Contents

1 Pure Spinors

The Pure Spinors action is

$$\begin{aligned}S_{PS} =&\frac{R^2}{2\pi}\int d^2z \mathrm{STr}\left[\frac{1}{2}J_2\bar{J}_2+\frac{1}{4}J_1\bar{J}_3+\frac{3}{4}\bar{J}_1J_3-\omega\bar{\nabla}\lambda+\hat{\omega}\nabla\hat{\lambda}-N\hat{N}\right]\\ =&S_{int}+S_{kin}+S'_{int}+S'_{kin},\end{aligned}$$

with

$$\begin{aligned}\nabla A =&\partial A+[J_0,A],\\ N =&\{\omega,\lambda\}.\end{aligned}$$

where we have divided to conquer: the S' are the ones with Ghost fields ω, λ, $\hat{\omega}$ and $\hat{\lambda}$, while the un-primed are the ones involving only the Js.

The fields are defined as

$$\begin{aligned}J_0 =&J_0^iT_i \quad ; \quad J_1 = J_1^\alpha T_\alpha \quad ; \quad J_2 = J_2^m T_m \quad ; \quad J_3 = J_3^{\hat{\alpha}}T_{\hat{\alpha}};\\ \lambda =&\lambda^A T_A \quad ; \quad \omega = \omega_A\eta^{A\hat{A}}T_{\hat{A}} \quad ; \quad \hat{\lambda} = \hat{\lambda}^{\hat{A}}T_{\hat{A}} \quad ; \quad \hat{\omega} = \hat{\omega}_{\hat{B}}\eta^{B\hat{B}}T_B.\end{aligned}$$

Note that A means α, but, due the few letters in the alphabet (the greek and latin togethers), we use a change of the definition.

The quantum expansion of the fields are

$$\begin{aligned}J_0 =&J_0+[J_1,x_3]+[J_2,x_2]+[J_3,x_1]+\frac{1}{2}\left([\partial x_1,x_3]+[\partial x_2,x_2]+[\partial x_3,x_1]\right.\\ &\left.+[[J_1,x_1],x_2]+[[J_1,x_2],x_1]+[[J_2,x_1],x_3]+[[J_2,x_3],x_1]+[[J_3,x_2],x_3]+[[J_3,x_3],x_2]\right),\\ J_1 =&J_1+\partial x_1+[J_2,x_3]+[J_3,x_2]+\frac{1}{2}\left([\partial x_2,x_3]+[\partial x_3,x_2]\right.\\ &\left.+[[J_1,x_1],x_3]+[[J_1,x_3],x_1]+[[J_1,x_2],x_2]+[[J_2,x_1],x_2]+[[J_2,x_2],x_1]+[[J_3,x_1],x_1]+[[J_3,x_3],x_3]\right),\\ J_2 =&J_2+\partial x_2+[J_1,x_1]+[J_3,x_3]+\frac{1}{2}\left([\partial x_1,x_1]+[\partial x_3,x_3]\right.\\ &\left.+[[J_1,x_2],x_3]+[[J_1,x_3],x_2]+[[J_2,x_2],x_2]+[[J_2,x_1],x_3]+[[J_2,x_3],x_1]+[[J_3,x_1],x_2]+[[J_3,x_2],x_1]\right),\\ J_3 =&J_3+\partial x_3+[J_2,x_1]+[J_1,x_2]+\frac{1}{2}\left([\partial x_1,x_2]+[\partial x_2,x_1]\right.\\ &\left.+[[J_1,x_1],x_1]+[[J_1,x_3],x_3]+[[J_2,x_2],x_3]+[[J_2,x_3],x_2]+[[J_3,x_3],x_1]+[[J_3,x_1],x_3]+[[J_3,x_2],x_2]\right),\\ \lambda =&\lambda+\delta\lambda,\\ \omega =&\omega+\delta\lambda.\end{aligned}$$

For the raising and lowering of index of the structure constants we used

$$f_{m\alpha\beta}=\eta_{\alpha\hat{\alpha}}f^{\hat{\alpha}}_{\beta m} \qquad \text{and} \qquad f_{m\hat{\alpha}\hat{\beta}}=-\eta_{\alpha\hat{\alpha}}f^{\alpha}_{\hat{\beta}m}.$$

With all this, we compute the system without Ghosts and with them.

2 Pure Spinors without Ghost fields

At second order in perturbation series the action is $S = S_{kin} + S_{int}$. Without the ghosts the interaction term is

$$\begin{aligned}S'_{int} =&\frac{R^2}{2\pi}\int d^2z\mathrm{STr}\left\{J_1\left(-\frac{3}{8}\left[\bar\nabla x_2,x_1\right]-\frac{5}{8}\left[\bar\nabla x_1,x_2\right]\right)+\bar J_1\left(\frac{1}{8}\left[\nabla x_1,x_2\right]-\frac{1}{8}\left[\nabla x_2,x_1\right]\right)+J_2\left(-\frac{1}{2}\left[\bar\nabla x_1,x_1\right]\right)\right.\\&+\bar J_2\left(-\frac{1}{2}\left[\nabla x_3,x_3\right]\right)+J_3\left(-\frac{1}{8}\left[\bar\nabla x_2,x_3\right]+\frac{1}{8}\left[\bar\nabla x_3,x_2\right]\right)+\bar J_3\left(-\frac{3}{8}\left[\nabla x_2,x_3\right]-\frac{5}{8}\left[\nabla x_3,x_2\right]\right)\\&+\frac{1}{2}\nabla x_2\bar\nabla x_2+\frac{1}{4}\nabla x_1\bar\nabla x_3+\frac{3}{4}\nabla x_3\bar\nabla x_1+[J_1,x_1]\left(\frac{1}{4}[\bar J_3,x_3]\right)+[J_1,x_2]\left(\frac{3}{8}[\bar J_2,x_3]+\frac{1}{2}[\bar J_3,x_2]\right)+\\&[J_1,x_3]\left(-\frac{3}{8}[\bar J_2,x_2]-\frac{1}{4}[\bar J_3,x_1]-\frac{1}{2}[\bar J_1,x_3]\right)+[\bar J_1,x_1]\left(-\frac{1}{4}[J_3,x_3]\right)+[\bar J_1,x_2]\left(-\frac{3}{8}[J_2,x_3]-\frac{1}{2}[J_3,x_2]\right)\\&+[\bar J_1,x_3]\left(-\frac{5}{8}[J_2,x_2]-\frac{3}{4}[J_3,x_1]\right)+[J_2,x_1]\left(\frac{1}{4}[\bar J_2,x_3]+\frac{3}{8}[\bar J_3,x_2]\right)+[J_2,x_2]\left(-\frac{1}{2}[\bar J_2,x_2]-\frac{3}{8}[\bar J_3,x_1]\right)\\&\left.+[J_2,x_3]\left(-\frac{1}{4}[\bar J_2,x_1]\right)+[\bar J_2,x_1]\left(-\frac{3}{8}[J_3,x_2]\right)+[\bar J_2,x_2]\left(-\frac{5}{8}[J_3,x_1]\right)-\frac{1}{2}[J_3,x_1][\bar J_3,x_1]\right\},\end{aligned}$$

and

$$S'_{kin} = \frac{R^2}{2\pi}\int d^2z\left[\frac{1}{2}\nabla x_2^m\bar\nabla x_2^n\eta_{mn}-\nabla x_1^\alpha\bar\nabla x_3^{\hat\alpha}\eta_{\alpha\hat\alpha}\right].$$

From now on, we drop the quantum J_0, because we don't will use it.
Computing the STr we obtain:

$$\begin{aligned}S'_{int} =&\frac{R^2}{2\pi}\int d^2z\left[\frac{1}{2}\bar\partial x_1^\alpha x_1^\beta J_2^m f_{m\alpha\beta}+\frac{1}{2}x_1^\alpha x_1^\beta J_3^{\hat\alpha}\bar J_3^{\hat\beta}f_{i\alpha\hat\alpha}f_{j\beta\hat\beta}g^{ij}+\frac{1}{8}\left(3x_1^\alpha\bar\partial x_2^m-5\bar\partial x_1^\alpha x_2^m\right)J_1^\beta f_{m\alpha\beta}\right.\\&+\frac{1}{8}x_1^\alpha x_2^m\left(-\partial\bar J_1^\beta f_{m\alpha\beta}+\left[3J_2^n\bar J_3^{\hat\alpha}+5\bar J_2^nJ_3^{\hat\alpha}\right]f_{i\hat\alpha\alpha}f_{jmn}g^{ij}+3\left[J_2^n\bar J_3^{\hat\alpha}-\bar J_2^nJ_3^{\hat\alpha}\right]f_{n\alpha\beta}f_{m\hat\beta\hat\alpha}\eta^{\beta\hat\beta}\right)\\&-\frac{1}{4}x_1^\alpha x_3^{\hat\alpha}\left(\left[\bar J_1^\beta J_3^{\hat\beta}-J_1^\beta\bar J_3^{\hat\beta}\right]f_{m\alpha\beta}f_{n\hat\alpha\hat\beta}\eta^{mn}+\left[J_1^\beta\bar J_3^{\hat\beta}+3\bar J_1^\beta J_3^{\hat\beta}\right]f_{i\hat\alpha\beta}f_{j\alpha\hat\beta}g^{ij}+J_2^m\bar J_2^n\left[f_{m\alpha\beta}f_{n\hat\alpha\hat\beta}-f_{n\alpha\beta}f_{m\hat\alpha\hat\beta}\right]\eta^{\beta\hat\beta}\right)\\&-\frac{1}{2}x_2^mx_2^n\left(\left[J_1^\alpha\bar J_3^{\hat\alpha}-\bar J_1^\alpha J_3^{\hat\alpha}\right]f_{m\alpha\beta}f_{n\hat\alpha\hat\beta}\eta^{\beta\hat\beta}+J_2^p\bar J_2^qf_{ipm}f_{jqn}g^{ij}\right)+\frac{1}{8}\left(3\partial x_2^mx_3^{\hat\alpha}-5x_2^m\partial x_3^{\hat\alpha}\right)\bar J_3^{\hat\beta}f_{m\hat\alpha\hat\beta}\\&+\frac{1}{8}x_2^mx_3^{\hat\alpha}\left(-\bar\partial J_3^{\hat\beta}f_{m\hat\alpha\hat\beta}+3\left[\bar J_1^\alpha J_2^n-J_1^\alpha\bar J_2^n\right]f_{m\alpha\beta}f_{n\hat\alpha\hat\beta}\eta^{\beta\hat\beta}+\left[3J_1^\alpha\bar J_2^n+5\bar J_1^\alpha J_2^n\right]f_{i\alpha\hat\alpha}f_{jmn}g^{ij}\right)\\&\left.+\frac{1}{2}\partial x_3^{\hat\alpha}x_3^{\hat\beta}\bar J_2^mf_{m\hat\alpha\hat\beta}+\frac{1}{2}x_3^{\hat\alpha}x_3^{\hat\beta}J_1^\alpha\bar J_1^\beta f_{i\alpha\hat\alpha}f_{j\beta\hat\beta}g^{ij}\right],\end{aligned}$$

with

$$\begin{aligned}g_{ij} =&\mathrm{STr}T_iT_j\\ \eta_{mn} =&\mathrm{STr}T_mT_n\\ \eta_{\alpha\hat\alpha} =&\mathrm{STr}T_\alpha T_{\hat\alpha}\end{aligned}$$

3 Pure Spinors with Ghosts

We now want to scare the reader, so we add ghosts fields to the calculations. Before we compute anything, is good to note that, because ω and λ are fermionic fileds, the covariant derivate simplifies to

$$\mathrm{STr}\left[\omega\nabla\lambda\right] =\mathrm{STr}\left[\omega\partial\lambda+\omega J_0\lambda-\omega\lambda J_0\right]=\mathrm{STr}\left[\omega\partial\lambda-NJ_0\right].$$

