Dynamos: helical & non-helical

Trouble with Euler potentials

Good examples with vector potential

Minimal tau approximation

Nonlinear dynamo theory

Axel Brandenburg (Nordita, Stockholm)

Struggle for the dynamo

- Larmor (1919): first qualitative ideas
- Cowling (1933): no → antidynamo theorem
- Larmor (1934): vehement response
  - 2-D not mentioned
- Parker (1955): cyclonic events, dynamo waves
- Herzenberg (1958): first dynamo
  - 2 small spinning spheres, slow dynamo ($\lambda \sim R_m^{-1}$)
- Steenbeck, Krause, Rädler (1966): $\alpha\omega$ dynamo
  - Many papers on this since 1970
- Kazantsev (1968): small-scale dynamo
  - Essentially unnoticed, simulations 1981, 2000-now

Mile stones in dynamo research

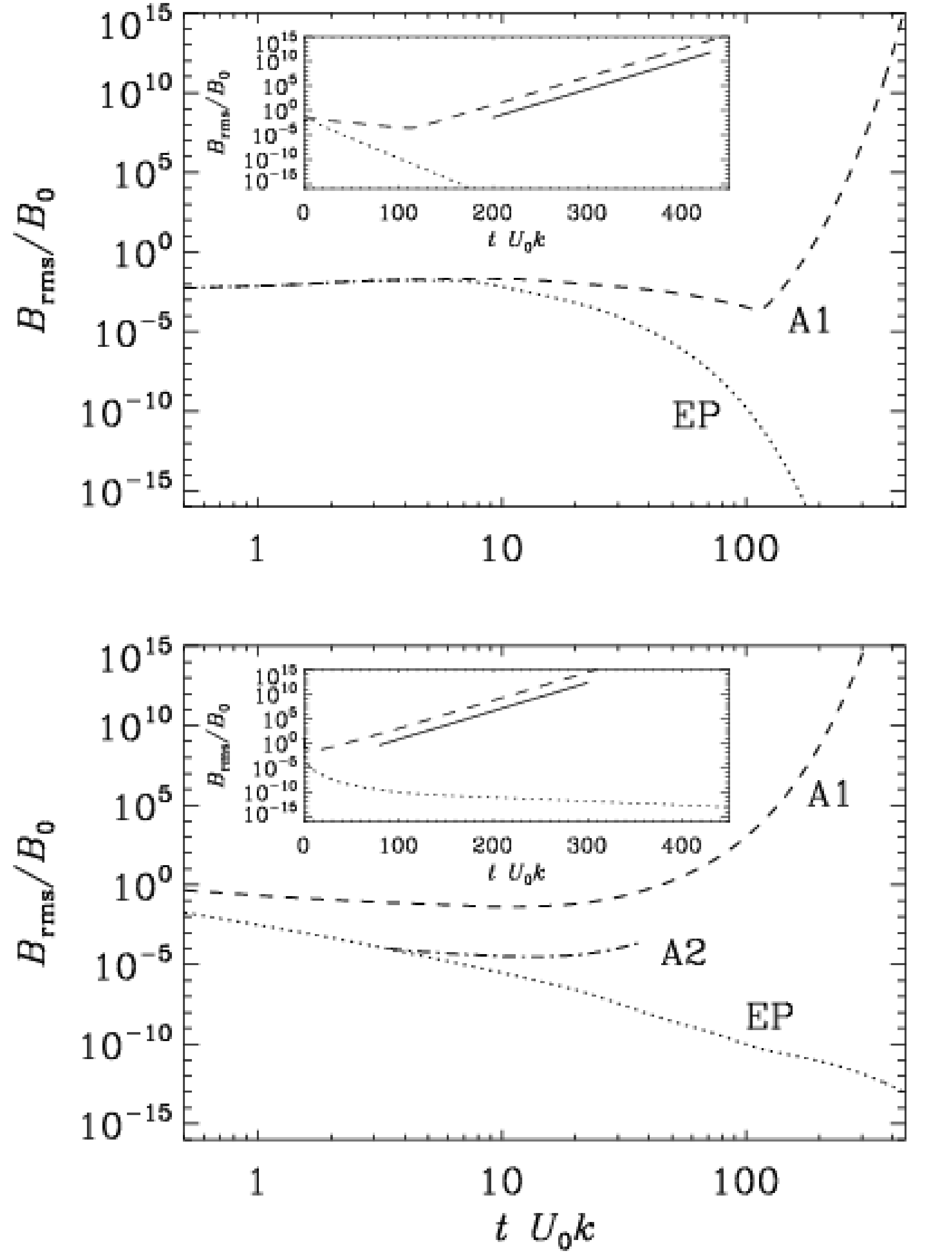
- 1970ies: mean-field models of Sun/galaxies
- 1980ies: direct simulations
- Gilman/Glatzmaier: poleward migration
- 1990ies: compressible simulations, MRI
  - Magnetic buoyancy overwhelmed by pumping
  - Successful geodynamo simulations
- 2000- magnetic helicity, catastr. quenching
  - Dynamos and MRI at low Pr_M=□□□

