School on Particle Physics in the LHC Era
Project on Extra Dimensions

Often, numerical information regarding the KK spectrum and wavefunctions is sufficient, and it is useful to have an "all-purpose" implementation that allows you to get this information. This project requires no more than Mathematica and a bit of thinking.

1. **A simple example of the "superpotential approach":** Derive the scalar profile, $\phi(y)$, and metric function, $A(y)$, for a *linear* superpotential written as

$$W(\Phi) = 2\sqrt{\frac{3kM_5^3}{L}}\,\Phi \; . \tag{1}$$

Write also the corresponding potential to see what you are dealing with.

2. **KK wavefunctions in arbitrary backgrounds:** in this exercise it will be important to keep in mind that, in a strongly warped background, the scale of KK resonances is set by

$$\tilde{k}_{\text{eff}} \equiv A'(L)\, e^{-A(L)} \; . \tag{2}$$

Also, all the orthonormality conditions take the form

$$\frac{1}{L}\int_0^L dy\, f^n(y) f^m(y) \;\; = \;\; \delta_{nm} \; , \tag{3}$$

i.e. all are "physical wavefunctions".

(a) **Gauge case:** The equation of motion is

$$\partial_y\left[e^{-2A}\partial_y f_V^n\right] + m_n^2 f_V^n \;\; = \;\; 0 \; . \tag{4}$$

The corresponding equation for f_5^n can be obtained by using $f_5^n = \partial_y f_V^n/m_n$, for $n \neq 0$. The boundary conditions that allow a gauge zero-mode are $\partial_y f_V^n|_{0,L} = f_5^n|_{0,L} = 0$.

i. Devise a general strategy for finding, numerically, m_n and f_V^n, in the absence of a closed solution to the above equations [but given an explicit $A(y)$].

ii. Convince yourself that trying to solve directly Eq. (4) in strongly warped backgrounds [$A(L) \gg 1$ with $A(0) = 0$] is delicate.

iii. I will guide you here by example. Show that the equation of motion for $\tilde{f}_5^n \equiv e^{-2A} f_5^n$ is $\partial_y^2 \tilde{f}_5^n + m_n^2 e^{2A} \tilde{f}_5^n = 0$ and that the boundary conditions are $\tilde{f}_5^n\Big|_{0,L} = 0$. This form is better for a numerical analysis, and should be useful to you.

iv. Ideally, your code should be such that, given the background $A(y)$, it should be easy to find a given eigenvalue and the corresponding (normalized) eigenfunction. Test it with the AdS_5 case, where explicit solutions are available. Apply your code also to the background of Problem 1, and compare the results to the AdS_5 case. For definiteness, assume $M_5 = k$.

(b) **Fermions:** The equations of motion are

$$\left(\partial_y + M_f - \frac{1}{2}A'\right) f_L^n = m_n e^A f_R^n \,, \tag{5}$$

$$\left(\partial_y - M_f - \frac{1}{2}A'\right) f_R^n = -m_n e^A f_L^n \,, \tag{6}$$

which imply the second order differential equations

$$f_L^{n\prime\prime} - 2A' f_L^{n\prime} + \left[\frac{3}{4}A'^2 - \frac{1}{2}A'' - M_f A' + M_f' - M_f^2 + e^{2A}m_n^2\right] f_L^n = 0 \,, \tag{7}$$

$$f_R^{n\prime\prime} - 2A' f_R^{n\prime} + \left[\frac{3}{4}A'^2 - \frac{1}{2}A'' + M_f A' - M_f' - M_f^2 + e^{2A}m_n^2\right] f_R^n = 0 \,. \tag{8}$$

It will be sufficient to implement the case where the 0-mode is LH, which is defined by the boundary conditions

$$f_L^{n\prime} + \left(M_f - \frac{1}{2}A'\right) f_L^n\Big|_{y=0,L} = 0 \,, \qquad f_R^n(0) = f_R^n(L) = 0 \,. \tag{9}$$

i. Clearly, a numerical solution requires to fix not only $A(y)$, but also the fermion mass M_f. For both the AdS$_5$ case and the background of Problem 1, it will be convenient to write

$$M_f(y) = cA'(y) \,. \tag{10}$$

Do you see a simple physical interpretation for this ansatz?

ii. Now devise a strategy that allows you to find a given m_n, and the two associated wavefunctions f_L^n and f_R^n, as you did for the gauge case.

(c) **Scalars:** The equation of motion is

$$f_n'' - 2A' f_n' + \left[A'' - 3A'^2 - M_s^2 + e^{2A}m_n^2\right] f_n = 0 \,, \tag{11}$$

For localized mass terms $\mathcal{L}_{0,L} = -\frac{1}{2}M_{0,L}\Phi^2$, the boundary conditions read:

$$f_n' + \left(A' \mp M_{0,L}\right) f_n\Big|_{0,L} = 0 \,. \tag{12}$$

Assume a constant scalar bulk mass, M_s, and define new parameters α, $m_{\rm UV}$ and $m_{\rm IR}$ via

$$M_s^2 = \left(\alpha^2 - 4\right) k_{\rm eff}^2 \,, \qquad M_{0,L} = \mp\left(\alpha - 2\right) k_{\rm eff} + m_{\rm UV,IR} \,, \tag{13}$$

where $k_{\rm eff} \equiv A'(L)$.

i. You know the drill: set up your code to tackle this problem. Test it in the two canonical examples of AdS$_5$ and the linear superpotential of Problem 1.

ii. By defining $\alpha \equiv c + 1/2$, with $c = M_f(L)/A'(L)$, compare to the fermion case. Assume first that $m_{\rm UV}$ and $m_{\rm IR}$ are small.

iii. Do you see a meaning for the parametrization (13), and for the two masses $m_{\rm UV}$ and $m_{\rm IR}$?