Fundamentals of magnetohydrodynamics
Part I

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What do we mean by magnetohydrodynamics?

- It is a fluid-like theoretical description for the dynamics of matter.

- Baryonic matter in the Universe is mostly hydrogen.

- At temperatures above $10^4$ K it becomes a hydrogen plasma, i.e. a gas made of protons and electrons.

- The large scale behavior of this gas can be described through fluidistic equations (Navier-Stokes).

- This fluid is made of electrically charged particles and therefore it suffers electric and magnetic forces.

- Not only that, these charges are sources of self-consistent electric and magnetic fields. Therefore, the fluid equations will couple to Maxwell’s equations.

- At small spatial scales (and fast timescales) non-fluid or kinetic effects become non-negligible.
Magnetic fields in Astrophysics

Earth and planets  
Sun and stars  
Interstellar medium

Pulsars  
Accretion disks  
Galaxies
Number of sunspots vs. time

It clearly shows an 11 yr period with irregularities in its maxima, its periods and rise-fall times.

Area covered by spots as a function of latitude and time.

At the beginning of each cycle, sunspots are born at latitudes of ±30° and migrate to the Equator.

Magnetic polarities are reversed from one cycle to the next and are different at different hemispheres (Hale’s law)
- Wolf Number vs. time

- Maunder minimum lasts from 1650 to 1700.

- There is evidence of more Maunder-like events (Beer 2000).

- N-S asymmetries were enhanced during the Maunder minimum (Ribes & Nesme-Ribes 1993).
The equations for the fluid are:

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \bar{u})
\]

\[
p = p_0 \left( \frac{\rho}{\rho_0} \right)^\gamma
\]

\[
\rho \frac{\partial \bar{u}}{\partial t} = -\rho (\bar{u} \cdot \nabla) \bar{u} - \nabla p + \frac{1}{4\pi} (\nabla \times \bar{B}) \times \bar{B} + \bar{F}_{ext} + \nabla \cdot \bar{\sigma}_{\text{visc}}
\]

The magnetic field is generated by the plasma, and satisfies the so-called induction equation.

\[
\frac{\partial \bar{B}}{\partial t} = \nabla \times (\bar{u} \times \bar{B}) + \eta \nabla^2 \bar{B}, \quad \nabla \cdot \bar{B} = 0
\]

It is obtained as a result of Ohm’s law (see below) and Faraday’s equation.

\[
\bar{E} + \frac{1}{\epsilon} \bar{u} \times \bar{B} = \frac{1}{\sigma} \bar{J}
\]
The MHD equations are:

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \bar{u})
\]

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p = p_0 \left( \frac{\rho}{\rho_0} \right)^\gamma
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\[
\frac{\partial \bar{B}}{\partial t} = \nabla \times (\bar{u} \times \bar{B}) + \eta \nabla^2 \bar{B}, \quad \nabla \cdot \bar{B} = 0
\]

These equations describe a large number of important plasma processes, such as:

- instabilities and wave propagation (Alfven and magnetosonic waves)
- dynamo mechanisms to generate magnetic fields
- MHD turbulence
- magnetic reconnection

Note that even though the electric field is not present, it does not mean that it is not relevant.

\[
\vec{E} + \frac{1}{c} \vec{u} \times \vec{B} = \frac{1}{\sigma} \vec{J}, \quad \eta = \frac{c^2}{4\pi\sigma}
\]
If we assume the magnetic field $B$ to be very small, the MHD equations decouple. We can first solve the equations of motion. For instance, in the incompressible limit

$$\frac{\partial \vec{u}}{\partial t} = - (\vec{u} \cdot \nabla) \vec{u} - \frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{u}, \quad \nabla \cdot \vec{u} = 0$$

Now that we know $\vec{u}(\vec{x}, t)$, we can solve the induction equation to obtain $\vec{B}(\vec{x}, t)$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{u} \times \vec{B}) + \eta \nabla^2 \vec{B}, \quad \nabla \cdot \vec{B} = 0$$

