\;\;\dot{\pi}=-\frac{\delta H}{\delta\eta}=(\vec{\sigma}^{*}\,\nabla)\pi+m\sigma_2\pi^{*}$$

If we introduce the new variable $\xi=\mathrm{i}\boldsymbol{\sigma}_2\,\boldsymbol{\pi}$, we reproduce the wave equation for a right-handed neutrino.

The canonical equations for $\boldsymbol{\eta}$ and $\boldsymbol{\pi}$ decouple. It means that one cannot reconstruct a Lagrangian.

Quantization of **c**-number Weyl spinors

$$\eta(x)=\frac{1}{2}\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{3/2}}\sqrt{1+\frac{E}{|\mathbf{p}|}}\left[\left(a_-w_- - \frac{m}{E+|\mathbf{p}|}a_+w_+\right)e^{-\mathrm{i}px}+\left(a_+^*w_- + \frac{m}{E+|\mathbf{p}|}a_-^*w_+\right)e^{\mathrm{i}px}\right]$$

$$\xi(x)=\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{3/2}}\sqrt{1+\frac{E}{|\mathbf{p}|}}\left[\left(b_+w_+ + \frac{m}{E+|\mathbf{p}|}b_-w_-\right)e^{-\mathrm{i}px}+\left(b_-^*w_+ - \frac{m}{E+|\mathbf{p}|}b_+^*w_-\right)e^{\mathrm{i}px}\right]$$

$$H=\frac{1}{4}\int d^3\mathbf{p}\,E\left(1+\frac{E}{|\mathbf{p}|}\right)\Big\{\Big\{a_-^*(\mathbf{p})b_-(\mathbf{p})+b_-^*(\mathbf{p})a_-(\mathbf{p})-a_+(\mathbf{p})b_+^*(\mathbf{p})-b_+(\mathbf{p})a_+^*(\mathbf{p})$$

$$+\left(\frac{m}{E+|\mathbf{p}|}\right)^2\left[a_-(\mathbf{p})b_-^*(\mathbf{p})+b_-(\mathbf{p})a_-^*(\mathbf{p})-a_+^*(\mathbf{p})b_+(\mathbf{p})-b_+^*(\mathbf{p})a_+(\mathbf{p})\right]$$

$$+\mathrm{i}\frac{m}{|\mathbf{p}|}e^{-2\mathrm{i}Et}\left[a_-(\mathbf{p})b_-(-\mathbf{p})+b_-(-\mathbf{p})a_-(\mathbf{p})+b_+(-\mathbf{p})a_+(\mathbf{p})+a_+(\mathbf{p})b_+(-\mathbf{p})\right]$$

$$+e^{-2\mathrm{i}Et}\left[a_-^*(\mathbf{p})b_-^*(-\mathbf{p})+b_-^*(-\mathbf{p})a_-^*(\mathbf{p})+b_+^*(-\mathbf{p})a_+^*(\mathbf{p})+a_+^*(\mathbf{p})b_+^*(-\mathbf{p})\right]\Big\}\Big\}$$

Two independent ways of quantization

(2) $a_{\pm}(\mathbf{p}) = b_{\mp}(\mathbf{p})$

(1) $a_{\pm}(\mathbf{p}) = b_{\pm}(\mathbf{p})$

$$a_{\pm} = \frac{1}{\sqrt{2}}\left(c_{-} \pm c_{+}^{*}\right)$$

$$\left[a_{\sigma}(\mathbf{k}), a_{\sigma}^{*}(\mathbf{p})\right]_{+} = \delta^{3}(\mathbf{k}-\mathbf{p})$$

$$\left[c_{\sigma}(\mathbf{k}), c_{\sigma}^{*}(\mathbf{p})\right]_{+} = \delta^{3}(\mathbf{k}-\mathbf{p})$$

$$H = \int \mathrm{d}^{3}\mathbf{p}\, E\left[\begin{pmatrix} a_{-}^{*} \\ c_{-}^{*} \end{pmatrix} \times \begin{pmatrix} a_{-} \\ c_{-} \end{pmatrix} + \begin{pmatrix} a_{+}^{*} \\ c_{+}^{*} \end{pmatrix} \times \begin{pmatrix} a_{+} \\ c_{+} \end{pmatrix}\right], \quad \mathbf{P} = \int \mathrm{d}^{3}\mathbf{p}\, \mathbf{p}\left[\begin{pmatrix} a_{-}^{*} \\ c_{-}^{*} \end{pmatrix} \times \begin{pmatrix} a_{-} \\ c_{-} \end{pmatrix} + \begin{pmatrix} a_{+}^{*} \\ c_{+}^{*} \end{pmatrix} \times \begin{pmatrix} a_{+} \\ c_{+} \end{pmatrix}\right]$$

Expressions (1) and (2) may be regarded as *quantum Majorana conditions*.