The ghost side of the action now reads

$$
\begin{aligned}
S' =& \frac{R^2}{2\pi}\int d^2z\left[-\omega\bar{\nabla}\lambda + \hat{\omega}\nabla\hat{\lambda} - N\hat{N}\right] \\
S'_{kin} =& \frac{R^2}{2\pi}\int d^2z\left[\omega_A\bar{\partial}\lambda^A + \hat{\omega}_{\hat{A}}\partial\hat{\lambda}^{\hat{A}}\right] \\
S'_{int} =& \frac{R^2}{2\pi}\int d^2z\Big[-\delta^2(N^i\hat{N}^j)g_{ij} + x_1^\alpha\left(\delta N^i\bar{J}_3^{\hat{\alpha}} - \delta\hat{N}^iJ_3^{\hat{\alpha}}\right)f_{i\alpha\hat{\alpha}} - x_2^m\left(\delta N^i\bar{J}_2^n - \delta\hat{N}^iJ_2^n\right)f_{imn} \\
&+ x_3^{\hat{\alpha}}\left(\delta N^i\bar{J}_1^\alpha - \delta\hat{N}^iJ_1^\alpha\right)f_{i\alpha\hat{\alpha}} + \frac{1}{2}x_1^\alpha x_1^\beta\left(N^i\bar{J}_2^m - \hat{N}^iJ_2^m\right)f_{m\alpha\mu}f_{i\beta\hat{\mu}}\eta^{\mu\hat{\mu}} \\
&+ \frac{1}{2}x_1^\alpha x_2^m\left(N^i\bar{J}_1^\beta - \hat{N}^iJ_1^\beta\right)\left(f_{ipm}f_{q\alpha\beta}\eta^{pq} + f_{i\alpha\hat{\mu}}f_{m\beta\mu}\eta^{\mu\hat{\mu}}\right) - \frac{1}{2}\left(\bar{\partial}x_1^\alpha x_3^{\hat{\alpha}} - x_1^\alpha\bar{\partial}x_3^{\hat{\alpha}}\right)N^if_{i\alpha\hat{\alpha}} \\
&+ \frac{1}{2}\left(\partial x_1^\alpha x_3^{\hat{\alpha}} - x_1^\alpha\partial x_3^{\hat{\alpha}}\right)\hat{N}^if_{i\alpha\hat{\alpha}} - \frac{1}{2}x_2^m\left(\bar{\partial}x_2^nN^i - \partial x_2^n\hat{N}^i\right)f_{imn} \\
&+ \frac{1}{2}x_2^mx_3^{\hat{\alpha}}\left(N^i\bar{J}_3^{\hat{\beta}} - \hat{N}^iJ_3^{\hat{\beta}}\right)\left(f_{ipm}f_{q\hat{\alpha}\hat{\beta}}\eta^{pq} - f_{i\hat{\alpha}\mu}f_{m\hat{\beta}\hat{\mu}}\eta^{\mu\hat{\mu}}\right) - \frac{1}{2}x_3^{\hat{\alpha}}x_3^{\hat{\beta}}\left(N^i\bar{J}_2^m - \hat{N}^iJ_2^m\right)f_{m\hat{\alpha}\hat{\mu}}f_{i\mu\hat{\beta}}\eta^{\mu\hat{\mu}}\Big],
\end{aligned}
$$

with

$$
\begin{aligned}
N^i =& \omega_A\lambda^B\eta^{A\hat{B}}f_{jB\hat{B}}g^{ij}, \\
\delta N^i =& (\delta\omega_A\lambda^B + \omega\delta\lambda)\eta^{A\hat{B}}f_{jB\hat{B}}g^{ij}, \\
\delta^2(N^i\hat{N}^j) =& \delta N^i\delta\hat{N}^j + \delta\omega_A\delta\lambda^B\eta^{A\hat{B}}f^i_{B\hat{B}}\hat{N}^j + N^i\delta\hat{\omega}_{\hat{A}}\delta\hat{\lambda}^{\hat{B}}\eta^{B\hat{A}}f^j_{B\hat{B}}.
\end{aligned}
$$

4 Schwinger-Dyson Equation

The Schwinger-Dyson equation in momentum space for this theory reads

$$
\begin{aligned}
G^{\alpha\Lambda} =& \frac{\eta^{\alpha\Lambda}}{|k|^2} + \frac{1}{|k|^2}(ik\bar{\partial} + i\bar{k}\partial + \Box)G^{\alpha\Lambda} - \frac{\eta^{\alpha\Omega}}{|k|^2}F_{\Sigma\Omega}G^{\Sigma\Lambda}, \\
G^{m\Lambda} =& \frac{\eta^{m\Lambda}}{|k|^2} + \frac{1}{|k|^2}(ik\bar{\partial} + i\bar{k}\partial + \Box)G^{\alpha\Lambda} - \frac{\eta^{m\Omega}}{|k|^2}F_{\Sigma\Omega}G^{\Sigma\Lambda}, \\
G^{\hat{\alpha}\Lambda} =& -\frac{\eta^{\hat{\alpha}\Lambda}}{|k|^2} + \frac{1}{|k|^2}(ik\bar{\partial} + i\bar{k}\partial + \Box)G^{\alpha\Lambda} + \frac{\eta^{\Omega\hat{\alpha}}}{|k|^2}F_{\Sigma\Omega}G^{\Sigma\Lambda}, \\
G_A^{\ \Lambda} =& \frac{i}{k}\delta_A^\Lambda + \frac{i}{k}\bar{\partial}G_A^{\ \Lambda} - \frac{i}{k}F_{\Sigma A}G^{\Sigma\Lambda}, \\
G^{B\Lambda} =& -\frac{i}{k}\delta_{B\Lambda} + \frac{i}{k}\bar{\partial}G^{B\Lambda} + \frac{i}{k}F_\Sigma^{\ B}G^{\Sigma\Lambda}, \\
G_{\hat{A}}^{\ \Lambda} =& \frac{i}{k}\delta_{\hat{A}}^\Lambda + \frac{i}{k}\partial G_{\hat{A}}^{\ \Lambda} - \frac{i}{k}F_{\Sigma\hat{A}}G^{\Sigma\Lambda}, \\
G^{\hat{B}\Lambda} =& -\frac{i}{k}\delta_{\hat{B}\Lambda} + \frac{i}{k}\partial G^{\hat{B}\Lambda} + \frac{i}{k}F_\Sigma^{\ \hat{B}}G^{\Sigma\Lambda},
\end{aligned}
$$

where all δ and η are reescaled by a $2\pi/R^2$

The Green's Function in coordinate space is related to its Fourier Transform by

$$
G(x,y) = \frac{1}{(2\pi)^2}\int d^2k\exp(ik(x-y))G(x,k).
$$

5 Equations of Motion

The first order terms in the perturbation gives the equation of motion:

$$\begin{aligned}
\partial\bar{J}_1^{\alpha} =& \left(-N^i\bar{J}_1^{\beta}+\hat{N}^iJ_1^{\beta}\right)f_{i\hat{\alpha}\beta}\eta^{\alpha\hat{\alpha}},\\
\bar{\partial}J_1^{\alpha} =& \partial\bar{J}_1^{\alpha}+\left(J_2^m\bar{J}_3^{\hat{\beta}}-\bar{J}_2^mJ_3^{\hat{\beta}}\right)f_{m\hat{\alpha}\hat{\beta}}\eta^{\alpha\hat{\alpha}},\\
\partial\bar{J}_2^m =& J_1^{\alpha}\bar{J}_1^{\beta}f_{n\alpha\beta}\eta^{mn}+\left(N^i\bar{J}_2^p-\hat{N}^iJ_2^p\right)f_{inp}\eta^{mn},\\
\bar{\partial}J_2^m =& -J_3^{\hat{\alpha}}\bar{J}_3^{\hat{\beta}}f_{n\hat{\alpha}\hat{\beta}}\eta^{mn}+\left(N^i\bar{J}_2^p-\hat{N}^iJ_2^p\right)f_{inp}\eta^{mn},\\
\partial\bar{J}_3^{\hat{\alpha}} =& \left(J_2^m\bar{J}_1^{\beta}-\bar{J}_2^mJ_1^{\beta}\right)f_{m\alpha\beta}\eta^{\alpha\hat{\alpha}}+\bar{\partial}J_3^{\hat{\alpha}},\\
\bar{\partial}J_3^{\hat{\alpha}} =& +\left(N^i\bar{J}_3^{\hat{\beta}}-\hat{N}^iJ_3^{\hat{\beta}}\right)f_{i\alpha\hat{\beta}}\eta^{\alpha\hat{\alpha}},\\
\bar{\partial}\lambda^A =& \hat{N}^i\lambda^Bf_{iB\hat{A}}\eta^{A\hat{A}}, \qquad \bar{\partial}\omega_B=-\hat{N}^i\omega_Af_{iB\hat{A}}\eta^{A\hat{A}},\\
\partial\hat{\lambda}^{\hat{A}} =& N^i\hat{\lambda}^{\hat{B}}f_{i\hat{B}A}\eta^{A\hat{A}}, \qquad \partial\hat{\omega}_{\hat{B}}=-N^i\hat{\omega}_{\hat{A}}f_{i\hat{B}A}\eta^{A\hat{A}},\\
\bar{\partial}N^k =& N^i\hat{N}^jf_{ij}^k, \qquad \partial\hat{N}^k=-\bar{\partial}N^k
\end{aligned}$$

6 Interaction Matrix

The interaction matrix is given by

$$F_{\Sigma\Omega}(x,y)=\frac{\overleftarrow{\delta}}{\overleftarrow{\delta}\,\Phi^{\Sigma}(y)}\frac{\delta S_{int}}{\delta\Phi^{\Omega}(x)}.$$

Because we are working in momentum space is useful to write also F in momentum space, for that reason the equation we work with is

$$F_{\Lambda\Omega}(x,k)f(x)=\int d^2y\frac{\overleftarrow{\delta}}{\overleftarrow{\delta}\,\Phi^{\Sigma}(y)}\frac{\delta S_{int}}{\delta\Phi^{\Omega}(x)}\exp(iky)f(y).$$

Note that the $f(y)$ stands for the previous Green's function and the exponential came from the Fourier Transform. The interaction matrix is,

$$\begin{aligned}
F_{\alpha\beta} =& -J_2^m(i\bar{k}+\bar{\partial})f_{m\alpha\beta}-\frac{1}{2}\bar{\partial}J_2^mf_{m\alpha\beta}-\frac{1}{2}J_3^{\hat{\alpha}}\bar{J}_3^{\hat{\beta}}\left(f_{i\alpha\hat{\alpha}}f_{j\beta\hat{\beta}}-f_{i\beta\hat{\alpha}}f_{j\alpha\hat{\beta}}\right)g^{ij}\\
&-\frac{1}{2}\left(N^i\bar{J}_2^m-\hat{N}^iJ_2^m\right)\left(f_{m\alpha\mu}f_{i\beta\hat{\mu}}-f_{m\beta\mu}f_{i\alpha\hat{\mu}}\right)\eta^{\mu\hat{\mu}}
\end{aligned}$$

$$\begin{aligned}
F_{\alpha m} =& J_1^{\beta}\left(i\bar{k}+\bar{\partial}\right)f_{m\alpha\beta}+\frac{1}{8}\left(\partial\bar{J}_1^{\beta}+3\bar{\partial}J_1^{\beta}\right)f_{m\alpha\beta}-\frac{1}{8}\left(3J_2^n\bar{J}_3^{\hat{\alpha}}+5\bar{J}_2^nJ_3^{\hat{\alpha}}\right)f_{i\hat{\alpha}\alpha}f_{jmn}g^{ij}\\
&-\frac{3}{8}\left(J_2^n\bar{J}_3^{\hat{\alpha}}-\bar{J}_2^nJ_3^{\hat{\alpha}}\right)f_{n\alpha\beta}f_{m\hat{\beta}\hat{\alpha}}\eta^{\beta\hat{\beta}}+\frac{1}{2}\left(N^i\bar{J}_1^{\beta}-\hat{N}^iJ_1^{\beta}\right)\left(f_{ipm}f_{q\alpha\beta}\eta^{pq}+f_{i\alpha\hat{\mu}}f_{m\beta\mu}\eta^{\mu\hat{\mu}}\right)
\end{aligned}$$

$$\begin{aligned}
F_{\alpha\hat{\alpha}} =& N^if_{i\alpha\hat{\alpha}}\left(i\bar{k}+\bar{\partial}\right)-\hat{N}^if_{i\alpha\hat{\alpha}}\left(ik+\partial\right)+\frac{1}{2}\left(\bar{\partial}N^i-\partial\hat{N}^i\right)f_{i\alpha\hat{\alpha}}+\frac{1}{4}\left(\bar{J}_1^{\beta}J_3^{\hat{\beta}}-J_1^{\beta}\bar{J}_3^{\hat{\beta}}\right)f_{m\alpha\beta}f_{n\hat{\alpha}\hat{\beta}}\eta^{mn}\\
&+\frac{1}{4}\left(J_1^{\beta}\bar{J}_3^{\hat{\beta}}+3\bar{J}_1^{\beta}J_3^{\hat{\beta}}\right)f_{i\hat{\alpha}\beta}f_{j\alpha\hat{\beta}}g^{ij}+\frac{1}{4}J_2^m\bar{J}_2^n\left(f_{m\alpha\beta}f_{n\hat{\alpha}\hat{\beta}}-f_{n\alpha\beta}f_{m\hat{\alpha}\hat{\beta}}\right)\eta^{\beta\hat{\beta}}
\end{aligned}$$