This particular and convenient approximation is known as the kinematic dynamo. Note that the induction equation is linear in $\vec{B}(\vec{x}, t)$ for any given $\vec{u}(\vec{x}, t)$. For stationary flows, there will be a dynamo solution whenever

$$\vec{B}(\vec{x}, t) = \vec{B}_0(\vec{x}) e^{\gamma t}, \quad \gamma > 0$$

What kind of permanent flows are ubiquitous in astrophysical objects?
Rotation and Convection

Rotation (macro)
- Radial differential rotation
- Latitudinal differential rotation
- Meridional flow

Convection (micro)
- Helicoidal convective turbulence
- Giant cells (driven by Coriolis)
- Regular and stochastic components

Omega effect

Alpha effect
The Sun rotates with a well documented differential rotation profile, obtained from helioseismology.

Radial: Almost solid body rotation in the interior at a rate

\[ \omega_{\text{core}} \approx 2 \times 10^{-6} \text{ s}^{-1} \]

There is an abrupt jump at the base of the convective zone (tachocline).

Latitudinal: Differential rotation at the surface is faster at the Equator and is given by (Beck 1999)

\[ \omega_{\text{surf}} (\theta) \equiv a + b \cos^2 (\theta) + c \cos^4 (\theta) \]

where \( \theta \) is the colatitude.
We assume an incompressible meridional flow with a surface speed of approx. 20 \( m\,s^{-1} \).

The mass flux is given by the following stationary stream function

\[
\psi(\theta, r) = -\psi_0 \sin^{3/2}(\theta) \cos(\theta) \left( r^{3/2} - (r_{min} - r) (r_{max} - r) \right)
\]

where \( r_{min} = 0.6 \, R_\theta \) and \( r_{max} = R_\theta \).

We adopt a density profile \( \rho(r) = \rho_0 \, r^{-1/2} \)

and perform a radial average of the poloidal velocity components.
MDI (SoHO) observations remarkably show the various components involved in convective and rotational motions (Beck et al. 1998).

Rising convective flows might become helical as a result of the Coriolis force. This process is relevant whenever the Rossby number is less than unity.

The Rossby number is

\[
Ro = \left| \frac{(\bar{u} \cdot \nabla)\bar{u}}{2\tilde{\Omega} \times \bar{u}} \right| \approx \frac{T_{\text{Sun}}}{T_{\text{vortex}}} 
\]

\[ Ro \approx \frac{T_{\text{Sun}}}{T_{\text{vortex}}} \leq 1 \quad \text{only for the giant cells.} \]

Note that with only 20-40 of these vortices we cover the whole solar surface.

Because of this poor statistics, we assume the alpha coefficient to have a regular and a stochastic part (Choudhuri 1992, Ossendrijver 1996).
We integrate the induction equation numerically, assuming axi-symmetry.


\[
\frac{\partial B_\phi}{\partial t} = -(U_r + \varepsilon \frac{\partial U_\theta}{\partial \theta})B_\phi - \varepsilon U_\theta \frac{\partial B_\phi}{\partial \theta} + (\Delta \omega \cos \theta - \sin \theta \frac{\partial \omega}{\partial \theta})A + \Delta \omega \sin \theta \frac{\partial A}{\partial \theta} + \frac{1}{\mathcal{R}} \nabla^2 B_\phi
\]
\[
\frac{\partial A}{\partial t} = -(U_r + \varepsilon U_\theta \cot \theta)A - \varepsilon U_\theta \frac{\partial A}{\partial \theta} + \alpha(B_\phi)B_\phi + \frac{1}{\mathcal{R}} \nabla^2 A
\]