Easy to simulate?

- Yes, but it can also go wrong
- *2 examples: manipulation with diffusion*
- Large-scale dynamo in periodic box
  - With hyper-diffusion curl$^{2n}\boldsymbol{B}$
  - amplitude by $(k/k_f)^{2n-1}$
- Euler potentials with artificial diffusion
  - Dα/Dt=□□α,
    Dβ/Dt=□□□□///□grad□□x□grad□

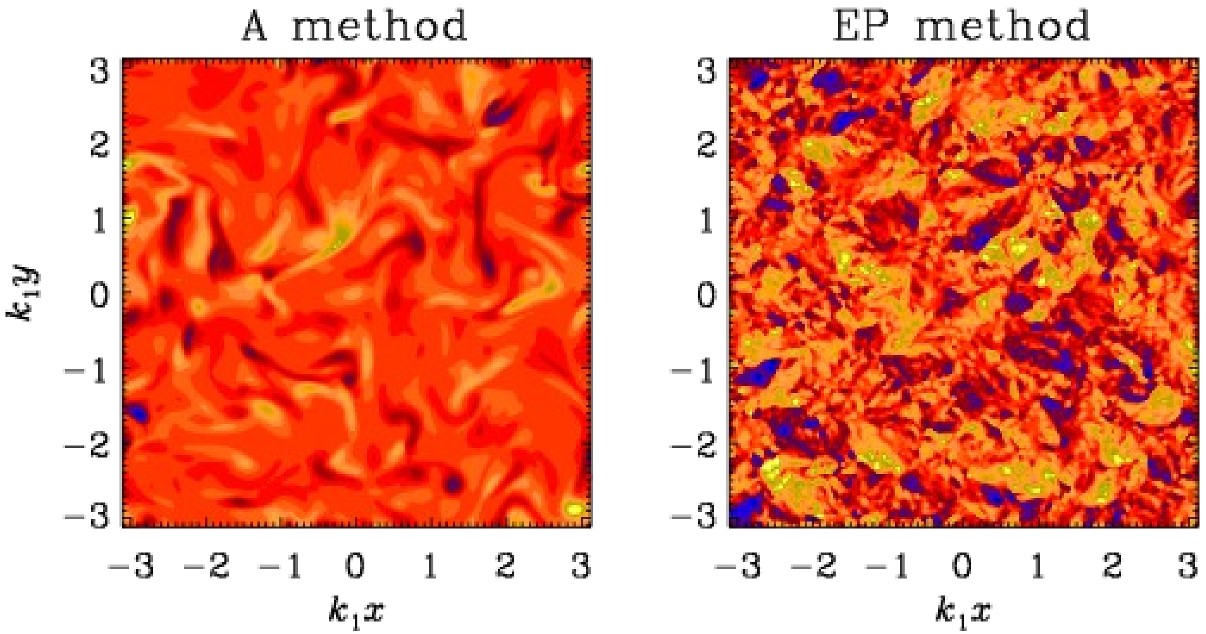
Dynamos with Euler Potentials

- ***B*** = grad□ x grad□
- ***A*** = □ grad□, so ***A.B***=0
- Here: Robert flow
  - Details MNRAS 401, 347
- Agreement for *t*<8
  - For smooth fields, not for □-correlated initial fields
- Exponential growth (A)
- Algebraic decay (EP)



