The Boulware-Deser mode in Zwei-Dreiben gravity

Max Bañados (Santiago)

in collaboration with Miguel Pino (Santiago).

–Work in progress–

- The papers say the Universe is accelerating!

- Einsein himself thought that a cosmological constant was "like adding mass to gravity". This is incorrect, but it makes some sense to explore to what extend Λ can be replaced by a massive graviton.

- I will not discuss the success/failure of this idea as a cosmological model

- I will discuss some features of massive gravity, specifically, the Bouware-Deser ghost.

- To make it simpler we shall do it in three dimensions "Zwei-Dreiben gravity" (3d bigravity), as discussed recently by Bergshoeff et al.

Fierz-Pauli theory

Adding mass to the graviton was started by Fierz and Pauli back in 1939! Let $h_{\mu\nu}$ a symmetric rank-2 tensor and

$$\begin{aligned} L(h_{\mu\nu}) &= -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h \\ &\quad -\frac{1}{2} m^2 (h_{\mu\nu}h^{\mu\nu} - h^2). \end{aligned}$$

- For $m^2 = 0$ this Lagrangian describes a massless graviton with <u>2 degrees of freedom</u>. And it is equal to Einstein-Hilbert Lagrangian linearized around flat space $\eta_{\mu\nu}$.
- For $m^2 \neq 0$, describes a massive graviton with <u>5 degrees of freedom</u>

Two challenges:

- Find a covariant form for the mass term
- Find an interacting non-linear theory, which is unitary.

Covariant interacting action, Isham-Salam-Strathdee, 1971

Consider a theory with two metrics, coupled by a potential U

$$I(g,f) = \int \Big(\sqrt{g} R(g) + \sqrt{f} R(f) - U(f,g) \Big).$$

$$U(f,g) = \frac{1}{2} m^2 (g_{\mu\nu} - f_{\mu\nu})(g_{\alpha\beta} - f_{\alpha\beta})(f^{\mu\alpha} f^{\nu\beta} - f^{\mu\nu} f^{\alpha\beta})$$

Linearizing around the flat space (and diagonalizing) we obtain the Lagrangian,

$$-\tfrac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \tfrac{1}{2}\partial_\lambda h \partial^\lambda h \ +$$

$$-\tfrac{1}{2}\partial_\lambda k_{\mu\nu}\partial^\lambda k^{\mu\nu} + \partial_\mu k_{\nu\lambda}\partial^\nu k^{\mu\lambda} - \partial_\mu k^{\mu\nu}\partial_\nu k + \tfrac{1}{2}\partial_\lambda k \partial^\lambda k - m^2(k_{\mu\nu}k^{\mu\nu} - k^2)$$

In words, this theory has two gravitons:

- $h_{\mu\nu}$ is massless carrying 2 degrees of freedom
- $k_{\mu\nu}$ is massive carrying 5 degrees of freedom

$\left.\right\}$ 7 degrees of freedom

The Boulware-Deser Ghost. Non linear dynamics

Now count the number of degrees of freedom in the full theory:

'Number of dynamical fields' – 'number of gauge symmetries'

Writing each metric in ADM form,

$$
\begin{aligned}
ds_g^2 &= -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \\
ds_f^2 &= -M^2 dt^2 + f_{ij}(dx^i + M^i dt)(dx^j + M^j dt)
\end{aligned}
$$

The counting goes as follows:

$$
\begin{aligned}
N, N_i; M, M_i &: \quad \text{4 Lagrange Multipliers/4 Auxiliary fields} \\
\mathcal{H}, \mathcal{H}_i &: \quad \text{4 first class constraints (overall diffeormorhisms)} \\
\{g_{ij}, \pi^{kl}\}, \{f_{ij}, p^{kl}\} &: \quad 6 \times 4 = 24 \;\; \text{dynamical fields}
\end{aligned}
$$

The number of degrees of freedom is

$$
\frac{1}{2}(24 - 2 \times 4) = 8 = \underline{2 + 5 + \mathbf{1}}
$$

- The extra mode (Boulware Deser) appears at non-linear level
- This mode is a ghost (negative kinetic energy)

Massive gravity is indeed not free of trouble

1. It has ghosts; Boulware-Deser mode just described
2. van Dam-Veltman-Zakharov discontinuity
   - The limit $m^2 \to 0$ does not give back general relativity

     $$ds^2 \approx \text{``} -\left(1 - \frac{2Me^{-mr}}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{Me^{-mr}}{r}} + r^2 d\Omega^2 \text{''}$$

     This problem might be solved by the Vainshtein mechanism (no-linearities)
3. Causality issues. There exists modes propagating faster than light (Osipov-Rubakov, 2008)

Getting rid of Boulware-Deser mode. A long history

$$I(g,f) = \int \left(\sqrt{g} R(g) + \sqrt{f} R(f) - U(f,g) \right).$$

- The degrees of freedom count shown before applies to the generic situation, for an arbitrary $U(g,f)$.
- Perhaps there exists particular $U(f,g)$ with special properties such that the Boulware-Deser mode does not show up?

