# Vasiliev's Equations, Deformed Oscillators and Topological Open Strings

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Workshop on Higher-Spin and Higher-Curvature Gravity

ICTP-SAIFR, São Paulo

Nov 6, 2013

Talk based on collaboration with

Boulanger and Valenzuela '12-'13
Boulanger, Colombo and Sezgin '10-'12
Engquist and Tamassia '05-'07

# Outline

- 3D fractional-spin Chern–Simons gravity
  Vasiliev '88; Bergshoeff, Blencowe, Stelle '89; also Plyushchay '93

- 3D and 4D fractional-spin Vasiliev gravity
  Prokuskin, Vasiliev '98

- Secret agenda: Testing ground for first- and second-quantized Poisson sigma models — topological open strings related to tensionless discretized closed strings.

# 3D CS FSGRA

Stress two technical details:

- Construction bi-graded associative fractional-spin algebra via "fusion" of polynomial and matrix sectors in the non-polynomial completion of the enveloping algebra of the deformed oscillator algebra.
- Bi-linear form via the canonical trace operation.

Leading up to the point we would like to make in this talk on the CS FSGRA:

- While the on-shell formulation can be fetched working entirely within standard Fock spaces, the off-shell formulation requires the star-product algebra realization (in the apparent absence of any meaningful regularization of Fock-space traces of monomials in deformed oscillators).

# 3D CS FSGRA: Deformed oscillator algebra

- Wigner's deformed oscillator algebra

$$[q_\alpha, q_\beta]_\star = 2i\epsilon_{\alpha\beta}(1+\nu k)\ , \quad \{k, q_\alpha\}_\star = 0\ , \quad k \star k = 1$$

- Its enveloping algebra $\mathtt{Env}(2,\nu)$ consists of arbitrary star-polynomials in $k$ and $q_\alpha$ modulo the above relations.
- Main theme: Non-polynomial extensions/completions of $\mathtt{Env}(2,\nu)$ with well-defined star-product compositions obeying associativity?

# 3D CS FSGRA: Non-polynomial completions

- One way of thinking of such a non-polynomial associative algebra, $\mathtt{A}(2,\nu)$ say, is as a direct sum of infinite-dimensional vector spaces, each with a specific basis,

$$\mathtt{A}(2,\nu)=\bigoplus_{\Sigma}\mathtt{A}_{\Sigma}\ ,$$

- Instead of requiring all possible combinations of star-products to be non-trivial, one may adopt a fusion rule

$$\mathtt{A}_{\Sigma}\star\mathtt{A}_{\Sigma'}=\sum_{\Sigma''}\mathcal{N}_{\Sigma\Sigma'}^{\Sigma''}\mathtt{A}_{\Sigma''}\ ,\quad \mathcal{N}_{\Sigma\Sigma'}^{\Sigma''}\in\{0,1\}\ ,$$

  such that the resulting nested star-products obeys associativity, which in particular requires

$$\sum_{\Sigma}\mathcal{N}_{\Sigma_1\Sigma_2}^{\Sigma}\mathcal{N}_{\Sigma\Sigma_3}^{\Sigma'}=\sum_{\Sigma}\mathcal{N}_{\Sigma_1\Sigma}^{\Sigma'}\mathcal{N}_{\Sigma_2\Sigma_3}^{\Sigma}\ .$$

- For extensions of minimal models, some nested products require prescriptions, but end result is associative.

- Non-polynomial extension in monomial basis

$$\mathtt{Mon}_\infty(2,\nu) := \left\{ Mo(k,q) = \Pi_+ \star Mo_+(q) + \Pi_- \star Mo_-(q) \right\}$$

$$Mo_\pm(q) := \sum_{n\geqslant 0} Mo^{\alpha_1\dots\alpha_n}_{\pm;(n)} q_{(\alpha_1} \star \cdots \star q_{\alpha_n)} \ , \quad \Pi_\pm = \tfrac{1}{2}(1 \pm k) \ ,$$

- Forms an associative algebra on its own provided sums and star products exchangeable.
- One-sided actions: the only non-trivial operator whose left-action is diagonal in monomial basis is $k$.

# 3D CS FSGRA: Matrix basis

- Diagonilize "Hamiltonian", $N$ say, yields new basis.
- Choose $N := \frac{1}{2}\{a^-, a^+\}_\star - \frac{1}{2}(1+\nu)$, which has a normalizable integer spectrum, *viz.*

$$(N-m)\star P^n_m(\sigma) = 0 = P^n_m(\sigma)\star(N-n) \ ,$$

$$\Pi_\sigma \star P^n_m(\sigma) = P^n_m(\sigma) \ ,$$

$$P^n_m(\sigma)\star P^{n'}_{m'}(\sigma') = \delta^n_{m'}\delta_{\sigma\sigma',(-1)^{m+n}} P^{n'}_m(\sigma) \ .$$

- $P^n_m(\sigma)$ have non-polynomial real-analytic Weyl-ordered symbols.
- Non-polynomial extension in matrix basis

$$\mathtt{Mat}_\infty(2,\nu;N) := \left\{ Ma(k,q) = \sum_{m,n\geqslant 0} Ma^n_m(\sigma) P^m_n(\sigma) \right\} \ .$$

- Forms an associative algebra on its own provided it is possible to exchange sums and star products.