Differential rotation

Meridional flow

Small-scale convection

Dissipation

where \( \mathcal{R} = \frac{U_0 \delta R}{\eta} \), \( \varepsilon = \frac{\delta R}{R} \), \( \Delta \omega = \omega_{\text{surf}}(\theta) - \omega_{\text{core}} \), \( \alpha = \frac{\alpha_0 + \delta \alpha}{1 + B_\phi^2/B_0^2} \sin(\theta) \cos(\theta) \)
Non-stochastic butterfly diagrams

- Toroidal field vs. latitude and time.
- Magnetic energy vs. latitude and time.
- Hale's law can clearly be observed.
- It is a proxy of Wolf's number.
We model \( \delta\alpha \) as a gaussian stochastic process, with spatial and temporal correlations corresponding to typical giant cells.

\[ \tau_{\text{corr}} \approx 30 \text{ days}, \quad \lambda_{\text{corr}} \approx 2 \times 10^5 \text{ km} \]

Toroidal magnetic field obtained from solar magnetograms, displaying the change of polarity in the polar regions.

Our results correctly reproduce the general behavior, although our butterflies arise at higher latitudes.
➢ Toroidal magnetic field for a long time integration (Gómez & Mininni 2006).

➢ A minimum of activity is observed at the center. After a few cycles, normal activity is restablished.

➢ Magnetic energy at mid-latitudes vs. time. Two Maunder-like events are observed.
The movie shows the solar magnetic cycle and the emergence of coronal loops as a result of differential rotation and magnetic buoyancy.
Most planets in the solar system have their own magnetic field. The impact of the solar wind on the planetary fields generate the so-called magnetospheres.

Mars and Venus do not have magnetic fields, but do have atmospheres.

The frozen-in condition of the magnetic field carried by the SW with atmospheric ions create the so-called induced magnetospheres.

In a stationary regime, the magnetic field should satisfy

$$\nabla \times (u \times B) = 0 \quad , \quad \nabla \cdot B = 0$$

The velocity field is the stationary and irrotational flow past a sphere. The Figure shows the corresponding streamlines for this flow.
Integration of $\nabla \times (\mathbf{u} \times \mathbf{B}) = 0$, $\nabla \cdot \mathbf{B} = 0$ seems straightforward, but it is not.

It can be integrated analytically using the method of characteristics along the flow streamlines.

Below I show the magnetic fieldlines for the 2D version. The 3D version is in the poster by Romanelli et al.

Let the magnetic field at infinity be tilted at an angle $\alpha$ with respect to the vertical.

\[ \alpha = 0^\circ \quad \alpha = 30^\circ \quad \alpha = 45^\circ \quad \alpha = 60^\circ \quad \alpha = 90^\circ \]
Today we presented the MHD equations as a valid description of the large-scale behavior of astrophysical plasmas.

As a first application, we presented the Alpha-Omega dynamos to describe the basic features of the solar dynamo.

Using empirical profiles of differential rotation and meridional flows, we manage to reproduce various observed aspects of the solar cycle, such as its period, rise-fall asymmetry and sunspot migration toward the Equator.

Moreover, considering a stochastic part for the Alpha effect, we not only reproduce the irregularities observed in the cycle, but also the potential occurrence of Maunder-like events where magnetic activity on the Sun switches off for several decades.

Finally, we solved the stationary induction equation for a stationary flow past a sphere, as a simple model to describe induced magnetospheres.
Fluid turbulence

- Energy cascade
  - energy flux toward high k
  - vortex breakdown

- Scale invariance
  - energy flux in k: \( \varepsilon_k \approx \frac{u_k^2}{\tau_k} \)
  - energy power spectrum: \( E_k \approx \frac{u_k^2}{k} \)

\[ \tau_k \approx \frac{1}{ku_k}, \quad \varepsilon_k \approx \frac{u_k^2}{\tau_k} = \text{const.} \]

- Therefore
  \[ E_k \approx \frac{u_k^2}{k} = \varepsilon^3 k^{-\frac{5}{3}} \]

Kolmogorov spectrum (K41)