Ponzano-Regge Model on Manifold with Torsion

Teofilo Vargas

Facultad de Ciencias Fisicas
Universidad Nacional Mayor de San Marcos, Lima-Peru

Workshop on Quantum Gravity in the Southern Cone VI
Sao Paulo, Brazil
11-14 September 2013

based on arXiv:1307.2624 (T Vargas)

Key features of Weitzenböck spacetime

There is an alternative approach to gravitation based on the Weitzenböck geometry:

This theory is known as teleparallel equivalent of general relativity (TEGR), where the gravitation is attributed to torsion.

GR uses the Levi-Civita connection $\Longrightarrow$ curvature but not torsion
TG uses the Weitzenböck connection $\Longrightarrow$ torsion but not curvature

Teleparallel gravity is a sector of Einstein-Cartan theories which describe gravity by means of a connection having both torsion and curvature.

Key features of Weitzenböck spacetime

In teleparallel gravity the dynamical object is the vierbein field:

$$e^a{}_\mu(x)$$

- This nontrivial tetrad field is used to define the linear Weitzenböck connection

$$\Gamma^\sigma{}_{\mu\nu}(x) = e_a{}^\sigma(x)\partial_\nu e^a{}_\mu(x),$$

  a connection presenting torsion, but no curvature.
- The Levi–Civita connection of the metric

$$g_{\mu\nu}(x) = \eta_{ab}\, e^a{}_\mu(x)\, e^b{}_\nu(x),$$

is given by

$$\overset{\circ}{\Gamma}{}^\sigma{}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left[\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right]$$

Key features of Weitzenböck spacetime

- The relation between these two connection is given by

$$\Gamma^{\sigma}{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\sigma}{}_{\mu\nu} + K^{\sigma}{}_{\mu\nu},$$

  where

$$K^{\sigma}{}_{\mu\nu} = \frac{1}{2}\left[T_{\mu}{}^{\sigma}{}_{\nu} + T_{\nu}{}^{\sigma}{}_{\mu} - T^{\sigma}{}_{\mu\nu}\right]$$

  is the contorsion tensor.
- The above relation means that the Weitzenböck four acceleration of a freely falling particle is not zero:

$$\frac{d^2x^{\sigma}}{d\tau^2} + \Gamma^{\sigma}{}_{\mu\nu}\frac{d^2x^{\mu}}{d\tau^2}\frac{d^2x^{\nu}}{d\tau^2} = K^{\sigma}{}_{\mu\nu}\frac{d^2x^{\mu}}{d\tau^2}\frac{d^2x^{\nu}}{d\tau^2}$$

The contorsion tensor can be regarded as a gravitational force which moves particles away from the Weitzenböck autoparallel lines.

Key features of Weitzenböck spacetime

- Since here we are interested in Euclidean three dimensional teleparallel gravity, we define the action on the manifold with torsion as

$$S = \int d^3x \, L_T \, .$$

- Here L_T is the teleparallel gravitational lagrangian given by

$$L_T = \frac{e}{16\pi G}\left[\frac{1}{4} T^{\rho}{}_{\mu\nu} T_{\rho}{}^{\mu\nu} + \frac{1}{2} T^{\rho}{}_{\mu\nu} T^{\nu\mu}{}_{\rho} - T_{\rho\mu}{}^{\rho} T^{\nu\mu}{}_{\nu}\right],$$

This Lagrangian is very appealing because it resembles the structure for a gauge field: it is quadratic in the torsion.

Only the discrete version of the action is required in both asymptotic approximations of $6j$ symbol and the partition

Discrete torsion and simplicial action

- In general, the Weitzenböck manifold is approximated by a D-dimensional polyhedra M^D. In this approach, the interior of each simplex is assumed to be flat, and this flat D-simplices are joined together at the $D-1$-hedral faces of their boundaries.
- The torsion turns out to be localized in the $D-2$-dimensional dislocation simplices (hinges) of the lattice, and the link lengths l between any pair of vertices serve as independent variables.
- Let us take a bundle of parallel dislocations (hinges) in M^3 and let $\mathbf{U}$ be a unity vector parallel to the dislocations.
- We test for the presence of torsion by carrying a vector $\mathbf{A}$ around a small loop of area vector $\mathbf{S} = S\mathbf{n}$.

Discrete torsion and simplicial action

- At the end of the test, if torsion is nonvanishing, **A** is found to have translated from the original position, along **U**, by the length $\mathbf{B} = N\mathbf{b}$, where N is the number of dislocations entangled by the loop, and **b** is the **Burgers vector**, which gives both **the length and direction of the closure failure for every dislocation**.

- In four dimension M^4, the **flux of dislocation lines through the loop of area** $S^{\alpha\beta}$ is

$$\Phi = \rho\,(\mathbf{US}) = \frac{1}{2}\,\rho_{\alpha\beta}\,S^{\alpha\beta},$$

ρ is the density of dislocation passing through the loop.
$\rho_{\alpha\beta} = \rho\,U_{\alpha\beta}$, $U^{\alpha\beta}$ a unity antisymmetric tensor: $U_{\alpha\beta}U^{\alpha\beta} = 2$.