# 3D CS FSGRA: Associative FS algebra

- $\mathbb{Z}_2$-graded associative algebra

$$\mathtt{Fs}(2,\nu;N) := \left\{ \begin{bmatrix} \Pi_+ \star \mathtt{Mon} \star \Pi_+ & \Pi_+ \star \mathtt{Mat} \star \Pi_- \\ \Pi_- \star \mathtt{Mat} \star \Pi_+ & \Pi_- \star \mathtt{Mat} \star \Pi_- \end{bmatrix} \right\}$$

- Product rule: $2 \times 2$-matrix product using

$$(\mathtt{Monomial}) \star (\mathtt{Monomial}) = \sum_{\text{finite}} (\mathtt{Monomial}) \,,$$

$$(\mathtt{Monomial}) \star (\mathtt{Matrix\ element}) = \sum_{\text{finite}} (\mathtt{Matrix\ element}) \,,$$

$$(\mathtt{Matrix\ element}) \star (\mathtt{Matrix\ element}) = \sum_{\text{finite}} (\mathtt{Matrix\ element}) \,,$$

and the real-analyticity of $P^n_m(\sigma)$ to expand

$$(\mathtt{Matrix\ element}) \star (\mathtt{Matrix\ element}) = \sum_{\text{infinite}} (\mathtt{Monomial}) \,.$$

- No need to expand basis of $\mathtt{Mon}(2,\nu)$ in basis of $\mathtt{Mat}(2,\nu;N)$.

# 3D CS FSGRA: Master one-form

- Add two Clifford algebras and define master one-form

$$\mathbb{A}_\varepsilon := \begin{bmatrix} W & \psi_\varepsilon \\ \overline{\psi}_\varepsilon & U \end{bmatrix} \in \mathtt{Fs}(2,\nu;N;\varepsilon)$$

$$\mathtt{Fs}(2,\nu;N;\varepsilon) := \left[\mathtt{Fs}(2,\nu;N) \otimes \mathtt{Cl}_1(\Gamma) \otimes \mathtt{Cl}_1(\xi)\right]/\mathbb{Z}_2(\varepsilon)$$

- Statistics and intrinsic parities of generating elements:

$$\epsilon_{\rm stat}(q_\alpha, k, \Gamma, \xi) = (0,0,0,1)$$

$$\pi_q(q,k,\Gamma,\xi) = (-q,k,\Gamma,\xi) \quad \textit{idem}\ \pi_k,\ \pi_\Gamma,\ \pi_k$$

- Statistics and intrinsic parities of master fields

$$\epsilon_{\rm stat}(\mathbb{A}_\varepsilon) = 0\ , \quad \varepsilon_{q,\xi}(W,\psi_\varepsilon,\overline{\psi}_\varepsilon,U) = (+1,\varepsilon,\varepsilon,+1)\ ,$$

  where $\varepsilon_{q,\xi}$ denotes the $\pi_q\pi_\xi$-parity.
- $(W,U)$ and $(\psi_-,\overline{\psi}_-)$ consist of bosonic component fields.
- $(\psi_+,\overline{\psi}_+)$ consist of fermionic component fields.

# 3D CS FSGRA: Trace on $\mathtt{Env}(2,\nu)\otimes\mathtt{Cl}_1(\Gamma)\otimes\mathtt{Cl}_1(\xi)$

- Trace operation on $\mathtt{Env}(2,\nu)\otimes\mathtt{Cl}_1(\Gamma)\otimes\mathtt{Cl}_1(\xi)$:

$$\mathrm{Tr}_\nu(f)=\mathrm{STr}_\nu(k\star f)|_{\xi=0=\Gamma}\ ,$$

  with supertrace $\mathrm{STr}_\nu$ operation on $\mathtt{Env}(2,\nu)$ fixed uniquely by its defining properties

$$\mathrm{STr}_\nu(f\star g)=(-1)^{\varepsilon_q(f)}\mathrm{STr}_\nu(g\star f)\ ,\quad \mathrm{STr}_\nu(1)=1$$

  where $\varepsilon_q(f)$ is the $q$-parity of $f$, *i.e.* $\varepsilon_q(f)f:=\pi_q(f)$.
- In the Weyl-ordered basis

$$\mathrm{STr}_\nu(f(k,q))=f_{0;(0)}-\nu f_{1;(0)}$$

- Extend $\mathrm{Tr}_\nu$ to non-polynomial completions of $\mathtt{Env}(2,\nu)$ whose Weyl-ordered symbols are real-analytic at $q_\alpha=0$.