After considerable work:

1. Georgi, Arkani-Hamed, Schwarz (2003)
2. Creminelli, Nicolis, Papucci, Trincherini (2005)
3. de Rham, Gabadadze (2010)
4. de Rham, Gabadadze, Tolley (2010): $U \sim \mathrm{Tr} \left(\sqrt{f^{\mu\nu} g_{\nu\rho}} \right)$
5. Hassan, Rosen (2010),(2011)...
6. Hinterbichler, Rosen (2011)

A special potential (apparently) does exist. It is best written in a first order formulation.

Hinterbichler-Rosen vielbein formulation (2011)

Let e^a and ℓ^a two independent 1-form vielbeins. Let R^{ab} and Q^{ab} the associated 2-form curvatures. Consider the bigravity action (wedge $\wedge$ symbols omitted)

$$I = \int \epsilon_{abcd} \Big(\underline{R^{ab} e^c e^d + \Lambda_1 e^a e^b e^c e^d} + \underline{Q^{ab} \ell^c \ell^d + \Lambda_2 \ell^a \ell^b \ell^c \ell^d} \\ + \underline{p_1 \, e^a e^b e^c \ell^d + p_2 \, e^a e^b \ell^c \ell^d + p_3 \, e^a \ell^b \ell^c \ell^d} \Big)$$

- This is a nice, geometrical action (Lovelock spirit)
- The interaction is severely restricted. Only three parameters (at $d = 4$).
- Easily generalized to any dimension, and any number of vielbeins (multigravity)

- This bigravity action is claimed to have no Boulware-Deser ghost
- We shall critically check this assertion in three dimensions, where the canonical structure is simpler and well understood

Massive gravity in three dimensions. A long story too

1. Massive graviton. What graviton? $\int \sqrt{g} R$ propagates noting!
2. Topologically Massive Gravity, TMG (Deser et al):

$$\int \sqrt{g} R + \frac{1}{\mu} \left(w d w + \frac{2}{3} w^3 \right)$$

   describes one helicity ± 2, depending on sign of μ.
3. New Massive Gravity, NMG (Bergshoeff, et al);

$$\int \sqrt{g} R + \frac{1}{m^2} \left(R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2 \right)$$

   describes two states ± 2
4. 3d Bigravity (MB & Theisen)

$$\int \sqrt{g} R + \sqrt{f} R(f) - m^2 (g - f)^2$$

   - linear theory: massless (0 states) plus a massive (two states) graviton;
   - non-linear theory: three states, Boulware-Deser mode.

Hinterbichler-Rosen theory in three dimensions

Let a^a and ℓ^b two 1-form dreibens in three dimensions

$$I[w, \pi, e, l] = \int \epsilon_{abc}\Big(\underline{R^{ab}e^c} + \underline{Q^{ab}\ell^c} + \underline{p_1\, e^a e^b \ell^c + p_2\, \ell^a \ell^b e^c}\Big)$$

For simplicity we do not incorporate cosmological constants at each sector.

- This action was called Zwei-Dreiben gravity in Bergshoeff et al (2013)
- Note that only two allowed terms in the potential are allowed (p_1 and p_2)
- It is argued in Bergshoeff et al that the Boulware-Deser mode is absent

On general grounds, the Boulware-Deser mode is expected.

Performing a 2+1 decomposition of forms

$$e^a_\mu \, dx^\mu = e^a_i dx^i + e^a_0 dt, \qquad \ell^a_\mu \, dx^\mu = \ell^a_i dx^i + \ell^a_0 dt$$

(the same for $\omega^a_\mu dx^\mu$,$\pi^a_\mu dx^\mu$) the action is "already" Hamiltonian

$$I = \int \epsilon^{ij} \Big(\frac{d\omega^a_i}{dt} \, e_{a\,j} + \frac{d\pi^a_i}{dt} \, \ell_{a\,j} \Big) + \omega^a{}_0 \phi_{1a} + \pi^a{}_0 \phi_{2a} + e^a{}_0 \, G_{1a} + \ell^a{}_0 \, G_{2a}$$

- $\{\omega^a_i, e^b_j\}$, $\{\pi^a_i, \ell^b_j\}$ are $6 \times 4 = 24$ canonical variables.
- $\omega^a{}_0, \ \pi^a{}_0, \ e^a{}_0, \ \ \ell^a{}_0$ are $3 \times 4 = 12$ Lagrangue Multipliers,
- $\phi_{1a}, \ \phi_{2a}, \ G_{1a}, \ G_{2a}$ are $3 \times 4 = 12$ constraints.