Lagrange and Hamilton formalisms for **g**-number Weyl spinors

- We start from the Lagrangian

$$\mathcal{L} = \mathrm{i}\eta^{\dagger}(\sigma^{\mu}\partial_{\mu})\eta - \frac{\mathrm{i}}{2}m\eta^{\mathrm{T}}\sigma_2\eta + \frac{\mathrm{i}}{2}m\eta^{\dagger}\sigma_2\eta^{*}$$

- This system has two second class constraints

$$\Phi_1 = \pi - \mathrm{i}\eta^{*} = 0,\ \Phi_2 = \pi^{*} = 0,$$

$$\pi = \frac{\partial_r \mathcal{L}}{\partial\dot{\eta}},\ \pi^{*} = \frac{\partial_r \mathcal{L}}{\partial\dot{\eta}^{*}}$$

- The Hamiltonian

$$\mathcal{H} = \mathrm{i}\eta^{\dagger}(\boldsymbol{\sigma}\nabla)\eta + \frac{\mathrm{i}}{2}m\eta^{\mathrm{T}}\sigma_2\eta - \frac{\mathrm{i}}{2}m\eta^{\dagger}\sigma_2\eta^{*}$$

- The extended Hamiltonian

$$\mathcal{H}_1 = \mathcal{H} + \Phi_1\lambda_1 + \lambda_2\Phi_2$$

- The Dirac bracket

$$\left\{\eta(\mathbf{x},t),\eta^{*}(\mathbf{y},t)\right\}_{\mathrm{D}} = \left\{\eta,\eta^{*}\right\} - \left\{\eta,\Phi_i\right\}C_{ij}\left\{\Phi_j,\eta^{*}\right\} = \delta^3(\mathbf{x}-\mathbf{y}),$$

$$C_{ij} = \left\{\Phi_i,\Phi_j\right\}^{-1}$$

- Wave equation in the equivalent form

$$\dot{\eta} = \left\{\eta,\mathcal{H}\right\}_{\mathrm{D}} = (\vec{\sigma}\nabla)\eta + \mathrm{i}\sigma_2 m\pi,$$

$$\dot{\pi} = \left\{\pi,\mathcal{H}\right\}_{\mathrm{D}} = (\vec{\sigma}^{\mathrm{T}}\nabla)\pi + \mathrm{i}\sigma_2 m\eta$$

Quantization of massive Weyl fermions (g-numbers)

The quantization of a fermionic field can be made by the replacement (Gitman & Tyutin 1990)

$$\eta \to \hat{\eta}, \quad \pi \to \hat{\pi}, \quad \left[\hat{\eta}(t,\mathbf{x}),\hat{\pi}(t,\mathbf{y})\right]_+ = \mathrm{i}\left\{\eta(t,\mathbf{x}),\pi(t,\mathbf{y})\right\}_\mathrm{D} = \mathrm{i}\delta^3(\mathbf{x}-\mathbf{y})$$

The realization of the quantized fields

$$\hat{\eta}(x) = \frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{3/2}}\sqrt{\frac{E+|\mathbf{p}|}{2E}}\left[\left(\hat{a}_- w_- - \frac{m}{E+|\mathbf{p}|}\hat{a}_+ w_+\right)e^{-\mathrm{i}px} + \left(\hat{a}_+^\dagger w_- + \frac{m}{E+|\mathbf{p}|}\hat{a}_-^\dagger w_+\right)e^{\mathrm{i}px}\right],$$

$$\left[\hat{a}_\sigma(\mathbf{k}),\hat{a}_{\sigma'}^\dagger(\mathbf{k}')\right]_+ = \delta_{\sigma\sigma'}\delta^3(\mathbf{k}-\mathbf{k}'), \quad \left[\hat{a}_\sigma(\mathbf{k}),\hat{a}_{\sigma'}(\mathbf{k}')\right]_+ = 0, \quad \left[\hat{a}_\sigma^\dagger(\mathbf{k}),\hat{a}_{\sigma'}^\dagger(\mathbf{k}')\right]_+ = 0.$$