# 3D CS FSGRA: Trace on $\mathtt{Fs}(2,\nu;N;\varepsilon)$

$$\mathrm{Tr}(\mathbb{M}_\varepsilon) := \mathrm{Tr}(Mo - \varepsilon Ma) \quad \text{for} \quad \mathbb{M}_\varepsilon = \left[\begin{array}{cc} Mo & Bi_\varepsilon \\ \overline{B}i_\varepsilon & Ma \end{array}\right] \in \mathtt{Fs}(2,\nu;N;\varepsilon)$$

- Cyclicity property

$$\mathrm{Tr}(\mathbb{M}_\varepsilon \star \mathbb{M}'_\varepsilon) = \mathrm{Tr}(\mathbb{M}'_\varepsilon \star \mathbb{M}_\varepsilon)$$

- Expand

$$Bi_\varepsilon = \sum_I Bi^I_\varepsilon \Theta^\varepsilon_I \ , \quad \epsilon_{\rm s}(\Theta^\varepsilon_I) = \epsilon_{\rm s}(Bi^I_\varepsilon) = -\varepsilon \ ,$$

  $\Theta^\varepsilon_I$ basis elements, $Bi^I_\varepsilon$ components *idem* $\overline{B}i_\varepsilon$.
- It follows from $\mathrm{Tr}(\Theta^\varepsilon_I \star \Theta^\varepsilon_J) = \mathrm{Tr}(\Theta^\varepsilon_J \star \Theta^\varepsilon_I)$ that $\mathrm{Tr}(\mathbb{M}_{1;\varepsilon} \star \mathbb{M}_{2;\varepsilon}) = \mathrm{Tr}(Mo_1 \star Mo_2 - \varepsilon Ma_1 \star Ma_2) - \varepsilon \sum_{I,J}(Bi^I_{1;\varepsilon}\overline{B}i^J_{2;\varepsilon} + Bi^I_{2;\varepsilon}\overline{B}i^J_{1;\varepsilon})\mathrm{Tr}(\Theta^\varepsilon_I \star \Theta^\varepsilon_J)$ which is manifestly symmetric under $1 \longleftrightarrow 2$ .

# 3D CS FSGRA: Chern–Simons action

- The Chern–Simons action

$$
\begin{aligned}
S[\mathbb{A}_\pm] &= \int \mathrm{Tr}\left(\tfrac{1}{2}\mathbb{A}_\pm \star d\mathbb{A}_\pm + \tfrac{1}{3}(\mathbb{A}_\pm)^{\star 3}\right) \\
&= \int \mathrm{STr}_\nu \Big(\tfrac{1}{2} W \star dW + \tfrac{1}{3} W^{\star 3} + W \star \psi_\pm \star \overline{\psi}_\pm \\
&\qquad \pm\big(\tfrac{1}{2} U \star dU + \tfrac{1}{3} U^{\star 3} + U \star \overline{\psi}_\pm \star \psi_\pm\big) \\
&\qquad + \tfrac{1}{2}\big(\psi_\pm \star d\overline{\psi}_\pm \pm \overline{\psi}_\pm \star d\psi_\pm\big)\Big)\Big|_{\Gamma=0=\xi} \qquad ,
\end{aligned}
$$

using the $\Pi_\pm$ projections and

$$
\mathrm{STr}_\nu(\overline{\psi}_\pm \star W \star \psi_\pm) = \pm \mathrm{STr}_\nu(W \star \psi_\pm \star \overline{\psi}_\pm) \ ,
$$

$$
\mathrm{STr}_\nu(\psi_\pm \star U \star \overline{\psi}_\pm) = \pm \mathrm{STr}_\nu(U \star \overline{\psi}_\pm \star \psi_\pm) \ .
$$

- In $\Pi_- \star \mathtt{Mat}(2,\nu;N) \star \Pi_- \ni U$ the $\mathrm{STr}_\nu$ reduces to matrix traces proportional to $\mathrm{STr}_\nu(P^0_0(\pm 1))$.
- Correlated $\nu$-dependent levels of tensorial and internal gauge theories.