$$\begin{aligned}
F_{\alpha B} =& -\omega_A\bar{J}_3^{\hat{\alpha}}A_{B\ \alpha\hat{\alpha}}^{A}=-F_{B\alpha}\\
F_{\alpha}{}^{A} =& -\lambda^B\bar{J}_3^{\hat{\alpha}}A_{B\ \alpha\hat{\alpha}}^{A}=-F^{A}_{\ \alpha}\\
F_{\alpha\hat{B}} =& \hat{\omega}_{\hat{A}}J_3^{\hat{\alpha}}A_{B\ \alpha\hat{\alpha}}^{A}=-F_{\hat{B}\alpha}\\
F_{\alpha}{}^{\hat{A}} =& \hat{\lambda}^{\hat{B}}J_3^{\hat{\alpha}}A_{B\ \alpha\hat{\alpha}}^{A}=-F^{\hat{A}}_{\ \alpha}
\end{aligned}$$

$$
\begin{aligned}
F_{m\alpha} =& J_1^\beta \left(i\bar{k}+\bar\partial\right) f_{m\alpha\beta} + \frac{1}{8}\left(5\bar\partial J_1^\beta - \partial \bar{J}_1^\beta\right) f_{m\alpha\beta} - \frac{1}{2}\left(N^i \bar{J}_1^\beta - \hat{N}^i J_1^\beta\right)\left(f_{ipm} f_{q\alpha\beta}\eta^{pq} + f_{i\alpha\hat\mu} f_{m\beta\mu}\eta^{\mu\hat\mu}\right) \\
&+ \frac{1}{8}\left(3J_2^n \bar{J}_3^{\hat\alpha} + 5\bar{J}_2^n J_3^{\hat\alpha}\right) f_{i\hat\alpha\alpha} f_{jmn} g^{ij} + \frac{3}{8}\left(J_2^n \bar{J}_3^{\hat\alpha} - \bar{J}_2^n J_3^{\hat\alpha}\right) f_{n\alpha\beta} f_{m\hat\beta\hat\alpha}\eta^{\beta\hat\beta}
\end{aligned}
$$

$$
\begin{aligned}
F_{mn} =& - N^i f_{imn}\left(i\bar{k}+\bar\partial\right) + \hat{N}^i f_{imn}\left(ik+\partial\right) - \frac{1}{2}\left(\bar\partial N^i - \partial \hat{N}^i\right) f_{imn} \\
&- \frac{1}{2}\left(J_1^\alpha \bar{J}_3^{\hat\alpha} - \bar{J}_1^\alpha J_3^{\hat\alpha}\right)\left(f_{m\alpha\beta} f_{n\hat\alpha\hat\beta} + f_{n\alpha\beta} f_{m\hat\alpha\hat\beta}\right)\eta^{\beta\hat\beta} - \frac{1}{2} J_2^p \bar{J}_2^q \left(f_{ipm} f_{jqn} + f_{ipn} f_{jqm}\right) g^{ij}
\end{aligned}
$$

$$
\begin{aligned}
F_{m\hat\alpha} =& \bar{J}_3^{\hat\beta} f_{m\hat\alpha\hat\beta}\left(ik+\partial\right) + \frac{1}{8}\left(5\partial \bar{J}_3^{\hat\beta} - \bar\partial J_3^{\hat\beta}\right) f_{m\hat\alpha\hat\beta} + \frac{3}{8}\left(\bar{J}_1^\alpha J_2^n - J_1^\alpha \bar{J}_2^n\right) f_{m\alpha\beta} f_{n\hat\alpha\hat\beta}\eta^{\beta\hat\beta} \\
&+ \frac{1}{8}\left(3J_1^\alpha \bar{J}_2^n + 5\bar{J}_1^\alpha J_2^n\right) f_{i\alpha\hat\alpha} f_{jmn} g^{ij} - \frac{1}{2}\left(N^i \bar{J}_3^{\hat\beta} - \hat{N}^i J_3^{\hat\beta}\right)\left(f_{ipm} f_{q\hat\alpha\hat\beta}\eta^{pq} - f_{i\hat\alpha\mu} f_{m\hat\beta\hat\mu}\eta^{\mu\hat\mu}\right)
\end{aligned}
$$

$$
\begin{aligned}
F_{mB} =& - \omega_A \bar{J}_2^n A^A_{B\ mn} = F_{Bm} \\
F_m{}^A =& - \lambda^B \bar{J}_2^n A^A_{B\ mn} = F^B{}_m \\
F_{m\hat{B}} =& \hat\omega_{\hat{A}} J_2^n A^A_{B\ mn} = F_{\hat{B}m} \\
F_m{}^{\hat{A}} =& \lambda^{\hat{B}} J_2^n A^A_{B\ mn} = F^{\hat{A}}{}_m
\end{aligned}
$$

$$
\begin{aligned}
F_{\hat\alpha\alpha} =& N^i f_{i\alpha\hat\alpha}\left(i\bar{k}+\bar\partial\right) - \hat{N}^i f_{i\alpha\hat\alpha}\left(ik+\partial\right) + \frac{1}{2}\left(\bar\partial N^i - \partial \hat{N}^i\right) f_{i\alpha\hat\alpha} - \frac{1}{4}\left(J_1^\beta J_3^{\hat\beta} - J_1^\beta \bar{J}_3^{\hat\beta}\right) f_{m\alpha\beta} f_{n\hat\alpha\hat\beta}\eta^{mn} \\
&- \frac{1}{4}\left(J_1^\beta \bar{J}_3^{\hat\beta} + 3\bar{J}_1^\beta J_3^{\hat\beta}\right) f_{i\hat\alpha\hat\beta} f_{j\alpha\beta} g^{ij} - \frac{1}{4} J_2^m \bar{J}_2^n \left(f_{m\alpha\beta} f_{n\hat\alpha\hat\beta} - f_{n\alpha\beta} f_{m\hat\alpha\hat\beta}\right)\eta^{\beta\hat\beta}
\end{aligned}
$$

$$
\begin{aligned}
F_{\hat\alpha m} =& \bar{J}_3^{\hat\beta} f_{m\hat\alpha\hat\beta}\left(ik+\partial\right) + \frac{1}{8}\left(3\partial \bar{J}_3^{\hat\beta} + \bar\partial J_3^{\hat\beta}\right) f_{m\hat\alpha\hat\beta} - \frac{3}{8}\left(\bar{J}_1^\alpha J_2^n - J_1^\alpha \bar{J}_2^n\right) f_{m\alpha\beta} f_{n\hat\alpha\hat\beta}\eta^{\beta\hat\beta} \\
&- \frac{1}{8}\left(3J_1^\alpha \bar{J}_2^n + 5\bar{J}_1^\alpha J_2^n\right) f_{i\alpha\hat\alpha} f_{jmn} g^{ij} + \frac{1}{2}\left(N^i \bar{J}_3^{\hat\beta} - \hat{N}^i J_3^{\hat\beta}\right)\left(f_{ipm} f_{q\hat\alpha\hat\beta}\eta^{pq} - f_{i\hat\alpha\mu} f_{m\hat\beta\hat\mu}\eta^{\mu\hat\mu}\right)
\end{aligned}
$$

$$
\begin{aligned}
F_{\hat\alpha\hat\beta} =& - \bar{J}_2^m \left(ik+\partial\right) f_{m\hat\alpha\hat\beta} - \frac{1}{2}\partial \bar{J}_2^m f_{m\hat\alpha\hat\beta} - \frac{1}{2} J_1^\alpha \bar{J}_1^\beta \left(f_{i\alpha\hat\alpha} f_{j\beta\hat\beta} - f_{i\beta\hat\alpha} f_{j\alpha\hat\beta}\right) g^{ij} \\
&+ \frac{1}{2}\left(N^i \bar{J}_2^m - \hat{N}^i J_2^m\right)\left(f_{m\hat\alpha\hat\mu} f_{i\mu\hat\beta} - f_{m\hat\beta\hat\mu} f_{i\mu\hat\alpha}\right)\eta^{\mu\hat\mu}
\end{aligned}
$$

$$
\begin{aligned}
F_{\hat\alpha B} =& - \omega_A \bar{J}_1^\alpha A^A_{B\ \alpha\hat\alpha} = -F_{B\hat\alpha} \\
F_{\hat\alpha}{}^A =& - \lambda^B \bar{J}_1^\alpha A^A_{B\ \alpha\hat\alpha} = -F^A{}_{\hat\alpha} \\
F_{\hat\alpha\hat{B}} =& \hat\omega_{\hat{A}} J_1^\alpha A^A_{B\ \alpha\hat\alpha} = -F_{\hat{B}\hat\alpha} \\
F_{\hat\alpha B} =& \lambda^{\hat{A}} J_1^\alpha A^A_{B\ \alpha\hat\alpha} = -F^{\hat{B}}{}_{\hat\alpha}
\end{aligned}
$$

$$
\begin{aligned}
F_B{}^A =& - \hat{N}_A^B = F^A{}_B \\
F_B{}^{\hat{A}} =& - \omega_A \hat\lambda^{\hat{B}} A^{A\hat{A}}_{B\hat{B}} = F^{\hat{A}}{}_B \\
F_{B\hat{B}} =& - \omega_A \hat\omega_{\hat{A}} A^{A\hat{A}}_{B\hat{B}} = F_{B\hat{A}} \\
F^A_{\hat{B}} =& - \lambda^B \hat\omega_{\hat{A}} A^{A\hat{A}}_{B\hat{B}} = F_{\hat{B}}{}^A \\
F^{A\hat{A}} =& - \lambda^B \hat\lambda^{\hat{B}} A^{A\hat{A}}_{B\hat{B}} = F^{\hat{A}A} \\
F_{\hat{B}}{}^{\hat{A}} =& - N^{\hat{A}}_{\hat{B}} = F^{\hat{A}}{}_{\hat{B}}
\end{aligned}
$$

We have defined

$$
\begin{aligned}
A^{B}_{Axy} =& f_{iA\hat{B}} f_{jxy} \eta^{B\hat{B}} g^{ij} \\
A^{\hat{B}}_{\hat{A}xy} =& f_{i\hat{A}B} f_{jxy} \eta^{B\hat{B}} g^{ij}
\end{aligned}
$$

with $\{x, y\} = \{\{m, n\}, \{\alpha\hat{\alpha}\}\}$

7 Green's Functions

With all the previous results, we start computing the Green's Functions.

7.1 G_1

$$
\begin{aligned}
G_{1A}{}^{B} =& \frac{i}{k} \delta^{B}_{A} = -G^{B}_{1\ A}, \\
G_{1\hat{A}}{}^{\hat{B}} =& \frac{i}{k} \delta^{\hat{B}}_{\hat{A}} = -G^{\hat{B}}_{1\ \hat{A}}.
\end{aligned}
$$

7.2 G_2

$$
\begin{aligned}
G^{\alpha\hat{\alpha}}_{2} =& \frac{1}{|k|^2} \eta^{\alpha\hat{\alpha}} = -G^{\hat{\alpha}\alpha}_{1}, \\
G^{mn}_{2} =& \frac{1}{|k|^2} \eta^{mn}, \\
G_{2A}{}^{B} =& -\frac{i}{k} \left(F_{A}{}^{C} G_{1C}{}^{B} \right) \\
=& -\frac{1}{k^2} \hat{N}^{B}_{A} = G^{B}_{2\ A}, \\
G_{2A\hat{A}} =& -\frac{i}{k} \left(F_{A\hat{C}} G^{\hat{C}}_{1\ \hat{A}} \right) \\
=& \frac{1}{|k|^2} \omega_B \hat{\omega}_{\hat{B}} A^{B\hat{B}}_{A\hat{A}} = G_{2\hat{A}A}, \\
G_{2A}{}^{\hat{B}} =& -\frac{i}{k} \left(F_{A}{}^{\hat{C}} G_{1\hat{C}}{}^{\hat{B}} \right) \\
=& -\frac{1}{|k|^2} \omega_B \hat{\lambda}^{\hat{A}} A^{B\hat{B}}_{A\hat{A}} = G^{\hat{B}}_{2\ A}, \\
G^{B}_{2\ \hat{A}} =& \frac{i}{k} \left(F^{B}_{\hat{C}} G^{\hat{C}}_{1\ \ \hat{A}} \right) \\
=& -\frac{1}{|k|^2} \lambda^{A} \hat{\omega}_{\hat{B}} A^{B\hat{B}}_{A\hat{A}} = G^{B}_{2\hat{A}}, \\
G^{B\hat{B}}_{2} =& \frac{i}{k} \left(F^{B\hat{C}} G_{1\hat{C}}{}^{\hat{B}} \right) \\
=& \frac{1}{|k|^2} \lambda^{A} \hat{\lambda}^{\hat{A}} A^{B\hat{B}}_{A\hat{A}} = G^{\hat{B}B}_{2}, \\
G_{2\hat{A}}{}^{\hat{B}} =& -\frac{i}{k} \left(F_{\hat{A}}{}^{\hat{C}} G_{1\hat{C}}{}^{\hat{B}} \right) \\
=& -\frac{1}{k^2} N^{\hat{B}}_{\hat{A}} = G^{\hat{B}}_{2\ \hat{A}},
\end{aligned}
$$