The total energy of quantized fields

$$E_\mathrm{tot} = \int \mathrm{d}^3\mathbf{r}\,\mathcal{H} = \int \mathrm{d}^3\mathbf{p}\,E\left(\hat{a}_-^\dagger\hat{a}_- + \hat{a}_+^\dagger\hat{a}_+\right) + \text{divergent terms}$$

Majorana neutrinos in external fields

- Now we generalize our treatment to arbitrary neutrino types. Thus wave functions acquire an index $\boldsymbol{a} = 1,2\ldots$
- We will be mainly interested in neutrino interactions with a background matter and an external electromagnetic field.
- Despite the neutrino charge is zero, a neutrino may participate in electromagnetic interaction owing to the possible presence of anomalous magnetic moments (Giunti & Studenikin, 2009).
- The interaction of active neutrinos with matter is diagonal in the flavor basis.
- We work with mass eigenstates since only in mass eigenstates basis we can say if neutrinos are Dirac of Majorana particles (Schechter & Valle, 1980).
- The effective potentials should be transformed to the mass eigenstates basis.
- The interaction with external fields is non-diagonal in the mass basis unless we discuss a specific case.

Lagrangian for Majorana neutrinos in background matter and electromagnetic fields

$$\mathcal{L} = \mathrm{i}\eta_a^\dagger(\sigma^\mu\partial_\mu)\eta_a - \frac{\mathrm{i}}{2}m_a\eta_a^{\mathrm{T}}\sigma_2\eta_a + \frac{\mathrm{i}}{2}m_a\eta_a^\dagger\sigma_2\eta_a^* - g_{ab}^\mu\eta_a^\dagger\sigma_\mu\eta_b$$

$$-\frac{1}{2}\left[\mu_{ab}\eta_a^\dagger\boldsymbol{\sigma}(\mathbf{B}-\mathrm{i}\mathbf{E})\sigma_2\eta_b^* + (\mu^\dagger)_{ab}\eta_a^{\mathrm{T}}\sigma_2(\mathbf{B}+\mathrm{i}\mathbf{E})\boldsymbol{\sigma}\eta_b\right]$$

$$-\frac{1}{2}\left[\varepsilon_{ab}\eta_a^\dagger\boldsymbol{\sigma}(\mathbf{E}+\mathrm{i}\mathbf{B})\sigma_2\eta_b^* + (\varepsilon^\dagger)_{ab}\eta_a^{\mathrm{T}}\sigma_2(\mathbf{E}+\mathrm{i}\mathbf{B})\boldsymbol{\sigma}\eta_b\right]$$

- The matrices of magnetic moments ($\boldsymbol{\mu_{ab}}$) and electric dipole moments ($\boldsymbol{\varepsilon_{ab}}$) are Hermitian and antisymmetric. No diagonal interaction with electromagnetic field is possible.
- The matrix of the interaction with background matter ($g^\mu{}_{ab}$) is Hermitian. The components of this this matrix for different oscillations channels were found by MD & Studenikin (2002). The zero component, $g^0{}_{ab}$, is proportional to the effective matter density. The spatial components, $\mathbf{g}_{ab}$, are the linear combinations of matter velocities and polarizations
- The complete system which involves terms nondiagonal in neutrino types can be analyzed only in frames of the perturbation theory. That is why we shall keep only diagonal interaction with external fields.

Propagators of massive Weyl fermions (g-numbers) in background matter

The evolution equations $\sigma_\mu \partial^\mu \eta - \mathrm{i}\sigma_2 m\pi + \mathrm{i}g_\mu \sigma_\mu \eta = 0,\ \sigma_\mu^* \partial^\mu \pi - \mathrm{i}\sigma_2 m\eta + \mathrm{i}g_\mu \sigma_\mu^* \pi = 0$