# 3D CS FSGRA: Summary so far

- Non-polynomial formally associative completion of $\mathtt{Env}(2,\nu)$ by fusing together monomial and matrix sectors.
- Universal trace operation yields Chern–Simons action

$$S[\mathbb{A}_\pm] = \int \mathrm{Tr}\left(\tfrac{1}{2}\mathbb{A}_\pm \star d\mathbb{A}_\pm + \tfrac{1}{3}(\mathbb{A}_\pm)^{\star 3}\right)$$

$$= \int \mathrm{STr}_\nu \left(\tfrac{1}{2} W \star dW + \tfrac{1}{3} W^{\star 3} + W \star \psi_\pm \star \overline{\psi}_\pm \right.$$
$$\pm(\tfrac{1}{2} U \star dU + \tfrac{1}{3} U^{\star 3} + U \star \overline{\psi}_\pm \star \psi_\pm)$$
$$\left. + \tfrac{1}{2}(\psi_\pm \star d\overline{\psi}_\pm \pm \overline{\psi}_\pm \star d\psi_\pm)\right)|_{\Gamma=0=\xi} \qquad ,$$

with fixed relative normalizations of all kinetic terms.

- On-shell formulation: fetched by Fock-space realization.
- Off-shell formulation: requires star-product realization.

# Inclusion of local fractional spin degrees of freedom?

Let's side-step important issues related to

- Real forms and positivity
- Dual CFTs and boundary anyons
- Critical limits

and instead take a look at generalizations of the basic ideas to Vasiliev's systems with local degrees of freedom.

# 3D and 4D Vasiliev fractional-spin gravities

Observations that we will make use of:

- Vasiliev's master-field formulation does not refer to any *a priori* Lorentz structure.
- Inequivalent models could therefore arise by expanding the dependence of the master fields on the fiber coordinates into inequivalent sets of sectors of fiber functions glued together into associative structures using fusion rules.
- Moreover, in each case, yet another choice governs the embedding of the Lorentz connection, after which the fully non-linear system should be written on manifestly Lorentz-covariant form using Vasiliev's deformed oscillator algebra.
- In particular, this allows for Lorentz structures with one-forms and zero-forms valued in fractional-spin algebras.

# 3D/4D Vasiliev FSGRA: Main technical steps

- Implementation of fusion rules using separation of variables
- (Non-canonical) embedding of Lorentz connection
- Manifestly Lorentz-covariant form of the equations
- Lorentz representation matrices in symbol calculus

# 3D/4D Vasiliev FSGRA: Master field equations

- Three overall bosonic master fields $(\widehat{A}, \widehat{B}, \widehat{J})$ of form degrees $(1, 0, 2)$ obeying

$$\widehat{d}\widehat{A} + \widehat{A} \star \widehat{A} + \widehat{B} \star \widehat{J} = 0 \ , \quad \widehat{d}\widehat{B} + \widehat{A} \star \widehat{B} - \widehat{B} \star \widehat{A} = 0 \ ,$$

$$\widehat{d}\widehat{J} = 0 \ , \quad [\widehat{J}, \widehat{A}]_\star = 0 = [\widehat{J}, \widehat{B}]_\star \ ,$$

on a correspondence space $\mathcal{C}$ with a bundle structure

$$\mathcal{F} \to \mathcal{C} \to \mathcal{B} \ ,$$

in which $\widehat{d}$ and $(\widehat{A}, \widehat{B}, \widehat{J})$ are horizontal.

# 3D/4D Vasiliev FSGRA: The fiber space

- The fiber space
$$\mathcal{F} = \mathcal{Y} \times \mathcal{I} \ ,$$
where $\mathcal{Y}$ and $\mathcal{I}$ are two non-commutative spaces.
- $\mathcal{Y}$ is a bosonic twistor space.
- $\mathcal{I}$ is internal; its coordinates generate a unital associative algebra $\mathtt{A}_{\mathcal{I}}$.
- Coordinatize $\mathcal{I}$ using generators of bosonic and fermionic Clifford algebras $\mathtt{Cl}_M(\Gamma^i)$ and $\mathtt{Cl}_N(\xi^r)$, respectively.

# 3D/4D Vasiliev FSGRA: The base manifold

- The base is generalized fiber bundle
$$\mathcal{B} \to \mathcal{M}$$
with non-commutative base $\mathcal{M}$ containing commutative space-time sub-manifolds.
- Over each point $p \in \mathcal{M}$ there is a non-commutative fiber $\mathcal{Z}_p$.
- $\mathcal{Z}_p$ looks locally like $\mathcal{Y}$ but its global geometry is determined by the field configuration $(\widehat{A}, \widehat{B}, \widehat{J})$.
- Thus, if $p \neq p'$ then $\mathcal{Z}_p$ need not be isomorphic to $\mathcal{Z}_{p'}$.
- Locally, charts $\mathcal{M}_I \subseteq \mathcal{M}$ embedded into sections
$$\mathcal{M}_I \xrightarrow{i} \mathcal{B} \ ,$$
such that $i^*(\widehat{A}) =: U$ and $i^*(\widehat{B}) =: B$ provide a set of boundary conditions on $\mathcal{Z}_p$ for each $p \in \mathcal{M}_I$.
- The deformation problem on $\mathcal{Z}_p$ requires additional gauge/boundary conditions on the $\mathcal{Z}$-space connection
$$\widehat{V} := \widehat{A} \cap \ker i^* \ .$$