7.3 G_3

$$
\begin{aligned}
G_3^{\alpha\beta} &= -\frac{\eta^{\alpha\hat{\alpha}}}{|k|^2}\left(F_{\hat{\beta}\hat{\alpha}}G_2^{\hat{\beta}\beta}\right)\\
&= -\frac{i}{|k|^2}\frac{\bar{J}_2^m}{\bar{k}}f_{m\hat{\alpha}\hat{\beta}}\eta^{\alpha\hat{\alpha}}\eta^{\beta\hat{\beta}},\\
G_3^{\alpha m} &= -\frac{\eta^{\alpha\hat{\alpha}}}{|k|^2}\left(F_{n\hat{\alpha}}G_2^{nm}\right)\\
&= -\frac{i}{|k|^2}\frac{\bar{J}_3^{\hat{\beta}}}{\bar{k}}f_{n\hat{\alpha}\hat{\beta}}\eta^{\alpha\hat{\alpha}}\eta^{mn} = -G_3^{m\alpha},\\
G_3^{\alpha\hat{\alpha}} &= -\frac{\eta^{\alpha\hat{\beta}}}{|k|^2}\left(F_{\beta\hat{\beta}}G_2^{\beta\hat{\alpha}}\right)\\
&= -\frac{i}{|k|^2}\left(\frac{N^i}{k}-\frac{\hat{N}^i}{\bar{k}}\right)f_{i\beta\hat{\beta}}\eta^{\alpha\hat{\beta}}\eta^{\beta\hat{\alpha}} = G_3^{\hat{\alpha}\alpha},\\
G_3^{\alpha}{}_{A} &= -\frac{\eta^{\alpha\hat{\alpha}}}{|k|^2}\left(F_{B\hat{\alpha}}G_1^{B}{}_{A}\right)\\
&= \frac{i}{|k|^2}\frac{\bar{J}_1^{\beta}}{\bar{k}}\omega_B A_A{}^{B}{}_{\beta\hat{\alpha}}\eta^{\alpha\hat{\alpha}} = -G_{3A}{}^{\alpha},\\
G_3^{\alpha B} &= -\frac{\eta^{\alpha\hat{\alpha}}}{|k|^2}\left(F^{A}{}_{\hat{\alpha}}G_{1A}{}^{B}\right)\\
&= -\frac{i}{|k|^2}\frac{\bar{J}_1^{\beta}}{\hat{k}}\lambda^A A_A{}^{B}{}_{\beta\hat{\alpha}}\eta^{\alpha\hat{\alpha}} = -G_3^{B\alpha},\\
G_3^{\alpha}{}_{\hat{A}} &= -\frac{\eta^{\alpha\hat{\alpha}}}{|k|^2}\left(F_{\hat{B}\hat{\alpha}}G_1^{\hat{B}}{}_{\hat{A}}\right)\\
&= -\frac{i}{|k|^2}\frac{J_1^{\beta}}{k}\hat{\omega}_{\hat{B}}A_{\hat{A}}{}^{\hat{B}}{}_{\beta\hat{\alpha}}\eta^{\alpha\hat{\alpha}} = -G_{3\hat{A}}{}^{\alpha},\\
G_3^{\alpha\hat{B}} &= -\frac{\eta^{\alpha\hat{\alpha}}}{|k|^2}\left(F^{\hat{A}}{}_{\hat{\alpha}}G_{1\hat{A}}{}^{\hat{B}}\right)\\
&= \frac{i}{|k|^2}\frac{J_1^{\beta}}{k}\hat{\lambda}^{\hat{A}}A_{\hat{A}}{}^{\hat{B}}{}_{\beta\hat{\alpha}}\eta^{\alpha\hat{\alpha}} = -G_3^{\hat{B}\alpha},
\end{aligned}
$$

$$
\begin{aligned}
G_3^{mn} =& -\frac{\eta^{mp}}{|k|^2}\left(F_{qp}G_2^{qn}\right)\\
=&\frac{i}{|k|^2}\left(\frac{N^i}{k}-\frac{\hat{N}^i}{\bar{k}}\right)f_{ipq}\eta^{mp}\eta^{nq},\\
G_3^{m\hat{\alpha}} =& -\frac{\eta^{mn}}{|k|^2}\left(F_{\alpha n}G_2^{\alpha\hat{\alpha}}\right)\\
=& -\frac{i}{|k|^2}\frac{J_1^{\beta}}{k}f_{n\alpha\beta}\eta^{\alpha\hat{\alpha}}\eta^{mn}=-G_3^{\hat{\alpha}m},\\
G_3{}^m{}_A =& -\frac{\eta^{mn}}{|k|^2}\left(F_{Bn}G_1{}^B{}_A\right)\\
=& -\frac{i}{|k|^2}\frac{\bar{J}_2^p}{\bar{k}}\omega_B A_A{}^B{}_{np}\eta^{mn}=-G_{3A}{}^m,\\
G_3^{mB} =& -\frac{\eta^{mn}}{|k|^2}\left(F^A{}_{\hat{\alpha}}G_{1A}{}^B\right)\\
=&\frac{i}{|k|^2}\frac{\bar{J}_2^p}{\hat{k}}\lambda^A A_A{}^B{}_{np}\eta^{mn}=G_3^{Bm},\\
G_3{}^m{}_{\hat{A}} =& -\frac{\eta^{mn}}{|k|^2}\left(F_{\hat{B}\hat{\alpha}}G_1{}^{\hat{B}}{}_{\hat{A}}\right)\\
=&\frac{i}{|k|^2}\frac{J_2^p}{k}\hat{\omega}_{\hat{B}}A_{\hat{A}}{}^{\hat{B}}{}_{np}\eta^{mn}=-G_{3\hat{A}}{}^m,\\
G_3^{m\hat{B}} =& -\frac{\eta^{mn}}{|k|^2}\left(F^{\hat{A}}{}_{\hat{\alpha}}G_{1\hat{A}}{}^{\hat{B}}\right)\\
=& -\frac{i}{|k|^2}\frac{J_2^p}{k}\hat{\lambda}^{\hat{A}}A_{\hat{A}}{}^{\hat{B}}{}_{np}\eta^{mn}=-G_3^{\hat{B}m},
\end{aligned}
$$

$$
\begin{aligned}
G_3^{\hat{\alpha}\hat{\beta}} =& -\frac{i}{|k|^2}\frac{J_2^m}{k}f_{m\alpha\beta}\eta^{\alpha\hat{\alpha}}\eta^{\beta\hat{\beta}}\\
G_3{}^{\hat{\alpha}}{}_A =&\frac{\eta^{\alpha\hat{\alpha}}}{|k|^2}\left(F_{B\hat{\alpha}}G_1{}^B{}_A\right)\\
=& -\frac{i}{|k|^2}\frac{J_3^{\hat{\beta}}}{\bar{k}}\omega_B A_A{}^B{}_{\hat{\beta}\alpha}\eta^{\alpha\hat{\alpha}}=-G_{3A}{}^{\hat{\alpha}},\\
G_3^{\hat{\alpha}B} =&\frac{\eta^{\alpha\hat{\alpha}}}{|k|^2}\left(F^A{}_{\hat{\alpha}}G_{1A}{}^B\right)\\
=&\frac{i}{|k|^2}\frac{J_3^{\hat{\beta}}}{\bar{k}}\lambda^A A_A{}^B{}_{\hat{\beta}\alpha}\eta^{\alpha\hat{\alpha}}=-G_3^{B\hat{\alpha}},\\
G_3{}^{\hat{\alpha}}{}_{\hat{A}} =&\frac{\eta^{\alpha\hat{\alpha}}}{|k|^2}\left(F_{\hat{B}\hat{\alpha}}G_1{}^{\hat{B}}{}_{\hat{A}}\right)\\
=&\frac{i}{|k|^2}\frac{J_3^{\hat{\beta}}}{k}\hat{\omega}_{\hat{B}}A_{\hat{A}}{}^{\hat{B}}{}_{\hat{\beta}\alpha}\eta^{\alpha\hat{\alpha}}=-G_{3\hat{A}}{}^{\hat{\alpha}},\\
G_3^{\hat{\alpha}\hat{B}} =&\frac{\eta^{\alpha\hat{\alpha}}}{|k|^2}\left(F^{\hat{A}}{}_{\hat{\alpha}}G_{1\hat{A}}{}^{\hat{B}}\right)\\
=& -\frac{i}{|k|^2}\frac{J_3^{\hat{\beta}}}{k}\hat{\lambda}^{\hat{A}}A_{\hat{A}}{}^{\hat{B}}{}_{\hat{\beta}\alpha}\eta^{\alpha\hat{\alpha}}=-G_3^{\hat{B}\hat{\alpha}},
\end{aligned}
$$

$$
\begin{aligned}
G_{3AC} =& -\frac{i}{\bar{k}}\left(F_{\hat{D}A}G_{2}^{\hat{D}}{}_{C}+F_{A}^{\hat{D}}G_{2\hat{D}C}\right)\\
&=\frac{i}{|k|^2}\frac{1}{\bar{k}}\omega_B\omega_D\hat{\lambda}^{\hat{A}}\hat{\omega}_{\hat{B}}\left[A^{B\hat{C}}_{A\hat{A}}A^{D\hat{B}}_{C\hat{C}}-A^{B\hat{B}}_{A\hat{C}}A^{D\hat{C}}_{C\hat{A}}\right],\\
G_{3A}{}^{B} =& \frac{i}{\bar{k}}\bar{\partial}G_{2A}{}^{B}-\frac{i}{\bar{k}}\left(F_{A}^{D}G_{2D}{}^{B}+F_{\hat{D}A}G_{2}^{\hat{D}B}+F_{A}^{\hat{D}}G_{2\hat{D}}{}^{B}\right)\\
&=-\frac{i}{\bar{k}^3}\left(\delta_A^D\bar{\partial}+\hat{N}_A^D\right)\hat{N}_D^B+\frac{i}{|k|^2}\frac{1}{\bar{k}}\omega_D\lambda^C\hat{\lambda}^{\hat{A}}\hat{\omega}_{\hat{B}}\left[A^{D\hat{B}}_{A\hat{C}}A^{B\hat{C}}_{C\hat{A}}-A^{D\hat{C}}_{A\hat{A}}A^{B\hat{B}}_{C\hat{C}}\right]\\
G_{3A\hat{A}} =& \frac{i}{\bar{k}}\bar{\partial}G_{2A\hat{A}}-\frac{i}{\bar{k}}\left(F_{\hat{D}A}G_{2}^{\hat{D}}{}_{\hat{A}}+F^{D}{}_{A}G_{2D\hat{A}}\right)\\
&=\frac{i}{|k|^2}\frac{1}{\bar{k}}\left(\delta_A^D\bar{\partial}+\hat{N}_A^D\right)\omega_B\hat{\omega}_{\hat{B}}A^{B\hat{B}}_{D\hat{A}}-\frac{i}{|k|^2}\frac{1}{k}\omega_B\hat{\omega}_{\hat{B}}N_{\hat{A}}^{\hat{D}}A^{B\hat{B}}_{A\hat{D}}\\
G_{3A}{}^{\hat{B}} =& \frac{i}{\bar{k}}\bar{\partial}G_{2A}{}^{\hat{B}}-\frac{i}{\bar{k}}\left(F_{A}^{D}G_{2D}{}^{\hat{B}}+F_{A}^{\hat{D}}G_{2\hat{D}}{}^{\hat{B}}\right)\\
&=-\frac{i}{|k|^2}\frac{1}{\bar{k}}\left(\delta_A^D\bar{\partial}+\hat{N}_A^D\right)\omega_B\hat{\lambda}^{\hat{A}}A^{B\hat{B}}_{D\hat{A}}-\frac{i}{|k|^2}\frac{1}{k}\omega_B\hat{\lambda}^{\hat{A}}N^{\hat{B}}_{\hat{D}}A^{B\hat{D}}_{A\hat{A}}
\end{aligned}
$$