Discrete torsion and simplicial action

- The closure failure is then found to be

$$B_\mu = \frac{1}{2}\,\rho_{\alpha\beta}\,S^{\alpha\beta}\,b_\mu .$$

- From differential geometry we know that, in the presence of torsion, infinitesimal parallelograms in spacetime do not close: **the closure failure** being equal to

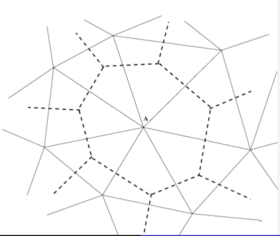
$$B_\mu = T_{\mu\nu\sigma}\,S^{\nu\sigma} .$$

- By comparing the last two equations we see that

$$T_{\mu\alpha\beta} = \frac{1}{2}\,\rho_{\alpha\beta}\,b_\mu \equiv \frac{1}{2}\,\rho\,U_{\alpha\beta}\,b_\mu .$$

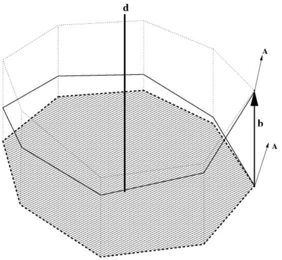
Discrete torsion and simplicial action

- Let us construct a polyhedral cells around each vertex, known in the literature as **Voronoi polygon**.
- The boundary of the Voronoi polygon is always perpendicular to the edges emanating from the vertex, and each **corner** of the Voronoi polygon lies at the **circuncentre** of any of the simplices of the **Delaunay geometry**, which shares the dislocation (bone).



Discrete torsion and simplicial action

- If we **parallel transport** a vector A **around** the perimeter of a **Voronoi polygon** of area Σ_d^*, it will return **dislocated** from its original position in a plane **parallel** to the **bone** by a length b_μ.



Discrete torsion and simplicial action

- It is well known that the Riemann scalar is proportional to the Gauss curvature, and in similar way, we define the simplicial torsion due to each dislocation by

$$T_{(d)\mu\nu\rho} = \sqrt{D(D-1)}\ \frac{b_{(d)\mu} U_{(d)\nu\rho}}{\Sigma_d^*} \equiv \sqrt{6}\ \frac{b_{(d)\mu} U_{(d)\nu\rho}}{\Sigma_d^*}.$$

- The D-volume Ω_d associated with each dislocation is proportional to the product of Σ_d, the two-dimensional volume of the dislocation, and Σ_d^*, the area of the Voronoi polygon.

$$\Omega_d \equiv \frac{2}{D(D-1)}\, \Sigma_d\, \Sigma_d^* = \frac{1}{3}\, \Sigma_d\, \Sigma_d^*.$$

Discrete torsion and simplicial action

- The invariant volume element $h\, d^3x$ is represented by Ω_d:

$$\int h\, d^3x \Longrightarrow \sum_{\text{dis}} \Omega_d = \frac{1}{3} \sum_{\text{dis}} l_d\, \Sigma_d^*,$$

- Let us take the lagrangian of teleparallel gravity, whose terms are proportional to the square of the torsion tensor. The first term is:

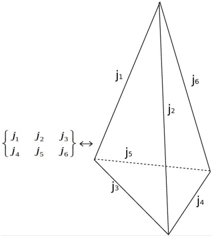
$$T_{(d)}^{\mu\nu\rho}\, T_{(d)\mu\nu\rho} = 6 \left(\frac{1}{\Sigma_d^*} \right)^2 b_{(d)}^{\mu}\, b_{(d)\mu}.$$

- The simplicial teleparallel action will be

$$S_T = \frac{1}{16\pi G} \sum_{\text{dis}} \left(\frac{b_d^2}{\Sigma_d^*} \right) l_d,$$

- Ponzano and Regge studied a model in which the simplicial bloks of three-dimensional Riemanian manifold are 3-dimensional tetrahedra formed from four angular momenta.

Each edge of a tetrahedron is labelled by a half integer j, corresponding to the $(2j+1)$-dimensional fundamental representation of the group $SU(2)$ such that:
$\sqrt{j(j+1)}\hbar \approx (j+\frac{1}{2})\hbar$ for large j.



Ponzano-Regge model

- There is a one-to-one correspondence betwen the number of edges of tetrahedron and the number of arguments of the 6j-symbol:

$$l_i = (j_i + \frac{1}{2})\hbar \,,\; i = 1, 2, ..., 6.$$

- These lengths must satisfy two conditions:
  The triangle inequalities corresponding to the triangular faces of the tetrahedron $|\, j_1 - j_2| \leq j_3 \leq j_1 + j_2$
  For each face j_1, j_2, j_3 are required to satisfy $j_1 + j_2 + j_3 =$integer.
- These inequalities for the angular momentum guarantees that the edges l_1, l_2, l_3 of tetrahedron form a closed triangle af non-zero surface area.