# 3D/4D Vasiliev FSGRA: Fusion rules

- Think of the full master fields as sections of a fiber bundle

$$\mathtt{A} \to E \to \mathcal{B} \ .$$

- $\mathtt{A} = \bigoplus_\Sigma \mathtt{A}_\Sigma$ is a unital associative algebra consisting of sectors $\mathtt{A}_\Sigma$ of functions on $\mathcal{F}$.
- $\mathtt{A}_\Sigma$ thus consists of composite operators built from the non-commutative coordinates of $\mathcal{F}$ and represented by suitable symbols.
- Use fusion rules to endow $\mathtt{A}$ with associative structure, *viz.*

$$\mathtt{A}_{\Sigma_1} \star \mathtt{A}_{\Sigma_2} \subseteq \sum_\Sigma \mathcal{N}_{\Sigma_1\Sigma_2}{}^\Sigma \mathtt{A}_\Sigma \ , \qquad \mathcal{N}_{\Sigma\Sigma'}{}^{\Sigma''} \in \{0,1\}$$

$$\mathcal{N}_{\Sigma_1\Sigma_2}{}^\Sigma \mathcal{N}_{\Sigma\Sigma_3}{}^{\Sigma'} = \mathcal{N}_{\Sigma_1\Sigma}{}^{\Sigma'} \mathcal{N}_{\Sigma_2\Sigma_3}{}^\Sigma \ .$$

# 3D/4D Vasiliev FSGRA: Matrix fusion and central element

$$\{\mathtt{A}_\Sigma\} = \left\{ \mathtt{A}_\lambda{}^{\lambda'} \right\} \ , \quad \mathcal{N}_{\Sigma_1\Sigma_2}{}^{\Sigma_3} = \delta^{\lambda'_1}_{\lambda_2}\delta^{\lambda_3}_{\lambda_1}\delta^{\lambda'_2}_{\lambda'_3} \ ,$$

$$\mathtt{A}_\lambda{}^{\lambda'} \star \mathtt{A}_{\lambda'}{}^{\lambda''} \subseteq \mathtt{A}_\lambda{}^{\lambda''} \qquad \text{(no sum!)} \ .$$

- If all $\mathtt{A}_\lambda{}^{\lambda'} = \mathtt{A}_0$ one has $\mathtt{A} = (\mathrm{Mat}_n(\mathtt{A}_0))$ for some $n$.
- In these cases, the central element

$$\widehat{J}_\lambda{}^{\lambda'} = \widehat{J}_0 \ \delta^{\lambda'}_\lambda \ , \qquad i^*(\widehat{J}_0) = 0 \ .$$

- Canonical coordinates $(y_\alpha, z_\alpha)$ of $\mathcal{Y} \times \mathcal{Z}$, obeying

$$[y_\alpha, y_\beta]_\star = 2i\epsilon_{\alpha\beta} \ , \quad [z_\alpha, z_\beta]_\star = -2i\epsilon_{\alpha\beta} \ .$$

- $\widehat{J}_0$ is two-form on undeformed $\mathcal{Z}$ built from function $\widehat{\kappa}$ on $\mathcal{Z} \times \mathcal{F}$ (and its hermitian conjugate) defining Klein element,

$$\widehat{J}_0 = i(bdz^\alpha dz_\alpha \widehat{\kappa} + \mathrm{h.c.}) \ ,$$

such that $\widehat{J} \star \widehat{B}$ cannot be removed by any $\star$-function field redefinition.

# 3D/4D Vasiliev FSGRA: Central element for $\texttt{Fs}$ models

- In the case of non-trivial fusion rules,

$$\widehat{J} = \sum_{\Sigma} \widehat{J}_{\Sigma} \ .$$

- In the case of matrix fusion rules, it is natural to use the factorization property

$$\widehat{\kappa} = \kappa_{\mathcal{Z}} \star \kappa_{\mathcal{F}} \ .$$

to take

$$\widehat{J}_{\lambda}{}^{\lambda'} = (\widehat{J}_0)_{\lambda} \ \delta_{\lambda}^{\lambda'} \ ,$$

$$(\widehat{J}_0)_{\lambda} = i(bdz^{\alpha} dz_{\alpha} \kappa_{\mathcal{Z}} \star (\kappa_{\mathcal{F}})_{\lambda} + \mathrm{h.c.}) \ .$$