$$
\begin{aligned}
G_{3}^{B}{}_{A} =& -\frac{i}{\bar{k}^3}\left(\delta_D^B\bar{\partial}+\hat{N}_D^B\right)\hat{N}_A^D+\frac{i}{|k|^2}\frac{1}{\bar{k}}\omega_D\lambda^C\hat{\lambda}^{\hat{A}}\hat{\omega}_{\hat{B}}\left[A^{D\hat{C}}_{A\hat{A}}A^{B\hat{B}}_{C\hat{C}}-A^{D\hat{B}}_{A\hat{C}}A^{B\hat{C}}_{C\hat{A}}\right]\\
G_{3}^{BD} =& \frac{i}{|k|^2}\frac{1}{\bar{k}}\lambda^A\lambda^C\hat{\lambda}^{\hat{A}}\hat{\omega}_{\hat{B}}\left[A^{B\hat{C}}_{A\hat{A}}A^{D\hat{B}}_{C\hat{C}}-A^{B\hat{B}}_{A\hat{C}}A^{D\hat{C}}_{C\hat{A}}\right]\\
G_{3}^{B}{}_{\hat{A}} =& -\frac{i}{\bar{k}}\frac{1}{|k|^2}\left(\delta_D^B\bar{\partial}+\hat{N}_D^B\right)\lambda^A\hat{\omega}_{\hat{B}}A^{B\hat{B}}_{A\hat{A}}+\frac{i}{|k|^2}\frac{1}{k}\lambda^A\hat{\omega}_{\hat{B}}N_{\hat{A}}^{\hat{D}}A^{B\hat{B}}_{A\hat{D}}\\
G_{3}^{B\hat{B}} =& \frac{i}{\bar{k}}\frac{1}{|k|^2}\left(\delta_D^B\bar{\partial}+\hat{N}_D^B\right)\lambda^A\hat{\lambda}^{\hat{A}}A^{B\hat{B}}_{A\hat{A}}+\frac{i}{|k|^2}\frac{1}{k}\lambda^A\hat{\lambda}^{\hat{A}}N^{\hat{B}}_{\hat{D}}A^{B\hat{D}}_{A\hat{A}}
\end{aligned}
$$

$$
\begin{aligned}
G_{3\hat{A}A} =& \frac{i}{|k|^2}\frac{1}{k}\left(\delta_{\hat{A}}^{\hat{D}}\partial+N_{\hat{A}}^{\hat{D}}\right)\omega_B\hat{\omega}_{\hat{B}}A^{B\hat{B}}_{A\hat{A}}-\frac{i}{|k|^2}\frac{1}{\bar{k}}\omega_B\hat{\omega}_{\hat{B}}\hat{N}_A^D A^{B\hat{B}}_{D\hat{A}}\\
G_{3\hat{A}}{}^{B} =& -\frac{i}{|k|^2}\frac{1}{k}\left(\delta_{\hat{A}}^{\hat{D}}\partial+N_{\hat{A}}^{\hat{D}}\right)\lambda^A\hat{\omega}_{\hat{B}}A^{B\hat{B}}_{A\hat{A}}-\frac{i}{|k|^2}\frac{1}{\bar{k}}\lambda^A\hat{\omega}_{\hat{B}}\hat{N}_C^B A^{C\hat{B}}_{A\hat{A}}\\
G_{3\hat{A}\hat{C}} =& \frac{i}{|k|^2}\frac{1}{k}\hat{\omega}_{\hat{B}}\hat{\omega}_{\hat{D}}\lambda^A\omega_B\left[A^{C\hat{B}}_{A\hat{A}}A^{B\hat{D}}_{C\hat{C}}-A^{B\hat{B}}_{C\hat{A}}A^{C\hat{D}}_{A\hat{C}}\right]\\
G_{3\hat{A}}{}^{\hat{B}} =& -\frac{i}{k^3}\left(\delta_{\hat{A}}^{\hat{D}}\partial+N_{\hat{A}}^{\hat{D}}\right)N_{\hat{D}}^{\hat{B}}+\frac{i}{|k|^2}\frac{1}{k}\hat{\omega}_{\hat{D}}\hat{\lambda}^{\hat{C}}\lambda^A\omega_B\left[A^{B\hat{D}}_{C\hat{A}}A^{C\hat{B}}_{A\hat{C}}-A^{C\hat{D}}_{A\hat{A}}A^{B\hat{B}}_{C\hat{C}}\right]
\end{aligned}
$$

$$
\begin{aligned}
G_{3}^{\hat{B}}{}_{A} =& -\frac{i}{|k|^2}\frac{1}{k}\left(\delta_{\hat{D}}^{\hat{B}}\partial-N_{\hat{D}}^{\hat{B}}\right)\omega_B\hat{\lambda}^{\hat{A}}A^{B\hat{D}}_{A\hat{A}}+\frac{i}{|k|^2}\frac{1}{\bar{k}}\omega_B\hat{\lambda}^{\hat{A}}\hat{N}_A^C A^{B\hat{B}}_{C\hat{A}}\\
G_{3}^{\hat{B}B} =& \frac{i}{|k|^2}\frac{1}{k}\left(\delta_{\hat{D}}^{\hat{B}}\partial-N_{\hat{D}}^{\hat{B}}\right)\lambda^A\hat{\lambda}^{\hat{A}}A^{B\hat{D}}_{A\hat{A}}+\frac{i}{|k|^2}\frac{1}{\bar{k}}\lambda^A\hat{\lambda}^{\hat{A}}\hat{N}_A^B A^{C\hat{B}}_{A\hat{A}}\\
G_{3}^{\hat{B}}{}_{\hat{A}} =& -\frac{i}{k^3}\frac{1}{k}\left(\delta_{\hat{D}}^{\hat{B}}\partial-N_{\hat{D}}^{\hat{B}}\right)N_{\hat{A}}^{\hat{D}}-\frac{i}{|k|^2}\frac{1}{k}\hat{\omega}_{\hat{D}}\hat{\lambda}^{\hat{C}}\lambda^A\omega_B\left[A^{C\hat{B}}_{A\hat{C}}A^{B\hat{D}}_{C\hat{A}}-A^{B\hat{B}}_{C\hat{C}}A^{C\hat{B}}_{A\hat{A}}\right]\\
G_{3}^{\hat{B}\hat{D}} =& \frac{i}{|k|^2}\frac{1}{k}\hat{\lambda}^{\hat{D}}\hat{\lambda}^{\hat{C}}\lambda^A\omega_B\left[A^{C\hat{B}}_{A\hat{A}}A^{B\hat{D}}_{C\hat{C}}-A^{B\hat{B}}_{C\hat{A}}A^{C\hat{D}}_{A\hat{C}}\right]
\end{aligned}
$$

7.4 G_4

$$
\begin{aligned}
G_4^{\alpha\beta} =& \frac{1}{|k|^2\bar{k}^2}\left(\bar{\partial}\bar{J}_2^m f_{m\hat{\alpha}\hat{\beta}}+\bar{J}_2^m\hat{N}^i\left[f_{i\mu\hat{\alpha}}f_{m\hat{\mu}\hat{\beta}}-f_{i\mu\hat{\beta}}f_{m\hat{\mu}\hat{\alpha}}\right]\eta^{\mu\hat{\mu}}+\bar{J}_3^{\hat{\mu}}\bar{J}_3^{\hat{\nu}}f_{m\hat{\alpha}\hat{\mu}}f_{n\hat{\beta}\hat{\nu}}\eta^{mn}\right)\eta^{\alpha\hat{\alpha}}\eta^{\beta\hat{\beta}}\\
&+\frac{1}{|k|^4}\left(\frac{1}{2}\partial\bar{J}_2^m f_{m\hat{\alpha}\hat{\beta}}+\frac{1}{2}J_1^{\mu}\bar{J}_1^{\nu}g^{ij}\left(f_{i\mu\hat{\alpha}}f_{j\nu\hat{\beta}}-f_{i\nu\hat{\alpha}}f_{j\mu\hat{\beta}}\right)\right.\\
&\left.+\frac{1}{2}\left[\bar{J}_2^m N^i+J_2^m\hat{N}^i\right]\left(f_{m\hat{\alpha}\hat{\mu}}f_{i\mu\hat{\beta}}-f_{m\hat{\beta}\hat{\mu}}f_{i\mu\hat{\alpha}}\right)\eta^{\mu\hat{\mu}}\right)\eta^{\alpha\hat{\alpha}}\eta^{\beta\hat{\beta}}
\end{aligned}
$$

$$
\begin{aligned}
G_4^{\alpha m} =& \frac{1}{|k|^2\bar{k}^2}\left[\bar{\partial}\bar{J}_3^{\hat{\beta}} f_{n\hat{\alpha}\hat{\beta}} + \bar{J}_3^{\hat{\beta}}\hat{N}^i\left(f_{p\hat{\alpha}\hat{\beta}} f_{inq}\eta^{pq} + f_{n\hat{\mu}\hat{\beta}} f_{i\mu\hat{\alpha}}\eta^{\mu\hat{\mu}}\right)\right]\eta^{mn}\eta^{\alpha\hat{\alpha}} \\
&+ \frac{1}{|k|^4}\left[\frac{1}{8}\left(3\partial\bar{J}_3^{\hat{\beta}} + \bar{\partial}J_3^{\hat{\beta}}\right) f_{n\hat{\alpha}\hat{\beta}} - \frac{1}{8}\left[3\bar{J}_1^{\beta}J_2^p + 5J_1^{\beta}\bar{J}_2^p\right] f_{n\beta\mu} f_{p\hat{\alpha}\hat{\mu}}\eta^{\mu\hat{\mu}} - \frac{1}{8}\left[5\bar{J}_1^{\beta}J_2^p + 3J_1^{\beta}\bar{J}_2^p\right] f_{i\beta\hat{\alpha}} f_{jnp} g^{ij}\right. \\
&\left. + \frac{1}{2}(3N^i\bar{J}_3^{\hat{\beta}} - \hat{N}^i J_3^{\hat{\beta}})\left(f_{ipn} f_{q\hat{\alpha}\hat{\beta}}\eta^{pq} - f_{i\hat{\alpha}\mu} f_{n\hat{\beta}\hat{\mu}}\eta^{\mu\hat{\mu}}\right)\right]\eta^{\alpha\hat{\alpha}}\eta^{mn}
\end{aligned}
$$

$$
\begin{aligned}
G_4^{\alpha\hat{\alpha}} =& \frac{1}{|k|^2k^2}\left[\partial N^i f_{i\beta\hat{\beta}} - N^iN^j f_{i\mu\hat{\beta}} f_{j\beta\hat{\mu}}\eta^{\mu\hat{\mu}}\right]\eta^{\alpha\hat{\beta}}\eta^{\beta\hat{\alpha}} + \frac{1}{|k|^2\bar{k}^2}\left[-\bar{\partial}\hat{N}^i f_{i\beta\hat{\beta}} - \hat{N}^i\hat{N}^j f_{i\mu\hat{\beta}} f_{j\beta\hat{\mu}}\eta^{\mu\hat{\mu}}\right]\eta^{\alpha\hat{\beta}}\eta^{\beta\hat{\alpha}} \\
&+ \frac{1}{|k|^4}\left[\frac{1}{2}\left(\bar{\partial}N^i - \partial\hat{N}^i\right) f_{i\beta\hat{\beta}} + \left(N^i\hat{N}^j + N^j\hat{N}^i\right) f_{i\mu\hat{\beta}} f_{j\hat{\mu}\beta}\eta^{\mu\hat{\mu}} + \frac{1}{4}J_2^m\bar{J}_2^n\left(3f_{m\mu\beta} f_{n\hat{\beta}\hat{\mu}} + f_{n\mu\beta} f_{m\hat{\beta}\hat{\mu}}\right)\eta^{\mu\hat{\mu}}\right. \\
&\left. + \frac{1}{4}J_1^{\mu}\bar{J}_3^{\hat{\mu}}\left(5f_{m\mu\beta} f_{n\hat{\mu}\hat{\beta}}\eta^{mn} - f_{i\mu\hat{\beta}} f_{j\hat{\mu}\beta} g^{ij}\right) - \frac{1}{4}\bar{J}_1^{\mu}J_3^{\hat{\mu}}\left(f_{m\mu\beta} f_{n\hat{\mu}\hat{\beta}}\eta^{mn} + 3f_{i\mu\hat{\beta}} f_{j\hat{\mu}\beta} g^{ij}\right)\right]\eta^{\alpha\hat{\beta}}\eta^{\beta\hat{\alpha}}
\end{aligned}
$$

$$
\begin{aligned}
G_4^{mn} =& \frac{1}{|k|^2\bar{k}^2}\left[\bar{\partial}\hat{N}^i f_{ipq} + \hat{N}^i\hat{N}^j f_{irp} f_{jsq}\eta^{rs}\right]\eta^{nq}\eta^{mp} + \frac{1}{|k|^2k^2}\left[-\partial N^i f_{ipq} + N^iN^j f_{irp} f_{jsq}\eta^{rs}\right]\eta^{nq}\eta^{mp} \\
&+ \frac{1}{|k|^4}\eta^{mp}\eta^{nq}\left[-\frac{1}{2}\left(\bar{\partial}N^i - \partial\hat{N}^i\right) f_{ipq} - \left(N^i\hat{N}^j + N^j\hat{N}^i\right) f_{irp} f_{jsq}\eta^{rs} + \frac{1}{2}J_2^r\bar{J}_2^s\left(f_{irp} f_{jsq} + f_{irq} f_{jsp}\right) g^{ij}\right. \\
&\left. - \frac{1}{2}\left(J_1^{\alpha}\bar{J}_3^{\hat{\alpha}} + \bar{J}_1^{\alpha}J_3^{\hat{\alpha}}\right)\left(f_{q\alpha\beta} f_{p\hat{\alpha}\hat{\beta}} + f_{p\alpha\beta} f_{q\hat{\alpha}\hat{\beta}}\right)\eta^{\beta\hat{\beta}}\right]
\end{aligned}
$$

