Homework Problems
for the String Theory Lectures

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1 Problems for Lecture 1

Problem 1

Consider a $p$-brane action of the form
\[ S_\sigma = -\frac{T_p}{2} \int d^{p+1}\sigma \sqrt{-h} h^{\alpha\beta} G_{\alpha\beta}(X) + \Lambda_p \int d^{p+1}\sigma \sqrt{-h}, \] (1)
where the induced metric is given by
\[ G_{\alpha\beta} = g_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu \quad \alpha, \beta = 0, \ldots, p \] (2)
where $\partial_\alpha$ denotes the partial derivative with respect to $\sigma^\alpha$, and $g_{\mu\nu}$ is the spacetime metric tensor in $D$ dimensions. Also, $h_{\alpha\beta}(\sigma)$ denotes a world-volume metric tensor in $p+1$ dimensions (with Lorentzian signature), $h^{\alpha\beta}$ denotes its inverse, and $h$ denotes its determinant.

a) Derive the equation of motion obtained by varying $h^{\alpha\beta}$.

b) Show that this equation is solved by $h_{\alpha\beta} = G_{\alpha\beta}$ provided that $\Lambda_p = c_p T_p$ for a certain choice of $c_p$. Verify that for this value of $c_1$ the $p = 1$ (or string) action is invariant under the Weyl transformation $h_{\alpha\beta}(\sigma) \to \exp(\phi(\sigma)) h_{\alpha\beta}(\sigma)$.

c) Use this solution to establish classical equivalence to the “volume action”
\[ S_p = -T_p \int \sqrt{-\det G_{\alpha\beta}} d^{p+1}\sigma, \] (3)

Problem 2

Consider an on-shell open-string state of the bosonic string theory in $D$-dimensional Minkowski spacetime of the form
\[ |\phi\rangle = (A\alpha_{-1} \cdot \alpha_{-1} + B\alpha_0 \cdot \alpha_{-2} + C(\alpha_0 \cdot \alpha_{-1})^2) |0; k\rangle, \]
where $A$, $B$ and $C$ are constants.

a) Determine the eigenvalue of $\alpha_0 \cdot \alpha_0$ and relations among the coefficients $A$, $B$ and $C$ so that $|\phi\rangle$ satisfies the physical-state conditions $(L_0 - 1)|\phi\rangle = L_1|\phi\rangle = L_2|\phi\rangle = 0$ for arbitrary $D$. Recall that $[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}$ and
\[ L_0 = \frac{1}{2}\alpha_0^2 + \alpha_{-1} \cdot \alpha_{-1} + \alpha_{-2} \cdot \alpha_{-2} + \ldots, \quad L_1 = \alpha_0 \cdot \alpha_1 + \alpha_{-1} \cdot \alpha_2 + \ldots, \quad L_2 = \frac{1}{2}\alpha_1 \cdot \alpha_1 + \alpha_0 \cdot \alpha_2 + \ldots \]

b) Compute the norm of $|\phi\rangle$. What conclusion can you draw from the result?
2 Problems for Lecture 2

Problem 3

Show that there are the same number of physical degrees of freedom in the NS and R sectors at the first massive level of the ten-dimensional open superstring after GSO projection.

Suggestion: The counting is most easily carried out in light-cone gauge. First, show that at the first massive level, the NS states have $N = 3/2$ and the R states have $N = 1$.

Problem 4

The conclusion that T-duality interchanges

$$X(\tau, \sigma) = X_L(\tau + \sigma) + X_R(\tau - \sigma)$$

and

$$\tilde{X}(\tau, \sigma) = X_L(\tau + \sigma) - X_R(\tau - \sigma)$$

can be understood from a world-sheet viewpoint. Consider the following world-sheet action:

$$\int \left( \frac{1}{2} V^\alpha \partial_\alpha - \epsilon^{\alpha \beta} X \partial_\beta V_\alpha \right) d^2 \sigma,$$

where an overall constant coefficient is omitted, because the considerations that follow are classical.