- For 3D models with $\texttt{A} = \texttt{Fs}(2,0;N,-1)$ (undeformed bosonic model), we use

$$\mathbb{J} = ibdz^{\alpha} dz_{\alpha} \kappa_{\mathcal{Z}} \star \left[ \begin{array}{cc} [2\pi\delta^2(y)]_{\mathrm{Weyl-order}} & 0 \\ 0 & [(-1)^N_{\star}]_{\mathrm{Fock-space}} \end{array} \right]$$

# 3D Vasiliev FSGRA: Statistics/twistor-space parity

- Correlate semi-classical statistics to twistor-space parity

$$\pi_y \pi_z \pi_\xi (\widehat{A}_\Sigma, \widehat{B}_\Sigma, \widehat{J}_\Sigma) = \sigma_\Sigma (\widehat{A}_\Sigma, \widehat{B}_\Sigma, \widehat{J}_\Sigma) \ , \quad \sigma_\Sigma \in \{1, -1\}$$

$$\mathcal{N}_{\Sigma\Sigma'}{}^{\Sigma''} (\sigma_\Sigma \sigma_{\Sigma'} - \sigma_{\Sigma''}) = 0 \ .$$

- Split the exterior derivative and connection one-form into

$$\widehat{d} = d + d' \ , \quad d := dX^M \partial_M \ , \quad d' := dz^\alpha \partial_\alpha \ ,$$

$$\widehat{A} = \widehat{U} + \widehat{V} \ , \quad \widehat{U} := dX^M \widehat{U}_M \ , \quad \widehat{V} := dz^\alpha \widehat{V}_\alpha \ .$$

- Expand in $\xi^r$ and symbols on $\mathcal{Y} \times \mathcal{Z}$ with definite parities

$$\varepsilon_{y,z} \left\{ \widehat{B}_{\Sigma;r[k]}, \widehat{U}_{\Sigma;M;r[k]}, \widehat{V}_{\Sigma;\alpha;r[k]} \right\} = (-1)^k \sigma_\Sigma \{+1, +1, -1\} \ ,$$

$$\epsilon_{\rm stat} \left\{ \widehat{B}_{\Sigma;r[k]}, \widehat{U}_{\Sigma;M;r[k]}, \widehat{V}_{\Sigma;\alpha;r[k]} \right\} = \tfrac{1}{2}(1 + (-1)^k) \{1, 1, 1\} \ .$$

# 3D Vasiliev FSGRA: Full Lorentz generators

- Embed an $sl(2;\mathbb{R})_{\rm Lor}$ connection Prokushkin, Vasiliev '98

$$\widehat{U} = \widehat{W} + \frac{1}{4i}\omega^{\alpha\beta} \star \widehat{M}_{\alpha\beta} \ , \qquad , \qquad \omega^{\alpha\beta} = dX^M \omega_M{}^{\alpha\beta} \ ,$$

$$\omega^{\alpha\beta} \star \widehat{f} = (-1)^{\deg(\widehat{f})} \widehat{f} \star \omega^{\alpha\beta} \qquad \forall \widehat{f} \in \widehat{\Omega}_{\rm hor}(\mathcal{C}) \star \mathtt{A} \ .$$

- The full Lorentz generators

$$\widehat{M}_{\alpha\beta} := \widehat{M}^{(0)}_{\alpha\beta} + \widehat{M}^{(S)}_{\alpha\beta} \ ,$$

consists of "orbital" plus "internal" parts, respectively.

# 3D Vasiliev FSGRA: Internal part and deformed oscillators

- Internal part

$$\widehat{M}^{(S)}_{\alpha\beta} := \widehat{S}_{(\alpha} \star \widehat{S}_{\beta)} \ , \quad \widehat{S}_\alpha := z_\alpha - 2i\widehat{V}_\alpha \ ,$$

from deformed oscillators

$$[\widehat{S}_\alpha, \widehat{S}_\beta]_\star = -2i\epsilon_{\alpha\beta}(1 + \frac{1}{2}\epsilon^{\gamma\delta}\widehat{J}_{\gamma\delta} \star \widehat{B}) \ ,$$

$$[\widehat{J}_{\alpha\beta}, \widehat{U}]_\star = [\widehat{J}_{\alpha\beta}, \widehat{B}]_\star = 0 = \{\widehat{J}_{\alpha\beta}, \widehat{S}_\gamma\}_\star \ .$$

- So $\widehat{S}_\alpha$ carries a Lorentz doublet index and hence, from what follows, so does $\widehat{V}_\alpha$.