The causal propagator (Gitman & Tyutin 1990)

$$S_c(x-y) = \left\langle 0 \middle| T\left\{\Xi(x)\Xi^\dagger(y)\right\} \middle| 0 \right\rangle,\ \ \Xi^{\mathrm{T}} = (\pi,\eta),\ \ \hat{K}S_c(x) = -\delta^4(x),\ \ \hat{K} = \begin{pmatrix} \partial_t - (\boldsymbol{\sigma}^{\mathrm{T}}\nabla) - \mathrm{i}g^\mu \sigma_\mu^* & -\mathrm{i}\sigma_2 m \\ -\mathrm{i}\sigma_2 m & \partial_t - (\boldsymbol{\sigma}\nabla) + \mathrm{i}g^\mu \sigma_\mu \end{pmatrix}$$

The modified propagator

$$\tilde{S}_c = -\mathfrak{S}_2 S_c \mathfrak{S}_2 \gamma^0\gamma^5,\ \mathfrak{S}_2 = \mathrm{diag}(\sigma_2,\sigma_0),\ \left[\Gamma^\mu(\mathrm{i}\partial_\mu + \mathrm{i}g_\mu \Gamma^5) - m\Gamma^5\right]\tilde{S}_c = \delta^4(x),\ \Gamma^n = \left(-\mathrm{i}\gamma^5\gamma^\mu, -\mathrm{i}\gamma^5\right),\ \left[\Gamma^k,\Gamma^n\right]_+ = 2\eta^{kn}$$

The path integral representation of a propagator

$$\tilde{S}_c = \exp\left(\mathrm{i}\Gamma^n \frac{\partial_l}{\partial\theta^n}\right)\int_0^\infty \mathrm{d}e_0 \int \mathrm{d}\chi_0 \int_{e_0} \mathcal{M}(e)\mathrm{D}e \int_{x_{\mathrm{in}}}^{x_{\mathrm{out}}} \mathrm{D}x \int \mathrm{D}\tilde{\pi} \int \mathrm{D}\nu \int_{\psi(0)+\psi(1)=\theta} \mathcal{D}\psi \exp\left\{\mathrm{i}(S_{\mathrm{cl}} + S_{\mathrm{GF}}) + \psi_n(1)\psi^n(0)\right\}\Bigg|_{\theta=0}$$

$$S_{\mathrm{cl}} = \int_0^1 \left[-\frac{z^2}{2e} - \frac{e}{2}M^2 + \dot{x}_\mu d^\mu + \mathrm{i}\chi\left(m\psi^5 + \frac{2}{3}\psi^\mu d_\mu\right) - \mathrm{i}\psi_n\dot{\psi}^n\right]\mathrm{d}\tau,$$

$$z^\mu = x^\mu + \mathrm{i}\chi\psi^\mu,\ M^2 = m^2 + g^2 + 16\partial_\mu g^\mu \psi^0\psi^1\psi^2\psi^3,\ d_\mu = 2\mathrm{i}\varepsilon_{\mu\nu\alpha\beta} g^\nu \psi^\alpha\psi^\beta$$

Wave equations for Weyl fermions (c-numbers) in a background matter and an external electromagnetic field

The wave equations for $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ can be derived directly from the Dirac equation. We should recall that the vector term $\sim \boldsymbol{g}^{\mu}{}_{ab}\,\gamma_{\mu}\,\psi_{b}$ of the matter interaction is absent for Majorana neutrinos and the axial vector term $\sim \boldsymbol{g}^{\mu}{}_{ab}\,\gamma_{\mu}\,\gamma^{5}\psi_{b}$ is twice the corresponding contribution for Dirac neutrinos.

$$\dot{\eta}_a - (\vec{\sigma}\,\nabla)\eta_a + m_a\sigma_2\eta_a^* - \mu_{ab}\vec{\sigma}\,(\mathbf{B}-\mathrm{i}\mathbf{E})\sigma_2\eta_b^* + \mathrm{i}(g_{ab}^0 + \vec{\sigma}\,\mathbf{g}_{ab})\eta_b = 0$$
$$\dot{\xi}_a + (\vec{\sigma}\,\nabla)\xi_a - m_a\sigma_2\xi_a^* + \mu_{ab}\vec{\sigma}\,(\mathbf{B}+\mathrm{i}\mathbf{E})\sigma_2\xi_b^* - \mathrm{i}(g_{ab}^0 - \vec{\sigma}\,\mathbf{g}_{ab})\xi_b = 0$$

The interaction with matter is characterized by a four vector $\boldsymbol{g}^{\mu}{}_{ab} = (\boldsymbol{g}^{0}{}_{ab}, \mathbf{g}_{ab})$. We discuss the situation when only active neutrinos are present and CP invariance is conserved. In this case the matrix $(\boldsymbol{g}^{\mu}{}_{ab})$ is symmetric.

