Mating Systems and Population Dynamics

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Southern-Summer School on Mathematical Biology
Introduction

1 Mating Systems\(^1\)
   - Monogamy
   - Polygamy
   - Polygyny

2 Negative Interactions
   - Predator-Prey
   - Competition
   - Parasite-Host
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   - Monogamy
   - Polygamy
   - Polygyny

2 Negative Interactions
   - Predator-Prey
   - Competition
   - Parasite-Host
Some Examples

* Images from Google.com
Do Mating Systems affect the outcome of Species Interactions?

Does sex-selective predation stabilize or destabilize predator-prey dynamics?[4]

**Mating Function**

\[
p(m, f, \theta) = 1 - \exp\left(-\frac{m}{\theta}\right)
\]

\[
p(m, f, \theta) = \frac{hm \exp\left(\frac{hm-f}{h\theta}\right) - hm}{hm \exp\left(\frac{hm-f}{h\theta}\right) - f}
\]

\[
p(m, f, \theta) = \min\left(\frac{hm}{f}, 1\right)
\]
Do Mating Systems affect the outcome of Species Interactions?

Does sex-selective predation stabilize or destabilize predator-prey dynamics?[^4]

### Mating Function

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p(m, f, \theta) = \min \left( \frac{hm}{f}, 1 \right)
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Sex-selective Predator-Prey Model

How sex-selective predation affects system stability?

\[
\begin{align*}
\frac{dm}{dt} & = \frac{b}{2} p(m, f, \theta) f - dm - \lambda_1 mx \\
\frac{df}{dt} & = \frac{b}{2} p(m, f, \theta) f - df - \lambda_2 fx \\
\frac{dx}{dt} & = -Mx + e_1 \lambda_1 mx + e_2 \lambda_2 fx
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...So it does, when it comes to Predator-Prey Interaction.
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...So it does, when it comes to Predator-Prey Interaction.
How about competition?

Lets consider two species: the structured species $A$, distinguishing males ($m$) and females ($f$); and an unstructured species $x$, the competitor.

\[
\frac{dm}{dt} = \frac{b}{2} p(m, f, \theta) f - dm - \lambda_1 mx - \alpha_1 (m + f)m
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\]

\[
\frac{dx}{dt} = rx \left(1 - \frac{x}{k}\right) - (\mu_1 m + \mu_2 f)x
\]

where:

- $b$: birth rate of species $A$.
- $p(m, f, \theta)$: mating function.
- $d$: mortality rate of species $A$.
- $\lambda_1, \lambda_2$: effect of competitor $x$ on males and females.
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Group IV (SSSMB)
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Now, changing the scale, we have:

\[
\frac{dm'}{d\tau} = \frac{b}{2r} p(m', f', \theta)f' - \frac{d}{r} m' - \frac{\alpha_1}{r} (m' + f') m' - \frac{\lambda_1 k}{r} m' x' \\
\frac{df'}{d\tau} = \frac{b}{2r} p(m', f', \theta)f' - \frac{d}{r} f' - \frac{\alpha_2}{r} (m' + f') f' - \frac{\lambda_2 k}{r} f' x' \\
\frac{dx'}{d\tau} = (1 - x') x' - \left( \frac{\mu_1}{r} m' + \frac{\mu_2}{r} f' \right) x'
\]

where all parameters are positive.
### Parameters in the simulations

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<th>$d^4$</th>
<th>$r$</th>
<th>$K$</th>
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$m_0 = 4, f_0 = 4, x_0 = 4$
Species Dynamics (sp. A goes extinct)

Limited Polyandry, $h=0.3$

- male w/o comp.
- female w/o comp.
- male w/ comp.
- female w/ comp.
- competitor

Parameters:
- $b=3$, $d=(0.2,0.2)$, $\lambda=(0.4,0.4)$, $\mu=0.4,0.4$, $\alpha=(0.2,0.1)$, $r=2$, $K=4$, $\text{initX}=(4,4,4)$

Time

density

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Species Dynamics (competitor \times \text{goes extinct})

Monogamy, $h=1$

- male w/o comp.
- female w/o comp.
- male w/ comp.
- female w/ comp.
- competitor

Parameters:
- $b=3$, $d=(0.2,0.2)$, $\lambda=(0.4,0.2)$, $\mu=0.4,0.4$, $\alpha=(0.2,0.1)$, $r=2$, $K=4$, $\text{initX}=(4,4,4)$

Time

Density

Group IV (SSSMB) Mating System SSSMB 10 / 19
Unlimited Polygyny, $h=NA$

- Male without competition
- Female without competition
- Male with competition
- Female with competition
- Competitor

Parameters:
- $b=3$, $d=(0.2,0.2)$, $\lambda=(0.4,0.2)$, $\mu=0.1,0.1$, $\alpha=(0.4,0.2)$, $r=2$, $K=4$, $\text{initX}=(4,4,4)$

Time and density trends are shown in the graph.
Some Results

- Only polyandry is excluded by competition with X
- Polyandry and monogamy are excluded by competition with X
- All mating systems are excluded by the competitor X
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Some Results

\[ \lambda \]

\[ \alpha_1 = \alpha_2 \]

\[ \alpha_1 < \alpha_2 \]

\[ \alpha_1 > \alpha_2 \]

A = (01; 01)  B = (02; 02)  C = (04; 04)  D = (01; 02)  E = (02; 04)  F = (04; 08)  G = (02; 01)  H = (04; 02)  I = (08; 04)

Group IV (SSSMB)  Mating System  SSSMB
Conclusions

- Polyandry is the least stable mating system in our parameter space.
- Sex-biased competition toward females leads to exclusion by the competitor.
- Increasing the effect of the competitor over males and females will drive all four mating system extinct.
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Unlimited Polygyny (stable if $b > 2d$)

$$m^* = \frac{b(b - 2d)}{4d(\alpha_2 - \alpha_1) + 2b(\alpha_1 + \alpha_2)}$$

$$f^* = \frac{(b - 2d)[2d(\alpha_2 - \alpha_1) + b\alpha_1]}{4d(\alpha_2 - \alpha_1) + 2b(\alpha_1 + \alpha_2)}$$

Limited Polygyny ($h > 1$), monogamy ($h = 1$) and limited polyandry ($0 < h < 1$) If $f^* < hm^*$, the equilibrium value is the same as the unlimited polygyny case, which is stable if $b > 2d$. If $f^* > hm$:

$$m^* = \frac{b(b - 2d)}{4d(\alpha_2 - \alpha_1) + 2b(\alpha_1 + \alpha_2)}$$

$$f^* = \frac{(b - 2d)[2d(\alpha_2 - \alpha_1) + b\alpha_1]}{4d(\alpha_2 - \alpha_1) + 2b(\alpha_1 + \alpha_2)}$$

which is stable if $bh > 2d$
Analytical Solution: w/ competitor, $\mu_1 = \mu_2$

- Unlimited Polygyny

\[ x^* = \frac{4\alpha_2 - (b - 2d)\mu}{\alpha_2 - 2\mu\lambda_2} \]