Fermions on Thick $D$-branes
Localisation and physical implications

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Outline

1. Brane-Worlds and Gravity
2. Localisation of Fermions
3. Analytic and Numerical Analysis
4. Results and Conclusions
Motivation

- $D$-branes appear in the AdS/CFT correspondence.
- May ‘solve’ the hierarchy problem.
- May be used as a new way of compactification.
- Different behaviour of chiralities in brane-world scenarios.
- May be thought as realization of ‘string’ phenomenology.
**Setup:** Couple of $D3$-branes.

**Interaction:** 5-dimensional Gravity.

**Hierarchy Problem**

\[
\left( M_p^{(4)} \right)^2 \propto \left( \frac{M_p^{(5)}}{k} \right)^3 \left( 1 - e^{-2\pi kr_c} \right)
\]

but

\[
M_p^{(5)} e^{-2\pi kr_c} \sim M_W.
\]
- **Setup**: Single $D_3$-branes.
- **Interaction**: 5-dimensional Gravity.

### 4-dimensional Gravity

The induced gravity on the brane is Newton’s plus consistent corrections with GR.
Domain Wall Scenarios

- Coupled Einstein-Scalar field.
- Self-interacting scalar field.
- $V(\phi)$ with degenerated minima.
- Types:
  - Static DW’s.
  - Dynamical DW’s.

Localisation of Gravity

Most of the Domain Walls share the property of gravity localisation, depending on the $\mathbb{Z}_2$ symmetry of the spacetime.
Early Stages

- Imposition that matter is localised. [ADD, 1998]
- Matter is not localised on RS scenarios. [Bajc and Gabadadze, 2000]
- Move to generalised models...
- Add new interactions. [BG, 2000]
General Set Up

- **Metric:**

\[ \hat{g}_{\hat{\mu}\hat{\nu}} = e^{2A(\xi)} \left( -dt_{\hat{\mu}} dt_{\hat{\nu}} + e^{2B(\xi, t)} dx_{\hat{\mu}} dx_{\hat{\nu}} \right) + e^{2C(\xi)} d\xi_{\hat{\mu}} d\xi_{\hat{\nu}}. \]

- **Fermions in curved spacetime,**

\[ \mathcal{L} = -\sqrt{\hat{g}} \bar{\Psi} \hat{\nabla} \Psi \]

- **Yukawa-like coupling with the scalar of the DW,**

\[ \mathcal{L}_I = \sqrt{\hat{g}} \bar{\Psi} (\lambda \phi^n - M \text{sgn}(\phi)) \Psi. \]

- \( n \in 2\mathbb{N} + 1. \)
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Procedure

1. Find the e.o.m. for the 5-dimensional spinor.

\[ \hat{\nabla} \Psi + M \Psi = \lambda \Phi \Psi \]

2. Separate the diff. op. in 4-dimensional part and extra terms.

3. Decompose,

\[ \Psi = f_+ \psi_+ + f_- \psi_- \]

4. Find the e.o.m. for \( f_\pm \)

Vielbein Formalism

When spinors in curve spacetimes are considered, the vielbein formalism is the right way to follow.
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\[ \hat{\nabla} = e^A(\xi) \nabla + e^C(\xi) \gamma^* \partial_\xi \]
Procedure

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4. Find the e.o.m. for \( f_\pm \)

\[ \nabla^{(4)} \psi_\pm \sim m \psi_\pm \]

\[ \gamma^* \psi_\pm = \pm \psi_\pm \]

Factorise them

Factorise the \( \psi_+ \) and \( \psi_- \)
Procedure

1. Find the e.o.m. for the 5-dimensional spinor.
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E.O.M.
Decouple and transform to Schrödinger-like equation.
Generalised Coupling

If general polynomial coupling is considered,

\[ L_I = \sqrt{\hat{g}} \lambda \bar{\Psi} \Phi^n \Psi, \]

the e.o.m. might be integrated for \( n \in \mathbb{Z} \).

Most Interesting Cases

For \( n \in 2\mathbb{N} + 1 \) the profile potential gives a condition on the coupling, \( \lambda \), for localisation to occur.
For \( n \in 2\mathbb{N} \) the profile potential gives a condition on the dimension of the extra dimension for localisation to occur.

Curiosity

The e.o.m. might be solved formally for \( n \in -\mathbb{N} \).
Possible Choices of E.O.M.

- $B(\xi = 0) = 0$ and $\bar{\theta}^{(4)} \psi_{\pm} = 0$.
- $B(\xi = 0) = 0$ and $\bar{\theta}^{(4)} \psi_{\pm} = -m \psi_{\mp}$.
- $\nabla^{(4)} \psi_{\pm} = 0$.
- $\nabla^{(4)} \psi_{\pm} = -m \psi_{\mp}$.

Profile solutions

$$f_{\pm}(\xi) = e^{-\frac{1}{4}(4A' + 3B')} \pm \lambda \int d\xi P(\phi) e^C.$$
Possible Choices of E.O.M.

- $B(\xi = 0) = 0$ and $\partial (4)\psi_\pm = 0$.
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- $\nabla (4)\psi_\pm = 0$.
- $\nabla (4)\psi_\pm = -m\psi_T$.

Profile solutions

$$f_\pm(\xi) = e^{-\frac{1}{4}(4A'+3B')}\pm \lambda \int d\xi P(\phi)e^C.$$  

$$f_\pm(\chi) = e^{-A\pm \lambda \int d\chi P(\phi)e^A}.$$
Possible Choices of E.O.M.