# 3D Vasiliev FSGRA: Orbital part

- Orbital part

$$\widehat{M}^{(0)}_{\alpha\beta} := M^{(\mathcal{Y})}_{\alpha\beta} + M^{(\mathcal{Z})}_{\alpha\beta} \ ,$$

$$M^{(\mathcal{Z})}_{\alpha\beta} := -z_{(\alpha} \star z_{\beta)} \ , \quad M^{(\mathcal{Y})}_{\alpha\beta} = \sum_{\Sigma} M_{\Sigma;\alpha\beta} \ .$$

- $M^{(\mathcal{Z})}_{\alpha\beta}$ transform the canonical coordinates of $\mathcal{Z}$ as doublets.
- $M^{(\mathcal{Y})}_{\alpha\beta}$ transform the elements of $\mathtt{A}$

$$[M^{(\mathcal{Y})}_{\alpha\beta}, f_\Sigma]_\star = L_{\Sigma;\alpha\beta} f_\Sigma \ ,$$

$$\sum_{\Sigma',\Sigma''} \left( \mathcal{N}_{\Sigma'\Sigma}{}^{\Sigma''} M_{\Sigma';\alpha\beta} \star f_\Sigma - \mathcal{N}_{\Sigma\Sigma'}{}^{\Sigma''} f_\Sigma \star M_{\Sigma';\alpha\beta} \right) |_{\mathtt{A}_{\Sigma''}} = L_{\Sigma;\alpha\beta} f_\Sigma \ ,$$

where $L_{\Sigma;\alpha\beta}$ are differential operators on $\mathcal{F}$ acting on symbols as to generate Lorentz representation matrices

$$[L_{\Sigma;\alpha\beta}, L_{\Sigma;\gamma\delta}] = 8i\epsilon_{\beta\gamma} L_{\Sigma;\alpha\delta} \ .$$

# 3D Vasiliev FSGRA: Full Lorentz algebra

- The above generators obey

$$[\widehat{M}^{(\mathcal{Y})}_{\alpha\beta}, \widehat{M}^{(\mathcal{Y})}_{\gamma\delta}]_\star = 8i\epsilon_{\beta\gamma}\widehat{M}^{(\mathcal{Y})}_{\alpha\delta} \ , \quad [\widehat{M}^{(\mathcal{Z})}_{\alpha\beta}, \widehat{M}^{(\mathcal{Z})}_{\gamma\delta}]_\star = 8i\epsilon_{\beta\gamma}\widehat{M}^{(\mathcal{Z})}_{\alpha\delta} \ ,$$

$$[\widehat{M}^{(0)}_{\alpha\beta}, \widehat{M}^{(0)}_{\gamma\delta}]_\star = 8i\epsilon_{\beta\gamma}\widehat{M}^{(0)}_{\alpha\delta} \ , \quad [\widehat{M}^{(S)}_{\alpha\beta}, \widehat{M}^{(S)}_{\gamma\delta}]_\star = -8i\epsilon_{\beta\gamma}\widehat{M}^{(S)}_{\alpha\delta} \ .$$

- It follows that

$$[\widehat{M}_{\alpha\beta}, \widehat{M}_{\gamma\delta}]_\star - 8i\epsilon_{\beta\gamma}\widehat{M}_{\alpha\delta} = [\widehat{M}_{\alpha\beta}, \widehat{M}^{(S)}_{\gamma\delta}]_\star - ((\alpha\beta) \leftrightarrow (\gamma\delta)) \ ,$$

has a field-dependent Lie algebra structure

$$[\widehat{M}_{\alpha\beta}, \widehat{M}_{\gamma\delta}]_\star - 8i\epsilon_{\beta\gamma}\widehat{M}_{\alpha\delta} = \mathcal{L}^{(\mathrm{cpts})}_{\alpha\beta}\widehat{M}_{\gamma\delta} - ((\alpha\beta) \leftrightarrow (\gamma\delta)) \ ,$$

where $\mathcal{L}^{(\mathrm{cpts})}_{\alpha\beta}$ generate Lorentz transformations on component fields (in accordance with the transformations dictated by $L_{\Sigma;\alpha\beta}$ and the fact that $\widehat{V}_{\Sigma;\alpha}$ is a doublet).

