Homework (L1)

The Local Potential Approximation (LPA)

Within the LPA, the truncated form of the scalar EAA reads

$$\boxed{\Gamma_k[\phi(\cdot)] = \int d^dx \left\{ -\frac{1}{2}\phi(x)\Box\phi(x) + \mathcal{U}_k(\phi(x)) \right\}}$$

Find the flow equation (a PDE) for the running potential $\mathcal{U}_k(\cdot)$.

Hint:

$$\int \frac{d^dp}{(2\pi)^d} F(p_\mu p^\mu) = 2 v_d \int_0^\infty dw\, w^{\frac{d}{2}-1} F(w)$$

$$v_d \equiv \left[2^{d+1} \pi^{d/2} \Gamma\left(\frac{d}{2}\right) \right]^{-1}$$

Homework (L2)

Write the running LPA potential in terms of a dimensionless function with a dimensionless argument:

$$\mathcal{U}_k(\phi) =: k^d \, u_k\big(\underbrace{k^{\frac{2-d}{2}}\phi}_{=:\varphi}\big)$$

Set likewise for the cutoff

$$R_k(p^2) =: k^2 \, R^{(0)}\left(\frac{p^2}{k^2}\right)$$

Find the flow equation and the fixed point condition on the truncated theory space

$$\mathcal{T}^{(d)}_{LPA} := \{\varphi \mapsto u(\varphi)\} \, .$$

Homework (L3)

The Gaussian fixed point (GFP) in scalar theories

1. Show that the fixed point condition on $J_{LPA}^{(d)}$ possess the solution $u_*(\varphi) = \text{const}$.

2. Linearize the RG flow near this fixed point:

$$u_k(\varphi) = u_* + e^{-\theta t}\, Y(\varphi)$$

Find the linear diff. eq. satisfied by the "eigenvalues" θ and "eigenvectors" $Y(\cdot)$.

3. Find the explicit form of the scaling fields Y in $d=2$.

4. For $d>2$, find the scaling dimensions θ of all polynomial solutions Y. Are there others?

(Hint: set $Y(\varphi) =: f\left(\sqrt{\frac{d-2}{2\sigma_d}}\,\varphi\right)$ and $f(x) =: e^{+\frac{x^2}{4}}\psi(x)$)

Homework (DE)

Derive the following derivative expansion for the scalar Laplacian D^2 (on $\mathcal{M}$ with $\partial\mathcal{M} = 0$):

$$\mathrm{Tr}\left[W(-D^2)\right] = \left(\frac{1}{4\pi}\right)^{d/2} \left\{ Q_{\frac{d}{2}}[W] \int d^dx \sqrt{g} + \frac{1}{6} Q_{\frac{d}{2}-1}[W] \int d^dx \sqrt{g}\, R + O(\partial^4 g) \right\}$$

with

$$Q_n[W] = \begin{cases} \frac{1}{\Gamma(n)} \int_0^\infty dz\, z^{n-1} W(z) & \text{for } n>0 \\ W(0) & \text{for } n=0 \\ (-1)^n \left(\frac{d}{dz}\right)^{|n|} W(z)\Big|_{z=0} & \text{for } n<0 \end{cases}$$

Hint:

Use the asymptotic expansion of the "traced" heat kernel:

$$\mathrm{Tr}\left[e^{-isD^2}\right] = \left(\frac{i}{4\pi s}\right)^{d/2} \int d^dx \sqrt{g} \left\{ 1 - \frac{1}{6} isR + O(\partial^4 g) \right\}$$

Homework (U)

Compute the value of the threshold function $\Phi^2_1(w)$ at $w = 0$ and show that it is <u>universal</u>, i.e. independent of the precise form of $R^{(0)}(\cdot)$.

Remark:

Anomalous dimension of Newton's constant in $d = 2 + \varepsilon$ - dimensional QEG:

$$\eta_N(g,\lambda) = \underbrace{-\frac{2}{3}\underbrace{\left[13 + 6 \underbrace{\Phi^2_1(0)}_{=1}\right]}_{=19} g}_{\overset{!}{=} -\frac{38}{3} \text{ is universal!}} + \underbrace{O(g^2,\lambda) + O(\varepsilon)}_{\text{not universal!}}$$