Homework Problems
for the String Theory Lectures

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1 Problems for Lecture 1

Problem 1

Consider a p-brane action of the form

$$S_\sigma = -\frac{T_p}{2}\int d^{p+1}\sigma\sqrt{-h}h^{\alpha\beta}G_{\alpha\beta}(X) + \Lambda_p\int d^{p+1}\sigma\sqrt{-h}, \tag{1}$$

where the induced metric is given by

$$G_{\alpha\beta} = g_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu \quad \alpha, \beta = 0, \ldots, p \tag{2}$$

where ∂_α denotes the partial derivative with respect to σ^α, and $g_{\mu\nu}$ is the spacetime metric tensor in D dimensions. Also, $h_{\alpha\beta}(\sigma)$ denotes a world-volume metric tensor in $p+1$ dimensions (with Lorentzian signature), $h^{\alpha\beta}$ denotes its inverse, and h denotes its determinant.

a) Derive the equation of motion obtained by varying $h^{\alpha\beta}$.

b) Show that this equation is solved by $h_{\alpha\beta} = G_{\alpha\beta}$ provided that $\Lambda_p = c_p T_p$ for a certain choice of c_p. Verify that for this value of c_1 the $p=1$ (or string) action is invariant under the Weyl transformation $h_{\alpha\beta}(\sigma) \to \exp(\phi(\sigma))h_{\alpha\beta}(\sigma)$.

c) Use this solution to establish classical equivalence to the "volume action"

$$S_p = -T_p\int \sqrt{-\det G_{\alpha\beta}}\, d^{p+1}\sigma, \tag{3}$$

Problem 2

Consider an on-shell open-string state of the bosonic string theory in D-dimensional Minkowski spacetime of the form

$$|\phi\rangle = \left(A\alpha_{-1}\cdot\alpha_{-1} + B\alpha_0\cdot\alpha_{-2} + C(\alpha_0\cdot\alpha_{-1})^2\right)|0;k\rangle,$$

where A, B and C are constants.

a) Determine the eigenvalue of $\alpha_0\cdot\alpha_0$ and relations among the coefficients A, B and C so that $|\phi\rangle$ satisfies the physical-state conditions $(L_0-1)|\phi\rangle = L_1|\phi\rangle = L_2|\phi\rangle = 0$ for arbitrary D. Recall that $[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}$ and

$$L_0 = \frac{1}{2}\alpha_0^2 + \alpha_{-1}\cdot\alpha_1 + \alpha_{-2}\cdot\alpha_2 + \ldots,\; L_1 = \alpha_0\cdot\alpha_1 + \alpha_{-1}\cdot\alpha_2 + \ldots,\; L_2 = \frac{1}{2}\alpha_1\cdot\alpha_1 + \alpha_0\cdot\alpha_2 + \ldots.$$

b) Compute the norm of $|\phi\rangle$. What conclusion can you draw from the result?

2 Problems for Lecture 2

Problem 3

Show that there are the same number of physical degrees of freedom in the NS and R sectors at the first massive level of the ten-dimensional open superstring after GSO projection.

Suggestion: The counting is most easily carried out in light-cone gauge. First, show that at the first massive level, the NS states have $N = 3/2$ and the R states have $N = 1$.

Problem 4

The conclusion that T-duality interchanges

$$X(\tau, \sigma) = X_L(\tau + \sigma) + X_R(\tau - \sigma)$$

and

$$\widetilde{X}(\tau, \sigma) = X_L(\tau + \sigma) - X_R(\tau - \sigma)$$

can be understood from a world-sheet viewpoint. Consider the following world-sheet action:

$$\int \left(\frac{1}{2} V^\alpha V_\alpha - \epsilon^{\alpha\beta} X \partial_\beta V_\alpha\right) d^2\sigma, \tag{4}$$

where an overall constant coefficient is omitted, because the considerations that follow are classical.

a) Show that varying X, which acts as a Lagrange multiplier, gives an equation of motion that is solved by $V_\alpha = \partial_\alpha \widetilde{X}$. This can then be used to obtain an action that only depends on $\widetilde{X}$. Alternatively, varying V_α in the original action gives an equation of motion that can be used to obtain an action that only depends on X. Show that the resulting actions for X and $\widetilde{X}$ have the same form. Thus, the (nonlocal) transformation from X to $\widetilde{X}$ is a symmetry.

b) Use the equations you found in part a) to deduce a relationship between the derivatives of $\widetilde{X}$ and X. Verify that this is exactly the relationship expected for T-duality.

3 Problems for Lecture 3

Problem 5

Let A be a positive definite $m \times m$ symmetric matrix and define

$$f(A) = \sum_{\{M\}} \exp\left(-\pi M^T A M\right) . \tag{5}$$

Here M represents a vector made of m integers $M_1, M_2, \ldots, M_m$ each of which is summed from $-\infty$ to $+\infty$. Derive the Poisson resummation formula:

$$f(A) = \frac{1}{\sqrt{\det A}} f(A^{-1}). \tag{6}$$

Suggestion: it is helpful to add dependence on m variables x^i and define

$$f(A, x) = \sum_{\{M\}} \exp\left(-\pi (M+x)^T A (M+x)\right) . \tag{7}$$

This function is periodic, with period 1, in each of the x^i. Therefore, it must have a Fourier series expansion of the form

$$f(A, x) = \sum_{\{N\}} C_N(A) \exp(2\pi i N^T x). \tag{8}$$

Evaluate the Fourier coefficients $C_N(A)$ explicitly. (Hint: the infinite sum of integrals over an interval of length one can be converted to a single integral over an infinite interval.) Finally, the desired formula is obtained by setting $x = 0$.

The Poisson resummation formula is crucial for establishing the modular invariance of one-loop closed-string amplitudes. Such amplitudes are given as an integral over conformal equivalence classes of a torus, which is the one-loop world sheet. These classes are parametrized by a modular parameter τ. The integrand (including the measure) should be invariant under $SL(2, \mathbb{Z})$ transformations of the form $\tau \to (a\tau + b)/(c\tau + d)$ in order that the integral not depend on the choice of a fundamental domain. The Poisson resummation formula is required for proving the symmetry for the S transformation $\tau \to -1/\tau$. The entire group is generated by the S transformation and the T transformation: $\tau \to \tau + 1$, which is usually trivial.

Problem 6

The type IIB theory has an infinite spectrum of half-BPS (p,q) strings in ten dimensions, which can be regarded as bound states of p fundamental strings and q D-strings. p and q are integer charges that measure the strengths with which the strings couple to the two-form gauge fields B_2 and C_2. These strings are stable whenever p and q are coprime. Their tensions (in the Einstein frame) are given by

$$T_{p,q} = |p + q\tau| T_F, \tag{9}$$

where T_F, the tension of the fundamental $(1,0)$ string, is

$$T_F = \frac{1}{2\pi l_S^2} = \frac{1}{2\pi\sqrt{\tau_2}\, l_B^2}. \tag{10}$$

Here l_S denotes the fundamental string length scale and l_B denotes the ten-dimensional Planck length in the type IIB superstring theory. τ is the vacuum value of the complex scalar field $C_0 + i\exp(-\Phi)$ in the type IIB theory, and τ_2 denotes the imaginary part of τ.

Verify that the τ dependence of these equations is consistent with the fact that the string charges p and q transform linearly under $SL(2,\mathbb{Z})$ transformations $\tau \to (a\tau + b)/(c\tau + d)$. This is required because the 2-form gauge fields also transform as a doublet. Remark: It is important that the Planck length l_B, and not the string length scale l_S, is $SL(2,\mathbb{Z})$ invariant.