Hamiltonian for Weyl spinors (c-numbers) in external fields

$$H = \int \mathrm{d}^3\mathbf{r} \left[\sum_a \left\{ \pi_a^{\mathrm{T}} (\vec{\sigma}\nabla)\eta_a - \eta_a^{\dagger}(\vec{\sigma}\nabla)\pi_a^* + m_a \left[\eta_a^{\dagger}\sigma_2\pi_a + \pi_a^{\dagger}\sigma_2\eta_a \right] \right\} \right.$$

$$\left. + \sum_{ab} \left\{ \mu_{ab} \left[\pi_a^{\mathrm{T}} \vec{\sigma}(\mathbf{B} - \mathrm{i}\mathbf{E})\sigma_2\eta_b^* + \eta_a^{\mathrm{T}}\sigma_2\vec{\sigma}(\mathbf{B} + \mathrm{i}\mathbf{E}) \right] - \mathrm{i} \left[\pi_a^{\mathrm{T}} \left(g_{ab}^0 + \vec{\sigma}\,\mathbf{g}_{ab} \right)\eta_b - \eta_a^{\dagger} \left(g_{ab}^0 + \vec{\sigma}\,\mathbf{g}_{ab} \right)\pi_b^* \right] \right\} \right]$$

$$\dot{\eta}_a = \frac{\delta H}{\delta \pi_a} = (\vec{\sigma}\nabla)\eta_a - m_a\sigma_2\eta_a^* + \mu_{ab}\vec{\sigma}(\mathbf{B} - \mathrm{i}\mathbf{E})\sigma_2\eta_b^* - \mathrm{i}\left(g_{ab}^0 + \vec{\sigma}\, g_{ab} \right)\eta_b,$$

$$\dot{\pi}_a = -\frac{\delta H}{\delta \eta_a} = (\vec{\sigma}^*\nabla)\pi_a + m_a\sigma_2\pi_a^* - \mu_{ab}\sigma_2\vec{\sigma}(\mathbf{B} + \mathrm{i}\mathbf{E})\pi_b^* + \mathrm{i}\left(g_{ab}^0 + \vec{\sigma}^* g_{ab} \right)\pi_b$$

Quantization of Weyl fields (c-numbers) in matter

We separate the total Hamiltonian into $\boldsymbol{H}_0$ and $\boldsymbol{H}_{\mathbf{int}}$ parts. $\boldsymbol{H}_0$ contains terms diagonal in neutrino types whereas $\boldsymbol{H}_{\mathbf{int}}$ is nondiagonal in neutrino types.

Let us discuss the case nonmoving and unpolarized matter, $\mathbf{g}_{ab} = \mathbf{0}$.

$$\eta_a^{(0)}(\mathbf{r},t) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{3/2}} \left\{ \left[a_a^- w_- e^{-\mathrm{i}E_a^- t} - \frac{m_a}{E_a^+ + |\mathbf{p}| - g_{aa}^0} a_a^+ w_+ e^{-\mathrm{i}E_a^+ t} \right] e^{\mathrm{i}\mathbf{pr}} + \left[\left(a_a^+\right)^* w_- e^{\mathrm{i}E_a^+ t} + \frac{m_a}{E_a^- + |\mathbf{p}| + g_{aa}^0} \left(a_a^-\right)^* w_+ e^{\mathrm{i}E_a^- t} \right] e^{-\mathrm{i}\mathbf{pr}} \right\}$$

$$\xi_a^{(0)}(\mathbf{r},t) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{3/2}} \left\{ \left[b_a^+ w_+ e^{-\mathrm{i}E_a^+ t} + \frac{m_a}{E_a^- + |\mathbf{p}| + g_{aa}^0} b_a^- w_- e^{-\mathrm{i}E_a^- t} \right] e^{\mathrm{i}\mathbf{pr}} + \left[\left(b_a^-\right)^* w_+ e^{\mathrm{i}E_a^- t} - \frac{m_a}{E_a^+ + |\mathbf{p}| - g_{aa}^0} \left(b_a^+\right)^* w_- e^{\mathrm{i}E_a^+ t} \right] e^{-\mathrm{i}\mathbf{pr}} \right\}$$