$$
\begin{aligned}
G_4^{\hat{\alpha}m} =& \frac{1}{|k|^2k^2}\left[-\partial J_1^{\beta} f_{n\alpha\beta} - J_1^{\beta}N^i\left(f_{ipn} f_{q\alpha\beta}\eta^{pq} + f_{n\mu\beta} f_{i\hat{\mu}\alpha}\eta^{\mu\hat{\mu}}\right)\right]\eta^{mn}\eta^{\alpha\hat{\alpha}} \\
&+ \frac{1}{|k|^4}\left[-\frac{1}{8}\left(3\bar{\partial}J_1^{\beta} + \partial\bar{J}_1^{\beta}\right) f_{n\alpha\beta} + \frac{1}{8}\left(3J_2^p\bar{J}_3^{\hat{\beta}} + 5\bar{J}_2^p J_3^{\hat{\beta}}\right) f_{i\alpha\hat{\beta}} f_{jnp} g^{ij} - \frac{1}{8}\left(5J_2^p\bar{J}_3^{\hat{\beta}} + 3\bar{J}_2^{\partial} J_3^{\hat{\beta}}\right) f_{p\alpha\mu} f_{n\hat{\beta}\hat{\mu}}\eta^{\mu\hat{\mu}}\right. \\
&\left. - \frac{1}{2}\left(N^i\bar{J}_1^{\beta} - 3\hat{N}^i J_1^{\beta}\right)\left(f_{ipn} f_{q\alpha\beta}\eta^{pq} + f_{i\alpha\hat{\mu}} f_{n\beta\mu}\eta^{\mu\hat{\mu}}\right)\right]\eta^{\alpha\hat{\alpha}}\eta^{nm}
\end{aligned}
$$

$$
\begin{aligned}
G_4^{\hat{\alpha}\hat{\beta}} =& \frac{1}{|k|^2k^2}\left[\partial J_2^m f_{m\alpha\beta} - J_2^m N^i\left(f_{m\alpha\mu} f_{i\beta\hat{\mu}} - f_{m\beta\mu} f_{i\alpha\hat{\mu}}\right)\eta^{\mu\hat{\mu}} + J_1^{\mu}J_1^{\nu} f_{m\alpha\mu} f_{n\beta\nu}\eta^{mn}\right]\eta^{\alpha\hat{\alpha}}\eta^{\beta\hat{\beta}} \\
&+ \frac{1}{|k|^4}\left[\frac{1}{2}\bar{\partial}J_2^m f_{m\alpha\beta} + \frac{1}{2}J_3^{\hat{\mu}}\bar{J}_3^{\hat{\nu}}\left(f_{i\hat{\mu}\alpha} f_{j\hat{\nu}\beta} - f_{i\hat{\mu}\beta} f_{j\hat{\nu}\alpha} g^{ij}\right)\right. \\
&\left. - \frac{1}{2}\left(N^i\bar{J}_2^m + \hat{N}^i J_2^m\right)\left(f_{m\beta\mu} f_{i\alpha\hat{\mu}} - f_{m\alpha\mu} f_{i\beta\hat{\mu}}\right)\eta^{\mu\hat{\mu}}\right]\eta^{\alpha\hat{\alpha}}\eta^{\beta\hat{\beta}}
\end{aligned}
$$

8 Current Algebra

We define $J = J_0 + K$, $K = J_1 + J_2 + J_3$, $X = x_1 + x_2 + x_3$.

$$
\langle K\rangle = \langle\bar{K}\rangle = \langle N\rangle = \langle\hat{N}\rangle = 0
$$

$$
\begin{aligned}
\langle J_0\rangle =& \frac{1}{2}\left(\{[N, T_{\hat{\alpha}}], T_{\alpha}\}\eta^{\alpha\hat{\alpha}} - \{[N, T_{\alpha}], T_{\hat{\alpha}}\}\eta^{\alpha\hat{\alpha}} + [[N, T_m], T_n]\eta^{mn}\right) \\
\langle\bar{J}_0\rangle =& -\frac{1}{2}\left(\{[\hat{N}, T_{\hat{\alpha}}], T_{\alpha}\}\eta^{\alpha\hat{\alpha}} - \{[\hat{N}, T_{\alpha}], T_{\hat{\alpha}}\}\eta^{\alpha\hat{\alpha}} + [[\hat{N}, T_m], T_n]\eta^{mn}\right)
\end{aligned}
$$

$$
\begin{aligned}
\langle X, J_0\rangle =& -[K, T_j]T_k g^{jk} \\
\langle X, \bar{J}_0\rangle =& -[\bar{K}, T_j]T_k g^{jk}
\end{aligned}
$$

$$
\begin{aligned}
\langle x_1, J_1\rangle =& -[J_2, T_{\hat\alpha}]T_\alpha \eta^{\alpha\hat\alpha}\\
\langle x_1, \bar J_1\rangle =& 0\\
\langle x_1, J_2\rangle =& -[J_3, T_m]T_n \eta^{mn}\\
\langle x_1, \bar J_2\rangle =& 0\\
\langle x_1, J_3\rangle =& [N, T_\alpha]T_{\hat\alpha} \eta^{\alpha\hat\alpha}\\
\langle x_1, \bar J_3\rangle =& -[\hat N, T_\alpha]T_{\hat\alpha} \eta^{\alpha\hat\alpha}
\end{aligned}
$$

$$
\begin{aligned}
\langle x_2, J_1\rangle =& [J_3, T_{\hat\alpha}]T_\alpha \eta^{\alpha\hat\alpha}\\
\langle x_2, \bar J_1\rangle =& 0\\
\langle x_2, J_2\rangle =& -[N, T_m]T_n \eta^{mn}\\
\langle x_2, \bar J_2\rangle =& [\hat N, T_m]T_n \eta^{mn}\\
\langle x_2, J_3\rangle =& 0\\
\langle x_2, \bar J_3\rangle =& [\bar J_1, T_\alpha]T_{\hat\alpha} \eta^{\alpha\hat\alpha}
\end{aligned}
$$

$$
\begin{aligned}
\langle x_3, J_1\rangle =& -[N, T_{\hat\alpha}]T_\alpha \eta^{\alpha\hat\alpha}\\
\langle x_3, \bar J_1\rangle =& [\hat N, T_{\hat\alpha}]T_\alpha \eta^{\alpha\hat\alpha}\\
\langle x_3, J_2\rangle =& 0\\
\langle x_3, \bar J_2\rangle =& -[\bar J_1, T_m]T_n \eta^{mn}\\
\langle x_3, J_3\rangle =& 0\\
\langle x_3, \bar J_3\rangle =& [\bar J_2, T_\alpha]T_{\hat\alpha} \eta^{\alpha\hat\alpha}
\end{aligned}
$$

$$
\langle X, N\rangle = \langle X, \hat N\rangle = 0
$$

$$
\begin{aligned}
\langle \omega, J\rangle = \langle \lambda, J\rangle =& 0\\
\langle \omega, \bar J_0\rangle = \langle \lambda, \bar J_0\rangle =& 0
\end{aligned}
$$

$$
\begin{aligned}
\langle \omega, \bar K\rangle =& -[\omega, T_i][\bar K, T_j]g^{ij}\\
\langle \lambda, \bar K\rangle =& -[\lambda, T_i][\bar K, T_j]g^{ij}
\end{aligned}
$$

$$
\begin{aligned}
\langle \omega, \lambda\rangle =& \langle \hat\omega, \hat\lambda\rangle = 0\\
\langle \omega, \hat\lambda\rangle =& -[\omega, T_i][\hat\lambda, T_j]g^{ij}\\
\langle \omega, \hat\omega\rangle =& -[\omega, T_i][\hat\omega, T_j]g^{ij}\\
\langle \lambda, \hat\lambda\rangle =& -[\lambda, T_i][\hat\lambda, T_j]g^{ij}\\
\langle \lambda, \hat\omega\rangle =& -[\lambda, T_i][\hat\omega, T_j]g^{ij}
\end{aligned}
$$

$$\langle\hat{\omega},\bar{J}\rangle = \langle\hat{\lambda},\bar{J}\rangle = 0$$
$$\langle\hat{\omega},J_0\rangle = \langle\hat{\lambda},J_0\rangle = 0$$

$$\langle\hat{\omega},K\rangle = -[\hat{\omega},T_i][K,T_j]g^{ij}$$
$$\langle\hat{\lambda},K\rangle = -[\hat{\lambda},T_i][K,T_j]g^{ij}$$

$$\langle\omega,N\rangle = \langle\lambda,N\rangle = 0$$

$$\langle\omega,\hat{N}\rangle = -[\omega,T_i][\hat{N},T_j]g^{ij}$$
$$\langle\lambda,\hat{N}\rangle = -[\lambda,T_i][\hat{N},T_j]g^{ij}$$

$$\langle\hat{\omega},\hat{N}\rangle = \langle\hat{\lambda},\hat{N}\rangle = 0$$

$$\langle\hat{\omega},N\rangle = -[\hat{\omega},T_i][N,T_j]g^{ij}$$
$$\langle\hat{\lambda},N\rangle = -[\hat{\lambda},T_i][N,T_j]g^{ij}$$

$$\begin{aligned}
\langle J_0,J_0\rangle =&[J_1,T_{\hat{\alpha}}][J_3,T_\alpha]\eta^{\alpha\hat{\alpha}} - [J_3,T_\alpha][J_1,T_{\hat{\alpha}}]\eta^{\alpha\hat{\alpha}} + [J_2,T_m][J_2,T_n]\eta^{mn}\\
\langle J_0,J_1\rangle =&[J_1,T_{\hat{\alpha}}][N,T_\alpha]\eta^{\alpha\hat{\alpha}} - [J_3,T_\alpha][J_2,T_{\hat{\alpha}}]\eta^{\alpha\hat{\alpha}} + [J_2,T_m][J_3,T_n]\eta^{mn}\\
\langle J_0,\bar{J}_1\rangle =&-[J_1,T_{\hat{\alpha}}][\hat{N},T_\alpha]\eta^{\alpha\hat{\alpha}}\\
\langle J_0,J_2\rangle =&-[J_3,T_\alpha][J_3,T_{\hat{\alpha}}]\eta^{\alpha\hat{\alpha}} + [J_2,T_m][N,T_n]\eta^{mn}\\
\langle J_0,\bar{J}_2\rangle =&[J_1,T_{\hat{\alpha}}][\bar{J}_1,T_\alpha]\eta^{\alpha\hat{\alpha}} - [J_2,T_m][\hat{N},T_n]\eta^{mn}\\
\langle J_0,J_3\rangle =&-[J_3,T_\alpha][N,T_{\hat{\alpha}}]\eta^{\alpha\hat{\alpha}}\\
\langle J_0,\bar{J}_3\rangle =&[J_3,T_\alpha][\hat{N},T_{\hat{\alpha}}]\eta^{\alpha\hat{\alpha}} + [J_1,T_{\hat{\alpha}}][\bar{J}_2,T_\alpha]\eta^{\alpha\hat{\alpha}} + [J_2,T_m][\bar{J}_1,T_n]\eta^{mn}
\end{aligned}$$

$$\begin{aligned}
\langle J_1,J_1\rangle =&\left([J_2,T_{\hat{\alpha}}][N,T_\alpha] - [N,T_\alpha][J_2,T_{\hat{\alpha}}]\right)\eta^{\alpha\hat{\alpha}} + [J_3,T_m][J_3,T_n]\eta^{mn}\\
\langle\bar{J}_1,\bar{J}_1\rangle =&0\\
\langle J_1,\bar{J}_1\rangle =&\frac{1}{2}[\partial J_2,T_{\hat{\alpha}}]T_\alpha\eta^{\alpha\hat{\alpha}} + \frac{1}{2}\left([J_1,T_i][\bar{J}_1,T_j] + [\bar{J}_1,T_i][J_1,T_j]\right)g^{ij}\\
&+\frac{1}{2}\left([\bar{J}_2,T_{\hat{\alpha}}][N,T_\alpha] - [J_2,T\hat{\alpha}][\hat{N},T_\alpha] - 3[N,T_\alpha][\bar{J}_2,T_{\hat{\alpha}}] - [\hat{N},T_\alpha][J_2,T_{\hat{\alpha}}]\right)\eta^{\alpha\hat{\alpha}}\\
\langle\bar{J}_1,J_1\rangle =&-\frac{1}{2}[\partial J_2,T_{\hat{\alpha}}]T_\alpha\eta^{\alpha\hat{\alpha}} + \frac{1}{2}\left([J_1,T_i][\bar{J}_1,T_j] + [\bar{J}_1,T_i][J_1,T_j]\right)g^{ij}\\
&+\frac{1}{2}\left([\bar{J}_2,T_{\hat{\alpha}}][N,T_\alpha] - [J_2,T_{\hat{\alpha}}][\hat{N},T_\alpha] + 3[\hat{N},T_\alpha][J_2,T_\alpha] + [N,T_\alpha][\bar{J}_2,T_{\hat{\alpha}}]\right)\eta^{\alpha\hat{\alpha}}
\end{aligned}$$

