Ponzano-Regge Model on Manifold with Torsion

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There is an alternative approach to gravitation based on the Weitzenbőck geometry:

This theory is known as teleparallel equivalent of general relativity (TEGR), where the gravitation is attributed to torsion.

GR uses the Levi-Civita connection $\rightarrow$ curvature but not torsion
TG uses the Weitzenbőck connection $\rightarrow$ torsion but not curvature

Teleparallel gravity is a sector of Einstein-Cartan theories which describe gravity by means of a connection having both torsion and curvature.
In teleparallel gravity the dynamical object is the vierbein field:

\[ e^a_\mu(x) \]

- This nontrivial tetrad field is used to define the linear Weitzenböck connection

\[ \Gamma^\sigma_{\mu\nu}(x) = e_a^\sigma(x) \partial_\nu e^a_\mu(x), \]

a connection presenting torsion, but no curvature.

- The Levi–Civita connection of the metric

\[ g_{\mu\nu}(x) = \eta_{ab} e^a_\mu(x) e^b_\nu(x), \]

is given by

\[ \Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left[ \partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right] \]
Key features of Weitzenböck spacetime

- The relation between these two connection is given by

\[ \Gamma^\sigma_{\mu\nu} = \tilde{\Gamma}^\sigma_{\mu\nu} + K^\sigma_{\mu\nu}, \]

where

\[ K^\sigma_{\mu\nu} = \frac{1}{2} [T^\mu_{\sigma\nu} + T^\nu_{\sigma\mu} - T^\sigma_{\mu\nu}] \]

is the contorsion tensor.

- The above relation means that the Weitzenböck four acceleration of a freely falling particle is not zero:

\[ \frac{d^2 x^\sigma}{d\tau^2} + \Gamma^\sigma_{\mu\nu} \frac{d^2 x^\mu}{d\tau^2} \frac{d^2 x^\nu}{d\tau^2} = K^\sigma_{\mu\nu} \frac{d^2 x^\mu}{d\tau^2} \frac{d^2 x^\nu}{d\tau^2} \]

The contorsion tensor can be regarded as a gravitational force which moves particles away from the Weitzenböck autoparallel lines.
Key features of Weitzenböck spacetime

- Since here we are interested in Euclidean three dimensional teleparallel gravity, we define the action on the manifold with torsion as

\[ S = \int d^3 x \, L_T. \]

- Here \( L_T \) is the teleparallel gravitational lagrangian given by

\[
L_T = \frac{e}{16\pi G} \left[ \frac{1}{4} T^\rho_{\mu\nu} T^{\mu\nu}_\rho + \frac{1}{2} T^\rho_{\mu\nu} T^{\nu\mu}_\rho - T_{\rho\mu}^\rho T^{\nu\mu}_\nu \right],
\]

This Lagrangian is very appealing because it resembles the structure for a gauge field: it is quadratic in the torsion.

Only the discrete version of the action is required in both asymptotic approximations of 6j symbol and the partition.
In general, the Weitzenböck manifold is approximated by a $D$-dimensional polyhedra $M^D$. In this approach, the interior of each simplex is assumed to be flat, and this flat $D$-simplices are joined together at the $D - 1$-hedral faces of their boundaries.

The torsion turns out to be localized in the $D - 2$-dimensional dislocation simplices (hinges) of the lattice, and the link lengths $l$ between any pair of vertices serve as independent variables.

Let us take a bundle of parallel dislocations (hinges) in $M^3$ and let $U$ be a unity vector parallel to the dislocations.

We test for the presence of torsion by carrying a vector $A$ around a small loop of area vector $S = Sn$. 
At the end of the test, if torsion is nonvanishing, $A$ is found to have translated from the original position, along $U$, by the length $B = N b$, where $N$ is the number of dislocations entangled by the loop, and $b$ is the Burgers vector, which gives both the length and direction of the closure failure for every dislocation.