a) Show that varying $X$, which acts as a Lagrange multiplier, gives an equation of motion that is solved by $V_\alpha = \partial_\alpha \tilde{X}$. This can then be used to obtain an action that only depends on $\tilde{X}$. Alternatively, varying $V_\alpha$ in the original action gives an equation of motion that can be used to obtain an action that only depends on $X$. Show that the resulting actions for $X$ and $\tilde{X}$ have the same form. Thus, the (nonlocal) transformation from $X$ to $\tilde{X}$ is a symmetry.

b) Use the equations you found in part a) to deduce a relationship between the derivatives of $\tilde{X}$ and $X$. Verify that this is exactly the relationship expected for T-duality.
3 Problems for Lecture 3

Problem 5

Let $A$ be a positive definite $m \times m$ symmetric matrix and define

$$f(A) = \sum_{\{M\}} \exp \left(-\pi M^T AM\right). \quad (5)$$

Here $M$ represents a vector made of $m$ integers $M_1, M_2, \ldots, M_m$ each of which is summed from $-\infty$ to $+\infty$. Derive the Poisson resummation formula:

$$f(A) = \frac{1}{\sqrt{\det A}} f(A^{-1}). \quad (6)$$

Suggestion: it is helpful to add dependence on $m$ variables $x^i$ and define

$$f(A, x) = \sum_{\{M\}} \exp \left(-\pi (M + x)^T A(M + x)\right). \quad (7)$$

This function is periodic, with period 1, in each of the $x^i$. Therefore, it must have a Fourier series expansion of the form

$$f(A, x) = \sum_{\{N\}} C_N(A) \exp(2\pi i N^T x). \quad (8)$$

Evaluate the Fourier coefficients $C_N(A)$ explicitly. (Hint: the infinite sum of integrals over an interval of length one can be converted to a single integral over an infinite interval.) Finally, the desired formula is obtained by setting $x = 0$.

The Poisson resummation formula is crucial for establishing the modular invariance of one-loop closed-string amplitudes. Such amplitudes are given as an integral over conformal equivalence classes of a torus, which is the one-loop world sheet. These classes are parametrized by a modular parameter $\tau$. The integrand (including the measure) should be invariant under $SL(2, \mathbb{Z})$ transformations of the form $\tau \rightarrow (a\tau + b)/(c\tau + d)$ in order that the integral not depend on the choice of a fundamental domain. The Poisson resummation formula is required for proving the symmetry for the S transformation $\tau \rightarrow -1/\tau$. The entire group is generated by the S transformation and the T transformation: $\tau \rightarrow \tau + 1$, which is usually trivial.
Problem 6

The type IIB theory has an infinite spectrum of half-BPS \((p, q)\) strings in ten dimensions, which can be regarded as bound states of \(p\) fundamental strings and \(q\) D-strings. \(p\) and \(q\) are integer charges that measure the strengths with which the strings couple to the two-form gauge fields \(B_2\) and \(C_2\). These strings are stable whenever \(p\) and \(q\) are coprime. Their tensions (in the Einstein frame) are given by

\[ T_{p,q} = |p + q\tau|T_F, \tag{9} \]

where \(T_F\), the tension of the fundamental \((1,0)\) string, is

\[ T_F = \frac{1}{2\pi l_S^2} = \frac{1}{2\pi \sqrt{\tau_2 l_B^2}}. \tag{10} \]

Here \(l_S\) denotes the fundamental string length scale and \(l_B\) denotes the ten-dimensional Planck length in the type IIB superstring theory. \(\tau\) is the vacuum value of the complex scalar field \(C_0 + i\exp(-\Phi)\) in the type IIB theory, and \(\tau_2\) denotes the imaginary part of \(\tau\).

Verify that the \(\tau\) dependence of these equations is consistent with the fact that the string charges \(p\) and \(q\) transform linearly under \(SL(2, \mathbb{Z})\) transformations \(\tau \rightarrow (a\tau + b)/(c\tau + d)\). This is required because the 2-form gauge fields also transform as a doublet. Remark: It is important that the Planck length \(l_B\), and not the string length scale \(l_S\), is \(SL(2, \mathbb{Z})\) invariant.