$$
\begin{aligned}
\langle J_1, J_2\rangle =& -[N,T_\alpha][J_2,T_{\hat\alpha}]\eta^{\alpha\hat\alpha} + [J_3,T_m][J_3,T_n]\eta^{mn}\\
\langle \bar J_1, \bar J_2\rangle =& 0\\
\langle J_1, \bar J_2\rangle =& \frac{1}{8}[5\partial\bar J_3 - \bar\partial J_3, T_m]T_n\eta^{mn} + \frac{1}{8}\left(11[J_2,T_{\hat\alpha}][\bar J_1,T_\alpha] + 5[\bar J_2,T_{\hat\alpha}][J_1,T_\alpha]\right)\eta^{\alpha\hat\alpha}\\
&+ \frac{1}{8}\left(5[\bar J_1,T_i][J_2,T_j] + 3[J_1,T_i][\bar J_2,T_j]\right)g^{ij} + \frac{1}{2}\Big([N,T_a][\bar J_3,T_{\hat\alpha}] - [\hat N,T_\alpha][J_3,T_{\hat\alpha}]\Big)\eta^{\alpha\hat\alpha}\\
&- \frac{1}{2}\Big(3[\bar J_3,T_m][N,T_n] + [J_3,T_m][\hat N,T_n]\Big)\eta^{mn}\\
\langle \bar J_1, J_2\rangle =& -\frac{1}{8}[3\partial\bar J_3 + \bar\partial J_3, T_m]T_n\eta^{mn} + \frac{3}{8}\left([J_2,T_{\hat\alpha}][\bar J_1,T_\alpha] - [\bar J_2,T_{\hat\alpha}][J_1,T_\alpha]\right)\eta^{\alpha\hat\alpha}\\
&+ \frac{1}{8}\left(5[\bar J_1,T_i][J_2,T_j] + 3[J_1,T_i][\bar J_2,T_j]\right)g^{ij} + \frac{1}{2}\Big(3[N,T_a][\bar J_3,T_{\hat\alpha}] + [\hat N,T_\alpha][J_3,T_{\hat\alpha}]\Big)\eta^{\alpha\hat\alpha}\\
&- \frac{1}{2}\Big([\bar J_3,T_m][N,T_n] - [J_3,T_m][\hat N,T_n]\Big)\eta^{mn}
\end{aligned}
$$

$$
\begin{aligned}
\langle J_1, J_3\rangle =& -[N,T_\alpha][N,T_{\hat\alpha}]\eta^{\alpha\hat\alpha}\\
\langle \bar J_1, \bar J_3\rangle =& -[\hat N,T_\alpha][\hat N,T_{\hat\alpha}]\eta^{\alpha\hat\alpha}\\
\langle J_1, \bar J_3\rangle =& \Big([N,T_\alpha][\hat N,T_{\hat\alpha}] + [\hat N,T_\alpha][N,T_{\hat\alpha}]\Big)\eta^{\alpha\hat\alpha} + \frac{1}{4}\left(3[\bar J_2,T\hat\alpha][J_2,T_a] + 5[J_2,T_{\hat\alpha}][\bar J_2,T_\alpha]\right)\eta^{\alpha\hat\alpha}\\
&+ \frac{1}{4}\left(5[\bar J_3,T_m][J_1,T_n] + 3[J_3,T_m][J_1,T_n]\right)\eta^{mn} + \frac{1}{4}\left([J_1,T_i][\bar J_3,T_j] + 3[\bar J_1,T_i][J_3,T_j]\right)g^{ij}\\
\langle \bar J_1, J_3\rangle =& \Big([N,T_\alpha][\hat N,T_{\hat\alpha}] + [\hat N,T_\alpha][N,T_{\hat\alpha}]\Big)\eta^{\alpha\hat\alpha} - \frac{1}{4}\left([\bar J_2,T\hat\alpha][J_2,T_a] - [J_2,T_{\hat\alpha}][\bar J_2,T_\alpha]\right)\eta^{\alpha\hat\alpha}\\
&+ \frac{1}{4}\left([\bar J_3,T_m][J_1,T_n] - [J_3,T_m][J_1,T_n]\right)\eta^{mn} + \frac{1}{4}\left([J_1,T_i][\bar J_3,T_j] + 3[\bar J_1,T_i][J_3,T_j]\right)g^{ij}
\end{aligned}
$$

$$
\begin{aligned}
\langle J_2, J_2\rangle =& [N,T_m][N,T_n]\eta^{mn}\\
\langle \bar J_2, \bar J_2\rangle =& [\hat N,T_m][\hat N,T_n]\eta^{mn}\\
\langle \bar J_2, J_2\rangle =& -\Big([N,T_m][\hat N,T_n] + [\hat N,T_m][N,T_n]\Big)\eta^{mn} + \frac{1}{2}\left([J_2,T_i][\bar J_2,T_j] + [\bar J_2,T_i][J_2,T_j]\right)g^{ij}\\
&- \frac{1}{2}\left([J_1,T_\alpha][\bar J_3,T_{\hat\alpha}] - 3[\bar J_3,T_{\hat\alpha}][J_1,T_\alpha] + 3[\bar J_1,T_\alpha][J_3,T_{\hat\alpha}] - [J_3,T_{\hat\alpha}][\bar J_1,T_\alpha]\right)\eta^{\alpha\hat\alpha}\\
\langle J_2, \bar J_2\rangle =& -\Big([N,T_m][\hat N,T_n] + [\hat N,T_m][N,T_n]\Big)\eta^{mn} + \frac{1}{2}\left([J_2,T_i][\bar J_2,T_j] + [\bar J_2,T_i][J_2,T_j]\right)g^{ij}\\
&+ \frac{1}{2}\left([\bar J_3,T_{\hat\alpha}][J_1,T_\alpha] - 3[J_1,T_\alpha][\bar J_3,T_{\hat\alpha}] + 3[J_1,T_\alpha][\bar J_3,T_{\hat\alpha}] - [\bar J_1,T_\alpha][J_3,T_{\hat\alpha}]\right)\eta^{\alpha\hat\alpha}
\end{aligned}
$$

$$
\begin{aligned}
\langle J_3, J_2\rangle =& 0\\
\langle \bar J_3, \bar J_2\rangle =& -[\hat N,T_{\hat\alpha}][\bar J_1,T_\alpha]\eta^{\alpha\hat\alpha} - [\bar J_1,T_m][\hat N,T_n]\eta^{mn}\\
\langle J_3, J_2\rangle =& \frac{1}{8}[5\bar\partial J_1 - \partial\bar J_1, T_m]Tn\eta^{mn} + \frac{1}{2}\Big([\bar J_1,T_m][N,T_n] + 3[J_1,T_m][\hat N,T_n]\Big)\eta^{mn}\\
&+ \frac{1}{2}\Big([\hat N,T_{\hat\alpha}][J_1,T_\alpha] - [N,T_{\hat\alpha}][\bar J_1,T_\alpha]\Big)\eta^{\alpha\hat\alpha} + \frac{1}{8}\left(3[\bar J_3,T_i][J_2,T_j] + 5[J_3,T_i][\bar J_2,T_j]\right)g^{ij}\\
&- \frac{1}{8}\left(5[J_2,T_\alpha][\bar J_3,T_{\hat\alpha}] + 11[\bar J_2,T_\alpha][J_3,T_{\hat\alpha}]\right)\eta^{\alpha\hat\alpha}\\
\langle J_3, J_2\rangle =& -\frac{1}{8}[3\bar\partial J_1 + \partial\bar J_1, T_m]Tn\eta^{mn} - \frac{1}{2}\Big([\bar J_1,T_m][N,T_n] - [J_1,T_m][\hat N,T_n]\Big)\eta^{mn}\\
&+ \frac{1}{2}\Big(3[\hat N,T_{\hat\alpha}][J_1,T_\alpha] + [N,T_{\hat\alpha}][\bar J_1,T_\alpha]\Big)\eta^{\alpha\hat\alpha} + \frac{1}{8}\left(3[\bar J_3,T_i][J_2,T_j] + 5[J_3,T_i][\bar J_2,T_j]\right)g^{ij}\\
&+ \frac{3}{8}\left([J_2,T_\alpha][\bar J_3,T_{\hat\alpha}] - [\bar J_2,T_\alpha][J_3,T_{\hat\alpha}]\right)\eta^{\alpha\hat\alpha}
\end{aligned}
$$

$$\begin{aligned}
\langle J_3, J_3\rangle =&0\\
\langle \bar{J}_3, \bar{J}_3\rangle =&\left([\bar{J}_2, T_\alpha][\hat{N}, T_{\hat{\alpha}}] - [\hat{N}, T_{\hat{\alpha}}][\bar{J}_2, T_\alpha]\right)\eta^{\alpha\hat{\alpha}} + [\bar{J}_1, T_m][\bar{J}_1, T_n]\eta^{mn}\\
\langle \bar{J}_3, J_3\rangle =&-\frac{1}{2}[\bar{\partial} J_2, T_\alpha]T_{\hat{\alpha}}\eta^{\alpha\hat{\alpha}} + \frac{1}{2}\left([J_3, T_i][\bar{J}_3, T_j] + [\bar{J}_3, T_i][J_3, T_j]\right)g^{ij}\\
&+\frac{1}{2}\left([N, T_{\hat{\alpha}}][\bar{J}_2, T_\alpha] - [\hat{N}, T_{\hat{\alpha}}][J_2, T_\alpha] - 3[\bar{J}_2, T_\alpha][N, T_{\hat{\alpha}}] - [J_2, T_\alpha][\hat{N}, T_{\hat{\alpha}}]\right)\eta^{\alpha\hat{\alpha}}\\
\langle J_3, \bar{J}_3\rangle =&\frac{1}{2}[\bar{\partial} J_2, T_\alpha]T_{\hat{\alpha}}\eta^{\alpha\hat{\alpha}} + \frac{1}{2}\left([J_3, T_i][\bar{J}_3, T_j] + [\bar{J}_3, T_i][J_3, T_j]\right)g^{ij}\\
&+\frac{1}{2}\left(3[N, T_{\hat{\alpha}}][\bar{J}_2, T_\alpha] + [\hat{N}, T_{\hat{\alpha}}][J_2, T_\alpha] - [\bar{J}_2, T_\alpha][N, T_{\hat{\alpha}}] + [J_2, T_\alpha][\hat{N}, T_{\hat{\alpha}}]\right)\eta^{\alpha\hat{\alpha}}
\end{aligned}$$

$$\begin{aligned}
\langle J, N\rangle =&0\\
\langle \bar{J}, \hat{N}\rangle =&0
\end{aligned}$$

$$\begin{aligned}
\langle \bar{K}, N\rangle =&-[\bar{K}, T_i][N, T_j]g^{ij}\\
\langle K, \hat{N}\rangle =&-[K, T_i][\hat{N}, T_j]g^{ij}
\end{aligned}$$

$$\langle N, \hat{N}\rangle = -[N, T_i][\hat{N}, T_j]g^{ij}$$

9 Super Jacobi Identities and other useful identities.

Let A, B and C be bosons, X, Y and Z fermions, then, the generalized Jacobi Identities are

$$\begin{aligned}
[A,[B,C]] + [B,[C,A]] + [C,[A,B]] =&0\\
[A,[B,X]] + [B,[X,A]] + [X,[A,B]] =&0\\
\{X,[Y,A]\} + \{Y,[X,A]\} + [A,\{X,Y\}] =&0\\
[X,\{Y,Z\}] + [Y,\{Z,X\}] + [Z,\{X,Y\}] =&0.
\end{aligned}$$

In this theory the Dual-Coxeter Number is 0, this implies

$$\begin{aligned}
[[A,T_i],T_j]g^{ij} + \{[A,T_\alpha],T_{\hat{\alpha}}\}\eta^{\alpha\hat{\alpha}} + [[A,T_m],T_n]\eta^{mn} - \{[A,T_{\hat{\alpha}}],T_\alpha\}\eta^{\alpha\hat{\alpha}} =&0\\
[[X,T_i],T_j]g^{ij} + [\{X,T_\alpha\},T_{\hat{\alpha}}]\eta^{\alpha\hat{\alpha}} + [[X,T_m],T_n]\eta^{mn} - [\{X,T_{\hat{\alpha}}\},T_\alpha]\eta^{\alpha\hat{\alpha}} =&0.
\end{aligned}$$