# 3D Vasiliev FSGRA: Manifestly Lorentz-covariant master-field equations

- Substituting $\widehat{U} = \widehat{W} + \frac{1}{4i}\omega^{\alpha\beta}\widehat{M}_{\alpha\beta}$ into the equations of motion for $\widehat{U}$, $\widehat{B}$ and $\widehat{S}_\alpha$ yields

$$\nabla\widehat{W} + \widehat{W}\star\widehat{W} + \frac{1}{4i}r^{\alpha\beta}\widehat{M}_{\alpha\beta} \approx 0\ ,$$

$$\nabla\widehat{B} + [\widehat{W},\widehat{B}]_\star \approx 0\ , \quad \nabla S_\alpha + [\widehat{W},\widehat{S}_\alpha]_\star \ \approx\ 0\ .$$

- Lorentz covariantized derivatives

$$\nabla\widehat{W} := d\widehat{W} + [\widehat{\omega}^{(0)},\widehat{W}]_\star\ , \quad \nabla\widehat{B} \ :=\ d\widehat{B} + [\widehat{\omega}^{(0)},\widehat{B}]_\star\ ,$$

$$\nabla\widehat{S}_\alpha := d\widehat{S}_\alpha - \omega_\alpha{}^\beta\widehat{S}_\beta + [\widehat{\omega}^{(0)},\widehat{S}_\alpha]_\star\ ,$$

$$\widehat{\omega}^{(0)} := \frac{1}{4i}\omega^{\alpha\beta}\widehat{M}^{(0)}_{\alpha\beta}\ , \quad r^{\alpha\beta} \ :=\ d\omega^{\alpha\beta} - \omega_\alpha{}^\gamma\omega_{\gamma\beta}\ ,$$

# 3D Vasiliev FSGRA: Three types of symmetries

- Lorentz transformations under which $\omega_{\alpha\beta}$ transforms as a spin connection and the component fields of $\widehat{W}$, $\widehat{B}$ and $\widehat{S}_\alpha$ in their canonical Lorentz representations;
- Lorentz-covariantized higher-spin symmetries

$$\delta_{\widehat{\epsilon}}\widehat{W} = \nabla\widehat{\epsilon} + [\widehat{W},\widehat{\epsilon}]_\star \ , \quad \delta_{\widehat{\epsilon}}\omega_{\alpha\beta} \ = \ 0 \ ,$$

$$\delta_{\widehat{\epsilon}}\widehat{B} = [\widehat{B},\widehat{\epsilon}]_\star \ , \quad \delta_{\widehat{\epsilon}}\widehat{S}_\alpha \ = \ [\widehat{S}_\alpha,\widehat{\epsilon}]_\star \ ;$$

- Shift symmetries with one-form parameters $\zeta_{\alpha\beta}$ acting as

$$\delta_\zeta\omega_{\alpha\beta} = \zeta_{\alpha\beta} \ , \quad \delta_\zeta\widehat{W} \ = \ -\frac{1}{4i}\zeta^{\alpha\beta}\star\widehat{M}_{\alpha\beta} \ , \qquad \delta_\zeta(\widehat{B},\widehat{S}_\alpha) \ = \ 0 \ .$$

# 3D Vasiliev FSGRA: Lorentz representation matrices in perturbative symbol calculus

- So far we have relied on separation of $\mathcal{Y}$ and $\mathcal{Z}$ coordinates.
- However, in order to obtain a sensible "metric-like" perturbation theory one needs to work with symbols defined with respect to a normal-ordered basis that mixes $\mathcal{Y}$ and $\mathcal{Z}$ coordinates in such a way that the symbol of the Klein element $\widehat{\kappa}$ is real-analytic at $y_\alpha = 0 = z_\alpha$.
- Let $\widehat{f}_\Sigma(y, z; \Gamma, \xi)$ denote the resulting normal-ordered symbols.
- The Lorentz representation matrices acting in this basis are defined by

$$[\widehat{M}^{(0)}_{\alpha\beta}, \widehat{f}_\Sigma]_\star = \widehat{L}_{\Sigma;\alpha\beta} \widehat{f}_\Sigma \ ,$$

where $\widehat{L}_{\Sigma;\alpha\beta}$ are differential operators on $\mathcal{Z} \times \mathcal{F}$ obeying

$$[\widehat{L}_{\Sigma;\alpha\beta}, \widehat{L}_{\Sigma;\gamma\delta}] = 8i\epsilon_{\beta\gamma} \widehat{L}_{\Sigma;\alpha\delta} \ .$$

# 3D Vasiliev FSGRA: Conclusions

We have stressed:

- Role of star-product quantization in FSGRA.
- The Lorentz covariantization of FSGRA with local degrees of freedom goes through formally as in the tensorial case.

Some basic questions:

- Extension of linearized on-mass-shell theorem to local propagating fractional spin fields?
- Switching on $\langle \widehat{B} \rangle = \nu$ in 4D deforms the higher-spin algebra such that maximal subalgebra is $so(2,2)$ or $so(1,3)$ — massive 3D boundary anyons?

Finally, besides holographic duals, it would be interesting to quantize fractional-spin gravities using first- and second-quantized Poisson sigma models — is FSGRA an example of a class of models for which the PSM supercedes the metric-like approach?