Energy levels of a neutrino in a background matter: $E_a^\pm = \sqrt{m_a^2 + \left(|\mathbf{p}| \mp g_{aa}^0\right)^2}$

$$a_a^\pm(\mathbf{p})\left(E_a^\pm + |\mathbf{p}| \mp g_{aa}^0\right) = 4 b_a^\pm(\mathbf{p})\left(|\mathbf{p}| \mp g_{aa}^0\right), \quad \left[a_a^\pm(\mathbf{k}), \left[a_a^\pm(\mathbf{p})\right]^* \right]_+ = \delta_{ab}\delta^3(\mathbf{k}-\mathbf{p})$$

$$H = \int \mathrm{d}^3\mathbf{p} \sum_a \left[E_a^- \left(a_a^-\right)^* a_a^- + E_a^+ \left(a_a^+\right)^* a_a^+ \right], \quad \mathbf{P} = \int \mathrm{d}^3\mathbf{p} \sum_a \mathbf{p} \left[\left(a_a^-\right)^* a_a^- + \left(a_a^+\right)^* a_a^+ \right]$$

Nondiagonal neutrino interaction

We work in the forward scattering regime, i.e. only the terms, which conserve the number of particles, should be left.

$$\delta^3(\mathbf{p}-\mathbf{k})\rho_{AB}(\mathbf{p}) = \left\langle a_B^*(\mathbf{p}) a_A(\mathbf{k}) \right\rangle, \quad A = (\pm, a) \qquad \mathrm{i}\dot{\rho} = \left[\rho, H_{\mathrm{int}}\right]_-$$

We discuss the case when $\mathbf{E} = 0$ and two ultrarelativistic neutrinos, $\boldsymbol{a} = 1{,}2$, with $\boldsymbol{\mu}_{12} = \mathrm{i}\boldsymbol{\mu}$, $\boldsymbol{\mu}_{21} = -\,\mathrm{i}\boldsymbol{\mu}$, and $\mathbf{k} \gg \max(\boldsymbol{m}_{\boldsymbol{a}}, \boldsymbol{g}^0{}_{\boldsymbol{aa}})$.

$$E_a^{\pm} = |\mathbf{k}| + \frac{m_a^2}{2|\mathbf{k}|} \mp g_{aa}^0 + \cdots \quad \rho_{\mathrm{qm}} = \mathcal{U}\rho\mathcal{U}^\dagger, \ \mathcal{U} = \mathrm{diag}\left(e^{-\mathrm{i}(\Phi+g_{11}^0)t}, e^{\mathrm{i}(\Phi-g_{22}^0)t}, e^{-\mathrm{i}(\Phi-g_{11}^0)t}, e^{\mathrm{i}(\Phi+g_{22}^0)t}\right), \ \Phi = \frac{m_1^2 - m_2^2}{2|\mathbf{k}|}$$

$$\mathrm{i}\dot{\rho}_{\mathrm{qm}} = \left[\mathcal{H}_{\mathrm{qm}}, \rho_{\mathrm{qm}}\right]_-, \ \mathcal{H}_{\mathrm{qm}} = \begin{pmatrix} \Phi + g_{11}^0 & g_{12}^0 & 0 & -\mathrm{i}\mu|\mathbf{B}|\sin\vartheta_{\mathbf{kB}} \\ g_{21}^0 & -\Phi + g_{22}^0 & \mathrm{i}\mu|\mathbf{B}|\sin\vartheta_{\mathbf{kB}} & 0 \\ 0 & -\mathrm{i}\mu|\mathbf{B}|\sin\vartheta_{\mathbf{kB}} & \Phi - g_{11}^0 & -g_{12}^0 \\ \mathrm{i}\mu|\mathbf{B}|\sin\vartheta_{\mathbf{kB}} & 0 & -g_{21}^0 & -\Phi - g_{22}^0 \end{pmatrix}$$

Here we reproduce the standard effective Hamiltonian for the description of neutrino flavor and spin-flavor oscillations in a background matter and a magnetic field (Lim & Marciano, 1988; Mannheim 1988, Giunti, *et al.*, 1992).