There are 6 gauge symmetries: 3 diffs + 3 Lorentz transformations.
So, the 12 constraints split into 6 first class + 6 second class

$$\tfrac{1}{2}\,(24 - 2 \times 6 - 6) = \underline{3 = 2 + 1}$$

The Boulware-Deser mode is still around..!!

Checking out the details. I.

$$\dot q^i = \frac{\partial \phi_\alpha}{\partial p_i}\lambda^\alpha$$

$$\dot p_i = -\frac{\partial \phi_\alpha}{\partial q^i}\lambda^\alpha$$

$$\phi_\alpha(p,q) = 0$$

Could there be other (secondary) constraints? No.

$$I[q^i, p_j, \lambda^\alpha] = \int dt \Big(p_i \dot q^i - \lambda^\alpha \phi_\alpha(q,p) \Big)$$

Consistency of constraints with time evolution imply,

$$0 = \frac{d\phi_\alpha}{dt} \;=\; \frac{\partial \phi_\alpha}{\partial q^i}\dot q^i + \frac{\partial \phi_\alpha}{\partial p_i}\dot p_i = [\phi_\alpha, \phi_\beta]\lambda^\beta = 0 \qquad (*)$$

Despite being algebraic, these equations are <u>not</u> constraints!

- If $[\phi_\alpha, \phi_\beta] = C^\gamma{}_{\alpha\beta}\phi_\gamma$, the constraints are preserved. There is a gauge symmetry and λ^α are arbitrary. Eq $(*)$ imposes nothing.
- If $[\phi_\alpha, \phi_\beta]$ is invertible, then Eq. $(*)$ implies $\lambda^\beta = 0$. No gauge symmetry. End of algorithm
- If $[\phi_\alpha, \phi_\beta]$ has a some non-zero eigenvalues, some Lagrange multipliers are fixed, some are arbitrary. End of algorithm

For this family of actions, $\frac{d\phi_\alpha}{dt} = 0$ never yields secondary constraints. Only conditions on the Lagrange multipliers, if any.

Bifurcations

To be fair, let us consider the consistency equation $[\phi_\alpha, \phi_\beta]\lambda^\beta = 0$ again and look at two toy models, say,

<u>Model A</u>

$$\begin{pmatrix} 0 & c_1 \\ -c_1 & 0 \end{pmatrix} \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix} = 0.$$

In this model, $c_1 \neq 0$ is a constant and assumed different from zero. The only solution to the consistency condition is

$$\lambda^1 = 0, \qquad \lambda^2 = 0.$$

<u>Model B</u>

$$\begin{pmatrix} 0 & p \\ -p & 0 \end{pmatrix} \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix} = 0.$$

- Branch I: For generic values of p, the matrix is invertible and implies $\lambda^1 = \lambda^2 = 0$.
- Branch II: Interpret this as an equation for p and impose a *secondary* constraint $p = 0$. (Implies $\dot{p} = 0$, and so on.)

A constraint system may have bifurcations, branches with different number of degrees of freedom.

Bifurcations and the Boulware-Deser mode

- The Boulware-Deser mode is present in 3d massive gravity.
- It can be hidden away, by choosing a branch with extra constraints (Bergshoeff et al, 2013). In other words, it can be set to zero by an initial condition.
- Nothing can prevent it to reappear under a generic perturbations

Checking out the details II. Consistency algorithm for Zwei-Dreiben gravity

$$I[w,\pi,e,l] = \int \epsilon_{abc}\Big(\underline{R^{ab}e^c} + \underline{Q^{ab}\ell^c} + \underline{p_1\, e^a e^b \ell^c + p_2\, \ell^a \ell^b e^c}\Big)$$

The equations of motion are,

$$\begin{aligned} R^{ab} &= -2p_1 e^a \ell^b - p_2 \ell^a \ell^b, \qquad & De^a = 0, \\ Q^{ab} &= -p_1 e^a e^b - 2p_2 e^a \ell^b, \qquad & \nabla \ell^a = 0. \end{aligned}$$