In four dimension $M^4$, the flux of dislocation lines through the loop of area $S^{\alpha\beta}$ is

$$\Phi = \rho (US) = \frac{1}{2} \rho_{\alpha\beta} S^{\alpha\beta},$$

$\rho$ is the density of dislocation passing through the loop.

$$\rho_{\alpha\beta} = \rho U_{\alpha\beta}, \quad U^{\alpha\beta} \text{ a unity antisymmetric tensor: } U_{\alpha\beta} U^{\alpha\beta} = 2.$$
The closure failure is then found to be

\[ B_\mu = \frac{1}{2} \rho_{\alpha\beta} S^{\alpha\beta} b_\mu. \]

From differential geometry we know that, in the presence of torsion, infinitesimal parallelograms in spacetime do not close: the closure failure being equal to

\[ B_\mu = T_{\mu\nu\sigma} S^{\nu\sigma}. \]

By comparing the last two equations we see that

\[ T_{\mu\alpha\beta} = \frac{1}{2} \rho_{\alpha\beta} b_\mu \equiv \frac{1}{2} \rho U_{\alpha\beta} b_\mu. \]
Let us construct a polyhedral cells around each vertex, known in the literature as **Voronoi polygon**.

The boundary of the Voronoi polygon is always perpendicular to the edges emanating from the vertex, and each **corner** of the Voronoi polygon lies at the **circumcentre** of any of the simplices of the **Delaunay geometry**, which shares the dislocation (bone).
If we **parallel transport** a vector $A$ **around** the perimeter of a **Voronoi polygon** of area $\Sigma^*_d$, it will return **dislocated** from its original position in a plane **parallel** to the **bone** by a length $b_\mu$. 
It is well known that the Riemann scalar is proportional to the Gauss curvature, and in similar way, we define the simplicial torsion due to each dislocation by

$$T_{(d)\mu\nu\rho} = \sqrt{D(D - 1)} \frac{b_{(d)\mu} U_{(d)\nu\rho}}{\Sigma^*_d} \equiv \sqrt{6} \frac{b_{(d)\mu} U_{(d)\nu\rho}}{\Sigma^*_d}.$$ 

The D-volume $\Omega_d$ associated with each dislocation is proportional to the product of $\Sigma_d$, the two-dimensional volume of the dislocation, and $\Sigma^*_d$, the area of the Voronoi polygon.

$$\Omega_d \equiv \frac{2}{D(D - 1)} \Sigma_d \Sigma^*_d = \frac{1}{3} \Sigma_d \Sigma^*_d.$$
The invariant volume element $h d^3x$ is represented by $\Omega_d$:

$$\int h d^3x \implies \sum_{\text{dis}} \Omega_d = \frac{1}{3} \sum_{\text{dis}} l_d \Sigma^*_d,$$

Let us take the lagrangian of teleparallel gravity, whose terms are proportional to the square of the torsion tensor. The first term is:

$$T_{(d)}^{\mu\nu\rho} T_{(d)\mu\nu\rho} = 6 \left( \frac{1}{\Sigma^*_d} \right)^2 b_{(d)}^\mu b_{(d)\mu}.$$

The simplicial teleparallel action will be

$$S_T = \frac{1}{16\pi G} \sum_{\text{dis}} \left( \frac{b_d^2}{\Sigma^*_d} \right) l_d,$$
Ponzano and Regge studied a model in which the simplicial blocks of three-dimensional Riemannian manifold are 3-dimensional tetrahedra formed from four angular momenta.

Each edge of a tetrahedron is labelled by a half integer $j$, corresponding to the $(2j + 1)$-dimensional fundamental representation of the group $SU(2)$ such that:

\[ \sqrt{j(j+1)} \hbar \approx (j + \frac{1}{2}) \hbar \text{ for large } j. \]
There is a one-to-one correspondence between the number of edges of tetrahedron and the number of arguments of the $6j$-symbol:

$$l_i = (j_i + \frac{1}{2})\hbar, \ i = 1, 2, ..., 6.$$  

These lengths must satisfy two conditions:

The triangle inequalities corresponding to the triangular faces of the tetrahedron $| j_1 - j_2 | \leq j_3 \leq j_1 + j_2$

For each face $j_1, j_2, j_3$ are required to satisfy $j_1 + j_2 + j_3 = \text{integer}$.