The self-interaction of neutrinos

Neutrinos can interact between themselves by exchanging a $\boldsymbol{Z}$-boson. This kind of interaction is significant when the neutrino density is high, e.g., in a supernova explosion.

$$H_{\mathrm{S}} = \int \mathrm{d}^3\mathbf{r} \sum_{abcd} G_{ab} G_{cd} \bar{\psi}_a \gamma_\mu^{\mathrm{L}} \psi_b \cdot \bar{\psi}_c \gamma_{\mathrm{L}}^\mu \psi_d \to \int \mathrm{d}^3\mathbf{r} \sum_{abcd} G_{ab} G_{cd} \eta_a^\dagger \sigma_\mu \eta_b \cdot \eta_c^\dagger \sigma^\mu \eta_d ,$$

In the forward scattering approximation for ultrarelativistic neutrinos we get the contribution to the effective quantum mechanical Hamiltonian:

$$\mathcal{H}_{\mathrm{qm}}(\mathbf{k}) = \mathrm{diag}\left(\mathcal{H}_{--}, \mathcal{H}_{++}\right), \ \rho_{\mathrm{qm}} = \begin{pmatrix} \rho_{--} & \rho_{-+} \\ \rho_{+-} & \rho_{++} \end{pmatrix}$$

$$\mathcal{H}_{--} = 2\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left(1 - \cos\vartheta_{\mathbf{kp}}\right) \left\{ G \mathrm{tr}\left[G\rho_{--}(\mathbf{p}) - G^{\mathrm{T}}\rho_{++}(\mathbf{p}) \right] + G\left[\rho_{--}(\mathbf{p}) - \rho_{++}^{\mathrm{T}}(\mathbf{p}) \right] G \right\},$$

$$\mathcal{H}_{++} = 2\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left(1 - \cos\vartheta_{\mathbf{kp}}\right) \left\{ G^{\mathrm{T}} \mathrm{tr}\left[G^{\mathrm{T}}\rho_{++}(\mathbf{p}) - G\rho_{--}(\mathbf{p}) \right] + G^{\mathrm{T}}\left[\rho_{++}(\mathbf{p}) - \rho_{--}^{\mathrm{T}}(\mathbf{p}) \right] G^{\mathrm{T}} \right\}$$

We can reproduce the case of Dirac neutrinos (Sigl & Raffelt, 1993) if we set $\mathcal{H}_{++} = 0$ and $\mathcal{H}_{--} \neq 0$, as well as introduce the antineutrino density matrix $\bar{\rho} = \rho_{++}^{\mathrm{T}}$.

Discussion: Weyl fields in vacuum

- We applied the Hamilton formalism to the description of classical (or c-number, or first-quantized) massive Weyl fields. Previously it was claimed by Schechter & Valle (1981) that a massive Weyl spinor cannot be desctibed in terms of a first-quantized field.
- We carried out a canonical quantization of a system. It was found that, owing to the degeneracy of the energy levels, two independent ways to quantize a Weyl field are possible.
- We introduced *quantum Majorana conditions*, which connects "particle", η, and "antipatricle", ξ, degrees of freedom. The classical Majorana condition, $\psi^c \sim \psi$, cannot be regarded as equality of "particles" and "antiparticles" since it is applied on a classical level *before* quantization.
- The description of massive Weyl fermions in terms of c-number spinors is consistent with the standard g-number description.

Discussion: Interacting Weyl fields

- We generalized our formalism to include the interaction with a background matter and an external electromagnetic field.
- We obtained a classical Hamiltonian which then was used to quantize the system in case when only the diagonal interaction with nonmoving and unpolarized matter is present.
- The account of the nondiagonal interaction with external fields, if we work in the forward scattering approximation, allowed us to reproduce the effective Hamiltonian for the description of neutrino spin-flavor oscillations in matter and magnetic field.
- We also studied the neutrino self-interaction. Again in the forward scattering limit, we derived the contribution to the effective Hamiltonian. It was shown that the self-interaction does not directly cause a spin-flip, but certainly it influences the process of spin-flavor oscillations of neutrinos. Although there is no direct correspondence to Dirac neutrinos case, we showed how one can reconstruct an effective Hamiltonian for Dirac neutrinos.
- We derived the most general representation of a propagator of a massive Weyl field interacting with a background matter. The characteristics of the background matter (number density, velocity, and polarization) can depend on time and spatial coordinates.

Acknowledgements

- I am thankful to A.E. Lobanov, J. Maalampi, G.G. Raffelt, and V.B. Semikoz for helpful discussions;
- FAPESP (Brazil) for a grant;
- The organizers of SILAFAE 2012 for the invitation.