Ponzano-Regge model

- Ponzano and Regge obtained the asymptotic limit of $6j$-symbol in the classically allowed region:

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} \simeq \sqrt{\frac{\hbar^3}{12\pi V(j)}} \cos\left[\frac{1}{\hbar} \sum_{i=1}^{6} (j_i + \frac{1}{2}) \varepsilon_i + \frac{\pi}{4} \right] ,$$

- On manifold with torsion:
  The torsion tensor is localized in one-dimensional dislocation line l_i called hinges.
  When torsion is present, it is detected a dislocation parallel to this hinge, and this dislocation is measured by the Burgers vector b_d.

Ponzano-Regge model

- From these set of l_i, let us choose six of them in such a way that they must satisfy triangle inequalities:
  Let l_1, l_2, l_3, l_4, l_5, l_6 be non-negative integers.
  An unordered triades of this family of dislocation lines (l_i, l_j, l_k) with $i \neq j \neq k$, is said to be admissible if they met the triangular inequalities $|\, l_j - l_k| < l_i < l_j + l_k$.
- These admissible l_i are the edge lengths of the terahedron and also they completely characterizes the tetrahedron in Euclidean 3-space:

$$V(l)^2 = \frac{1}{288}\begin{vmatrix} 0 & l_4^2 & l_5^2 & l_6^2 & 1 \\ l_4^2 & 0 & l_3^2 & l_2^2 & 1 \\ l_5^2 & l_3^2 & 0 & l_1^2 & 1 \\ l_6^2 & l_2^2 & l_1^2 & 0 & 1 \end{vmatrix}.$$

Ponzano-Regge model

- Then, the simplicial teleparallel action for the tetrahedron reduces to

$$S_T = \frac{1}{16\pi G} \sum_{i=1}^{6} l_i \left(\frac{b_i^2}{\Sigma_i^*} \right) ,$$

  l_i, b_i are the edge length and the closure failure or gap at the edge.
- Σ_i^* is the area of a Voronoi polygon orthogonal to the edge.
- The Regge action may be re-expressed as the sum of the gravitational contribution from each edge of the tetrahedron:

$$S_T = \frac{1}{16\pi G} \sum_{i=1}^{6} (j_i + \frac{1}{2}) \left(\frac{b_i^2}{\Sigma_i^*} \right) .$$

Ponzano-Regge model

- For complex of tetrahedra with N internal edges the action is:

$$S_T = \frac{1}{16\pi G} \sum_{i=1}^{N} (j_i + \frac{1}{2}) \left(\frac{b_i^2}{\Sigma_i^*} \right) .$$

- The **discrete action** is then a function of the **angular momentum**, the **Burgers vector of dislocation** and the **area of Voronoi polygon**.
- The **Euclidean Einstein-Hilbert action** is then a function of the **angular momentum** on the edges and is given by **summing the simplicial action over all the tetrahedra in** M.

Ponzano-Regge model

- Having identified the edges l_i of the tetrahedron with the angular momenta, the asymptotic form of 6j symbol for large values of j_i is given by:

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} \simeq \sqrt{\frac{\hbar^3}{12\pi V(j)}} \cos\left[\frac{1}{8\pi G\hbar} \sum_{i=1}^{6} (j_i + \frac{1}{2}) \left(\frac{b_i^2}{\Sigma_i^*} \right) + \frac{\pi}{4} \right].$$

- This is the Ponzano and Regge asymptotic formula for the Wigner 6j symbol on simplicial manifold with torsion.
- $V(j)$ is the three dimensional volume of the tetrahedron.
- b_i is the Burgers vector which gives both the length and direction of the gap for every dislocation in the tetrahedron corresponding to the edge $j_i + \frac{1}{2}$.

Ponzano-Regge model

- Following Ponzano and Regge, we also defined a partition function by summing over all possible edge lengths simillar to Regge calculus and by taking the product of the $6j$ symbols over all fixed number of tetrahedra and connectivity of the simplicial manifold.
- Let us remark that the sum of contributions to S_T from all tetrahedra in a tesselation approaches to the action of teleparallel gravity S , provided the number of edges and vertices in the simplical manifold becomes very large:

$$\lim_{N\to\infty} \sum_{j=i}^{N} (j_i + \frac{1}{2}) \left(\frac{b_i^2}{\Sigma_i^*} \right) \simeq 16\pi G S = \int d^3x \, L_T \, .$$

Conclusion

- We considered the connection between angular momentum in quantum mechanics and geometric objects, namely the relation between angular momentum and tetrahedra on manifold with torsion without the cosmological term.
- First, we noticed the relation between the $6j$ symbol and Regge's discrete version of the action functional of Euclidean three dimensional gravity with torsion.
- Then we considered the Ponzano and Regge asymptotic formula for the Wigner $6j$ symbol on this simplicial manifold with torsion.