These inequalities for the angular momentum guarantees that the edges $l_1, l_2, l_3$ of tetrahedron form a closed triangle with non-zero surface area.
Ponzano-Regge model

- Ponzano and Regge obtained the asymptotic limit of 6\(j\)-symbol in the classically allowed region:

\[
\left\{ j_1 \ j_2 \ j_3 \ \ j_4 \ \ j_5 \ \ j_6 \right\} \sim \sqrt{\frac{\hbar^3}{12\pi V(j)}} \cos \left[ \frac{1}{\hbar} \sum_{i=1}^{6} (j_i + \frac{1}{2})\varepsilon_i + \frac{\pi}{4} \right],
\]

- On manifold with torsion:
The torsion tensor is localized in one-dimensional dislocation line \(\ell_i\) called hinges.
When torsion is present, it is detected a dislocation parallel to this hinge, and this dislocation is measured by the Burgers vector \(b_d\).
From these set of \( l_i \), let us choose six of them in such a way that they must satisfy triangle inequalities:

Let \( l_1, l_2, l_3, l_4, l_5, l_6 \) be non-negative integers. An unordered triades of this family of dislocation lines \((l_i, l_j, l_k)\) with \( i \neq j \neq k \), is said to be admissible if they met the triangular inequalities \(| l_j - l_k | < l_i < l_j + l_k\).

These admissible \( l_i \) are the edge lengths of the terahedron and also they completely characterizes the tetrahedron in Euclidean 3-space:

\[
V(l)^2 = \frac{1}{288} \begin{vmatrix}
0 & l_4^2 & l_5^2 & l_6^2 & 1 \\
l_4^2 & 0 & l_3^2 & l_2^2 & 1 \\
l_5^2 & l_3^2 & 0 & l_1^2 & 1 \\
l_6^2 & l_2^2 & l_1^2 & 0 & 1
\end{vmatrix}.
\]
Then, the simplicial teleparallel action for the tetrahedron reduces to

\[ S_T = \frac{1}{16\pi G} \sum_{i=1}^{6} l_i \left( \frac{b_i^2}{\Sigma_i^*} \right), \]

where \( l_i \) and \( b_i \) are the edge length and the closure failure or gap at the edge.

\( \Sigma_i^* \) is the area of a Voronoi polygon orthogonal to the edge.

The Regge action may be re-expressed as the sum of the gravitational contribution from each edge of the tetrahedron:

\[ S_T = \frac{1}{16\pi G} \sum_{i=1}^{6} \left( j_i + \frac{1}{2} \right) \left( \frac{b_i^2}{\Sigma_i^*} \right). \]
For complex of tetrahedra with $N$ internal edges the action is:

$$S_T = \frac{1}{16\pi G} \sum_{i=1}^{N} (j_i + \frac{1}{2}) \left( \frac{b_i^2}{\Sigma^*} \right).$$

The discrete action is then a function of the angular momentum, the Burgers vector of dislocation and the area of Voronoi polygon.

The Euclidean Einstein-Hilbert action is then a function of the angular momentum on the edges and is given by summing the simplicial action over all the tetrahedra in $M$. 
Having identified the edges $l_i$ of the tetrahedron with the angular momenta, the asymptotic form of $6j$ symbol for large values of $j_i$ is given by:

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} \simeq \sqrt{\frac{\hbar^3}{12\pi V(j)}} \cos \left[ \frac{1}{8\pi G\hbar} \sum_{i=1}^{6} (j_i + \frac{1}{2}) \left( \frac{b_i^2}{\Sigma_i^*} \right) + \frac{\pi}{4} \right].$$

This is the Ponzano and Regge asymptotic formula for the Wigner $6j$ symbol on simplicial manifold with torsion.

- $V(j)$ is the three dimensional volume of the tetrahedron.
- $b_i$ is the Burgers vector which gives both the length and direction of the gap for every dislocation in the tetrahedron corresponding to the edge $j_i + \frac{1}{2}$. 
Following Ponzano and Regge, we also defined a partition function by summing over all possible edge lengths similar to Regge calculus and by taking the product of the $6j$ symbols over all fixed number of tetrahedra and connectivity of the simplicial manifold.

Let us remark that the sum of contributions to $S_T$ from all tetrahedra in a tessellation approaches to the action of teleparallel gravity $S_T$, provided the number of edges and vertices in the simplicial manifold becomes very large:

$$\lim_{N \to \infty} \sum_{j=i}^{N} (j_i + \frac{1}{2}) \left( \frac{b_i^2}{\Sigma^*_i} \right) \simeq 16\pi G S = \int d^3 x L_T.$$
We considered the connection between angular momentum in quantum mechanics and geometric objects, namely the relation between angular momentum and tetrahedra on manifold with torsion without the cosmological term.

First, we noticed the relation between the 6j symbol and Regge’s discrete version of the action functional of Euclidean three dimensional gravity with torsion.

Then we considered the Ponzano and Regge asymptotic formula for the Wigner 6j symbol on this simplicial manifold with torsion.