Using Cartan equations one finds integrability algebraic relations like

$$(p_1 e^a + p_2 \ell^a) e_b \ell^b \;\; = \;\; 0 \;\; \Leftrightarrow \;\; [\phi_\alpha, \phi_\beta]\lambda^\beta = 0$$

(plus others). These are exactly equal to the constraint consistency conditions. They can be solved by:

- Impose further constraints as $e_a \ell^a = 0$ (Bergoshoeff et al (2013)) $\Rightarrow$ <u>2 degrees of freedom.</u>
- Interpret as equations for the Lagrange multipliers $e^a_0, \ell^b_0, \ldots$ (MB & Pino (2013)) $\Rightarrow$ <u>3 degrees of freedom.</u>

Is it "natural" to impose $e_a\ell^a = 0$?

The constraint $e_a\ell^a$ imposed by Bergoshoeff et al is consistent with but not a consequence of the equations of motion. The following field,

$$
\begin{aligned}
e^a{}_\mu &= \begin{pmatrix} r & 0 & 0 \\ 0 & \frac{\sqrt{p_2}}{\sqrt{2}p_1}\frac{1}{r} & 0 \\ 0 & 0 & r \end{pmatrix}, \\
\ell^a{}_\mu &= \begin{pmatrix} -\frac{p_1}{p_2}r & 0 & 0 \\ 0 & \frac{c}{r} & \frac{r}{p_2}\sqrt{2c^2p_1p_2^2 - p_1^2} \\ 0 & 0 & -\frac{p_1}{p_2}r \end{pmatrix}.
\end{aligned}
$$

solves all equations of motion (de Sitter space) and yet

$$
e_a\ell^a = -\frac{\sqrt{2c^2p_1p_2^2 - p_1^2}}{\sqrt{2p_2}p_1} \neq 0
$$

is not zero. Imposing $e_a\ell^a = 0$ does kill interesting solutions.

Checking details III. Maximum rank

$$
\begin{aligned}
[\phi_1(\xi), \phi_1(\chi)] &= -\epsilon^{ab}{}_c \xi_a \chi_b D e^c, \qquad [\phi_1(\xi), \phi_2(\chi)] = 0 \\
[\phi_2(\xi), \phi_2(\chi)] &= -\epsilon^{ab}{}_c \xi_a \chi_b \nabla l^c \\
[G_1(\xi), G_1(\chi)] &= 2p_1 \epsilon_{abc} \xi^a \chi^b D l^c \\
[G_2(\xi), G_2(\chi)] &= 2p_2 \epsilon_{abc} \xi^a \chi^b \nabla e^c \\
[G_1(\xi), G_2(\chi)] &= -2\epsilon_{abc} \left(D\xi^a \chi^b + \xi^a \nabla \chi^b \right) \left(p_1 e^c + p_2 l^c \right) \\
[G_1(\xi), \phi_1(\chi)] &= \epsilon^a{}_{bc} \xi^b \chi_a R^c - 2p_1 \epsilon_{abc} \epsilon^{ad}{}_e \xi^b \chi_d l^c e^e \\
[G_2(\xi), \phi_2(\chi)] &= \epsilon^a{}_{bc} \xi^b \chi_a Q^c - 2p_2 \epsilon_{abc} \epsilon^{ad}{}_e \xi^b \chi_d e^c l^e \\
[G_1(\xi), \phi_2(\chi)] &= -2\epsilon_{abc} \epsilon^{ad}{}_e \xi^b \chi_d \left(p_1 e^c + p_2 l^c \right) l^e \\
[G_2(\xi), \phi_1(\chi)] &= -2\epsilon_{abc} \epsilon^{ad}{}_e \xi^b \chi_d \left(p_2 l^c + p_1 e^c \right) e^e
\end{aligned}
$$

This is a 12×12 matrix.

- Evaluating on generic solution $e_a \ell^a \neq 0 \Rightarrow$ rank $= 6$, as expected. Confirms 3 degrees of freedom.
- Evaluating on solutions with $e_a \ell^a = 0 \Rightarrow$ rank $= 4$; 2 degrees of freedom. (No hidden symmetry! Further constraints arise from $\frac{d}{dt} e_a \ell^a = 0$.)

Conclusions

- Massive gravity in its first order formulation is an attractive theory.
- Just like Lovelock choice avoids ghosts in higher curvature gravity, one could have expected that massive gravity would be unitary. Apparently it is not.
- In four dimensions the calculation is more complicated because the spin connection ω^{ab} does not have the same number of components as the vielbein e^{a}. Work in progress.