- $B(\xi = 0) = 0$ and $\bar\partial (4) \psi_\pm = 0$.
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- $\nabla (4) \psi_\pm = 0$.
- $\nabla (4) \psi_\pm = -m \psi_\mp$.

Profile solutions

\[ f_\pm(\xi) = e^{-\frac{1}{4}(4A' + 3B')} \pm \lambda \int d\xi \mathcal{P}(\phi) e^C. \]

\[ f_\pm(\chi) = e^{-A} \pm \lambda \int d\chi \mathcal{P}(\phi) e^A. \]
For massive cases

The initial e.o.m. is,

\[ f'_\pm + A' f_\pm \mp \lambda \mathcal{P}(\phi) e^A f_\pm = \pm m f_\mp. \]

Decoupling the profiles equations, and changing,

\[ f_\pm = e^{-A} u_\pm, \]

these equations are rewritten in a Schrödinger-like form,

\[ \left[ -\partial^2_x + V_{qm}^\pm \right] u_\pm = m^2 u_\pm, \]

with

\[ V_{qm}^\pm = \left( \lambda \mathcal{P}(\phi) e^A \right)^2 \pm \partial_x \left( \lambda \mathcal{P}(\phi) e^A \right). \]
Numerov’s Method

Numerov’s method is a numerical method to solve ordinary differential equations of second order in which the first-order term does not appear.

The algorithm

For a ODE.,

\[
\left( \frac{\partial^2}{\partial x^2} + f(x) \right) y(x) = 0,
\]

Numerov’s solution is,

\[
y_{n+1} = \frac{\left( 2 - \frac{5h^2}{6} f_n \right) y_n - \left( 1 + \frac{h^2}{12} f_{n-1} \right) y_{n-1}}{\left( 1 + \frac{h^2}{12} f_{n+1} \right)}
\]
Implementation

- Numerov algorithm: for solving the ODE.
- Swept of energy: for finding the approx. eigen-energy.
- Bisection algorithm: For refining the energy eigenvalue.

Moduli Space

The five dimensional mass takes values from $0.00$ to $1.00$ in steps of $0.02$, while $\lambda \in [1.45, 5.00]$ in steps of $0.05$. Additionally, the computations have been restricted to values of $s = 1, 3, 5$. 
Double Domain Walls

Solution:

\[ \phi(\xi) = \frac{\sqrt{6s - 3}}{s} \arctan (\alpha \xi)^s, \]
\[ A(\xi) = C(\xi) = -\frac{1}{2s} \ln (1 + (\alpha \xi)). \]

Parameters: \( n, s \in 2\mathbb{N} + 1. \)

Results

- Existence of a \( \lambda_* \) below which no positive chirality is localised. \( (n = 1) \)
- Existence of a \( \lambda_c \) above which no negative chirality is localised.
- \( \lambda_* < \lambda_c. \)
Physical Implications

Case $n = s = 1$:

Interpretation

- Lightest particles have no detectable right-handed chirality. (Neutrinos)
- Not so light particles might localise both chiralities. (Electrons)
Physical Implications

Case \( n, s \neq 1 \):

**Interpretation**

- Not so light particles might localise both chiralities.
- Mass difference could favour the See-Saw mechanism.
Additional Configurations

- **Kink Domain Wall:**

\[ A(\xi) = -\frac{2}{3} \delta \left[ \ln \left( \cosh \left( \frac{\alpha \xi}{\delta} \right) \right) + \frac{1}{4} \tanh^2 \left( \frac{\alpha \xi}{\delta} \right) \right], \]

\[ B(\xi) = C(\xi) = 0, \]

\[ \phi(\xi) = \sqrt{3\delta} \tanh \left( \frac{\alpha \xi}{\delta} \right). \]
Sine-Gordon Wall:

\[ A(\xi) = -\delta \ln \left( \cosh \left( \frac{\alpha \xi}{\delta} \right) \right), \]
\[ B(\xi) = C(\xi) = 0, \]
\[ \phi(\xi) = \sqrt{3} \delta \arctan \left( \sinh \left( \frac{\alpha \xi}{\delta} \right) \right). \]
Asymmetric Domain Wall:

\[
A(\xi) = \alpha \xi - \delta \exp \left( -2e^{-\frac{\beta \xi}{\delta}} \right) + \delta \text{Ei} \left( -2e^{-\frac{\beta \xi}{\delta}} \right),
\]

\[
B(\xi) = C(\xi) = 0,
\]

\[
\phi(\xi) = 2\sqrt{3}\delta \left( \exp \left( -e^{-\frac{\beta \xi}{\delta}} \right) - \varepsilon \right).
\]

with

\[
\text{Ei}(x) = \int_{-\infty}^{x} dt \frac{e^t}{t}.
\]
Results

Moduli space for the asymmetric domain wall, with \( n = 1, 3, 5 \). Negative and positive chirality respectively.
Results

Moduli space for the asymmetric domain wall, with $n = 1, 3, 5$. Negative and positive chirality respectively.
Moduli space for the asymmetric domain wall, with $n = 1, 3, 5$. Negative and positive chirality respectively.
Conclusions

- More general profile equations were found. These are useful when more complex systems are considered.
- Restrictions to the moduli space of parameters were found, by restricting the lower-dimensional theory to be tachyon-free.
- Due to the mass difference between localised chiralities, it is interesting to consider the See-Saw mechanism.
- The mass difference between neutrinos and electrons makes natural the absence of right-handed part of the former, while the analog of the latter is present in the low-dimensional theory.
- Similar behaviour for diverse DW’s. Except the appearance of a sort of band structure of the moduli space.