It is also true that $f_{m\alpha\beta}f_{n\hat{\alpha}\hat{\beta}}\eta^{mn}\eta^{\alpha\hat{\alpha}} = f_{i\alpha\hat{\beta}}f_{j\hat{\alpha}\beta}g^{ij}\eta^{\alpha\hat{\alpha}} = 0$. This implies that

$$\begin{aligned}
[[J_{1,3},T_i],T_j]g^{ij} = [[J_{1,3},T_n],T_m]\eta^{mn} = \{[J_{1,3},T_\alpha],T_{\hat{\alpha}}\}\eta^{\alpha\hat{\alpha}} = \{[J_{1,3},T_{\hat{\alpha}}],T_\alpha\}\eta^{\alpha\hat{\alpha}} =&0\\
[[\omega+\lambda+\hat{\omega}+\hat{\lambda},T_i],T_j]g^{ij} = [[\omega+\lambda+\hat{\omega}+\hat{\lambda},T_n],T_m]\eta^{mn} =&0\\
[\{\omega+\lambda+\hat{\omega}+\hat{\lambda},T_\alpha\},T_{\hat{\alpha}}]\eta^{\alpha\hat{\alpha}} = [\{\omega+\lambda+\hat{\omega}+\hat{\lambda},T_{\hat{\alpha}}\},T_\alpha]\eta^{\alpha\hat{\alpha}} =&0
\end{aligned}$$

10 $\langle T\rangle$

$$\begin{aligned}
T =&\mathrm{STr}\left(\frac{1}{2}J_2J_2 + J_1J_3 - \omega\nabla\lambda\right)\\
\bar{T} =&\mathrm{STr}\left(\frac{1}{2}\bar{J}_2\bar{J}_2 + \bar{J}_1\bar{J}_3 - \hat{\omega}\bar{\nabla}\hat{\lambda}\right)
\end{aligned}$$

$$
\begin{aligned}
\langle T\rangle =&\mathrm{STr}\left(\frac{1}{2}\langle J_2,J_2\rangle+\langle J_1,J_3\rangle+N\langle J_0\rangle\right)\\
=&\mathrm{SRr}\left(\frac{1}{2}[N,T_m][N,T_n]\eta^{mn}-[N,T_\alpha][N,T_{\hat\alpha}]\eta^{\alpha\hat\alpha}+\frac{1}{2}N\left(\{[N,T_{\hat\alpha}],T_\alpha\}\eta^{\alpha\hat\alpha}-\{[N,T_\alpha],T_{\hat\alpha}\}\eta^{\alpha\hat\alpha}+[[N,T_m],T_n]\eta^{mn}\right)\right)\\
=&0
\end{aligned}
$$

$$
\begin{aligned}
\langle \bar T\rangle =&\mathrm{STr}\left(\frac{1}{2}\langle \bar J_2,\bar J_2\rangle+\langle \bar J_1,\bar J_3\rangle-\hat N\langle \bar J_0\rangle\right)\\
=&\mathrm{SRr}\left(\frac{1}{2}[\hat N,T_m][\hat N,T_n]\eta^{mn}-[\hat N,T_\alpha][\hat N,T_{\hat\alpha}]\eta^{\alpha\hat\alpha}+\frac{1}{2}\hat N\left(\{[\hat N,T_{\hat\alpha}],T_\alpha\}\eta^{\alpha\hat\alpha}-\{[\hat N,T_\alpha],T_{\hat\alpha}\}\eta^{\alpha\hat\alpha}+[[\hat N,T_m],T_n]\eta^{mn}\right)\right)\\
=&0
\end{aligned}
$$

11 Conserved Currents

11.1 j

$$
\begin{aligned}
j =&g\left(J_2+\frac{3}{2}J_3+\frac{1}{2}J_1+2N\right)g^{-1}=gAg^{-1}\\
\bar j =&g\left(\bar J_2+\frac{1}{2}\bar J_3+\frac{3}{2}\bar J_1+2\hat N\right)g^{-1}=g\bar A g^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\langle j\rangle =&\langle g\rangle A_0g_0^{-1}+\langle g,A\rangle g_0^{-1}+\langle g,A_0,g^{-1}\rangle+g_0\langle A,g^{-1}\rangle+g_0A_0\langle g^{-1}\rangle\\
\langle g\rangle =&\frac{1}{2}g_0\left(T_mT_n\eta^{mn}+T_{\hat\alpha}T_\alpha\eta^{\alpha\hat\alpha}-T_\alpha T_{\hat\alpha}\eta^{\alpha\hat\alpha}\right)\\
\langle g^{-1}\rangle =&\frac{1}{2}\left(T_mT_n\eta^{mn}+T_{\hat\alpha}T_\alpha\eta^{\alpha\hat\alpha}-T_\alpha T_{\hat\alpha}\eta^{\alpha\hat\alpha}\right)g_0^{-1}\\
\langle g,A_0,g^{-1}\rangle =&g_0\left(T_\alpha A_0T_{\hat\alpha}\eta^{\alpha\hat\alpha}-T_{\hat\alpha}A_0T_\alpha\eta^{\alpha\hat\alpha}-T_mA_0T_n\eta^{mn}\right)g_0^{-1}\\
\langle g\rangle A_0g_0^{-1}+\langle g,A_0,g^{-1}\rangle+g_0A_0\rangle g^{-1}\rangle =&\frac{1}{2}g_0\left([[A_0,T_m],T_n]\eta^{mn}+\{[A_0,T_{\hat\alpha}],T_\alpha\}\eta^{\alpha\hat\alpha}-\{[A_0,T_\alpha],T_{\hat\alpha}\}\eta^{\alpha\hat\alpha}\right)g_0^{-1}\\
\langle A\rangle =&0\\
g_0^{-1}\langle g,J_1\rangle+\langle J_1,g^{-1}\rangle g_0 =&-\{[J_2,T_{\hat\alpha}],T_\alpha\}\eta^{\alpha\hat\alpha}-\{[J_3,T_{\hat\alpha}],T_\alpha\}\eta^{\alpha\hat\alpha}-\{[N,T_{\hat\alpha}],T_\alpha\}\eta^{\alpha\hat\alpha}\\
g_0^{-1}\langle g,J_2\rangle+\langle J_2,g^{-1}\rangle g_0 =&-[[J_3,T_m],T_n]\eta^{mn}-[[N,T_m],T_n]\eta^{mn}\\
g_0^{-1}\langle g,J_3\rangle+\langle J_3,g^{-1}\rangle g_0 =&\{[N,T_\alpha],T_{\hat\alpha}\}\eta^{\alpha\hat\alpha}\\
g_0^{-1}\langle g,N\rangle+\langle N,g^{-1}\rangle g_0 =&0\\
\Rightarrow\langle j\rangle =&0
\end{aligned}
$$

$$
\begin{aligned}
\langle \bar j\rangle =&\langle g\rangle \bar A_0g_0^{-1}+\langle g,\bar A\rangle g_0^{-1}+\langle g,\bar A_0,g^{-1}\rangle+g_0\langle \bar A,g^{-1}\rangle+g_0\bar A_0\langle g^{-1}\rangle\\
g_0^{-1}\langle g,\bar J_1\rangle+\langle \bar J_1,g^{-1}\rangle g_0 =&\{[\hat N,T_{\hat\alpha}],T_\alpha\}\eta^{\alpha\hat\alpha}\\
g_0^{-1}\langle g,\bar J_2\rangle+\langle \bar J_2,g^{-1}\rangle g_0 =&-[[\bar J_1,T_m],T_n]\eta^{mn}+[[\hat N,T_m],T_n]\eta^{mn}\\
g_0^{-1}\langle g,\bar J_3\rangle+\langle \bar J_3,g^{-1}\rangle g_0 =&\{[\bar J_1,T_\alpha],T_{\hat\alpha}\}\eta^{\alpha\hat\alpha}+\{[\bar J_2,T_\alpha],T_{\hat\alpha}\}\eta^{\alpha\hat\alpha}-\{[\hat N,T_\alpha],T_{\hat\alpha}\}\eta^{\alpha\hat\alpha}\\
g_0^{-1}\langle g,\hat N\rangle+\langle \hat N,g^{-1}\rangle g_0 =&0\\
\Rightarrow\langle \bar j\rangle =&0
\end{aligned}
$$

11.2 b

$$
\begin{aligned}
b =&(\lambda\hat\lambda)^{-1}\mathrm{STr}\left(\hat\lambda[J_2,J_3]+\{\omega,\hat\lambda\}[\lambda,J_1]\right)-\mathrm{STr}\left(\omega J_1\right)\\
\bar b =&(\lambda\hat\lambda)^{-1}\mathrm{STr}\left(\lambda[\bar J_2,\bar J_1]+\{\hat\omega,\lambda\}[\hat\lambda,\bar J_3]\right)-\mathrm{STr}\left(\hat\omega \bar J_3\right)\\
(\lambda\hat\lambda) =&\lambda^A\hat\lambda^{\hat A}\eta_{A\hat A}
\end{aligned}
$$

$$
\begin{aligned}
\langle b\rangle =&(\lambda\hat\lambda)^{-1}\mathrm{STr}\langle\hat\lambda[J_2,J_3]+\{\omega,\hat\lambda\}[\lambda,J_1]\rangle-\mathrm{STr}\langle\omega J_1\rangle-(\lambda\hat\lambda)^{-2}\langle\lambda\hat\lambda\rangle\mathrm{STr}\left(\hat\lambda[J_2,J_3]+\{\omega,\hat\lambda\}[\lambda,J_1]\right)\\
&-(\lambda\hat\lambda)^{-2}\langle(\lambda\hat\lambda),\mathrm{STr}\left(\hat\lambda[J_2,J_3]+\{\omega,\hat\lambda\}[\lambda,J_1]\right)\rangle
\end{aligned}
$$

$$
\begin{aligned}
\langle(\lambda\hat\lambda)\rangle =&-\lambda^A\hat\lambda^{\hat A}f_{Ai}^{B}f_{\hat A j}^{\hat B}g^{ij}\eta_{B\hat B}=0\\
\mathrm{STr}\langle\hat\lambda[J_2,J_3]\rangle =&-\mathrm{STr}\left([\hat\lambda,T_i]([[J_2,T_j],J_3]+[J_2,[J_3,T_j]])g^{ij}\right)\\
=&-\mathrm{STr}\left([\hat\lambda,T_i][T_j,[J_2,J_3]]g^{ij}\right)=-\mathrm{STr}\left([[\hat\lambda,T_i],T_j][J_2,J_3]g^{ij}\right)=0\\
\mathrm{STr}\langle\{\omega,\hat\lambda\}[\lambda,J_1]\rangle =&-\mathrm{STr}\left(\{[\omega,T_i],[\hat\lambda,T_j]\}[\lambda,J_1]+\{\omega,[\hat\lambda,T_i]\}[[\lambda,T_j],J_1]+\{\omega,[\hat\lambda,T_i]\}[\lambda,[J_1,T_j]]\right)g^{ij}\\
=&-\mathrm{STr}\left(\{\omega,[\hat\lambda,T_i]\}([T_j,[\lambda,J_1]]+[[\lambda,T_j],J_1]+[\lambda,[J_1,T_j]])\right)g^{ij}=0\\
\lambda^A[\hat\lambda^{\hat A},T_i]\eta_{A\hat A} =&-[\lambda^A,T_i]\hat\lambda^{\hat A}\eta_{A\hat A}=\{\lambda,\hat\lambda\}^jg_{ij}\\
\langle(\lambda\hat\lambda),\mathrm{STr}\left(\hat\lambda[J_2,J_3]\right)\rangle =&\mathrm{STr}\left([\hat\lambda,\{\lambda,\hat\lambda\}][J_2,J_3]\right)+\mathrm{STr}\left(\hat\lambda[[J_2,J_3],\{\lambda,\hat\lambda\}]\right)=0\\
\langle(\lambda\hat\lambda),\mathrm{STr}\left(\{\omega,\hat\lambda\}[\lambda,J_1]\right)\rangle =&\mathrm{STr}\left(\{\omega,[\hat\lambda,\{\lambda,\hat\lambda\}]\}[\lambda,J_1]-\{[\omega,\{\lambda,\hat\lambda\}],\hat\lambda\}[\lambda,J_1]+[\{\omega,\hat\lambda\},\{\lambda,\hat\lambda\}][\lambda,J_1]\right)\\
=&2\mathrm{STr}\left(\{\omega,[\hat\lambda,\{\lambda,\hat\lambda\}]\}[\lambda,J_1]\right)=0
\end{aligned}
$$

For $\langle\bar b\rangle$ one needs to use the same relations from above.