Reasons for disagreement

- because dynamo field is helical?
- because field is three-dimensional?
- none of the two: it is because η is finite



Is this artificial diffusion kosher?

$$\left(\frac{\mathrm{D}\alpha}{\mathrm{D}t}-\eta\nabla^2\alpha\right)\nabla\beta-\left(\frac{\mathrm{D}\beta}{\mathrm{D}t}-\eta\nabla^2\beta\right)\nabla\alpha=\mathbf{R}+\nabla\phi$$

$$\mathbf{R}=\eta\left(\nabla\alpha\cdot\nabla\right)\nabla\beta-\eta\left(\nabla\beta\cdot\nabla\right)\nabla\alpha$$

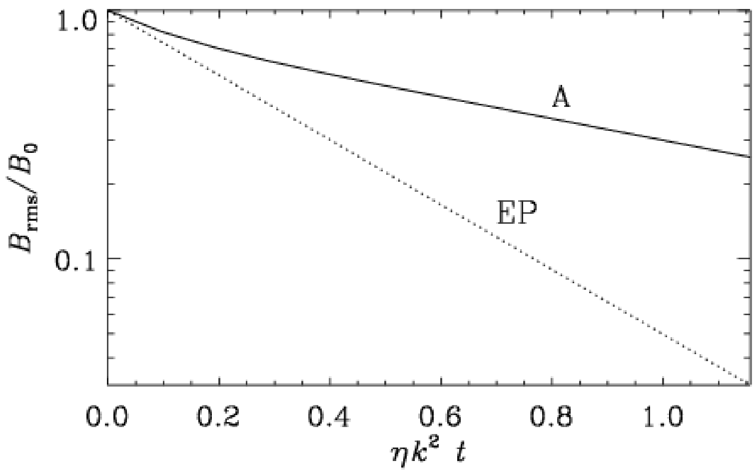
Make □ very small, it is artificial anyway,
Surely, the R term cannot matter then?

Problem already in 2-D nonhelical

$$\mathbf{B}=(0,0,\sin x\sin^2 y)$$

$$\alpha=-\cos y,\quad \beta=\cos x\sin y$$

$$\nabla\alpha=\begin{pmatrix}0\\ \sin y\\ 0\end{pmatrix},\quad \nabla\beta=\begin{pmatrix}-\sin x\sin y\\ \cos x\cos y\\ 0\end{pmatrix}$$



- Works only when α and β are not functions of the same coordinates

Alternative possible in 2-D

$$\alpha=\tfrac{1}{2}y-\tfrac{1}{4}\sin 2y,\quad \beta=\cos x$$

Remember: $\mathbf{R}=\eta(\nabla\alpha\cdot\nabla)\nabla\beta-\eta(\nabla\beta\cdot\nabla)\nabla\alpha$

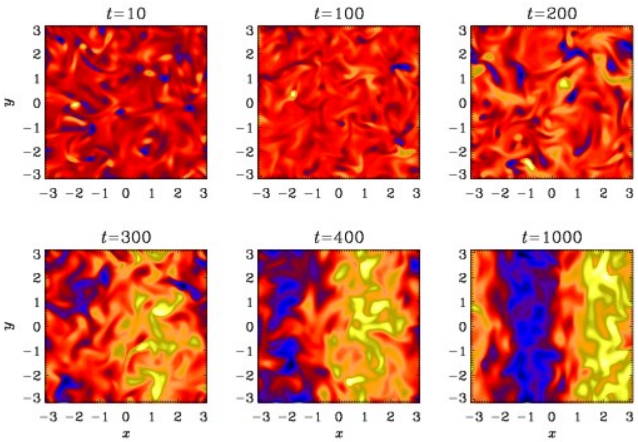
Method of choice? No, thanks

required and the corresponding terms can be switched off. The Euler potential approach shows in all tests a considerably higher accuracy than previous magnetic SPH formulations and is our method of choice for our future astrophysical applications of the MAGMA code.

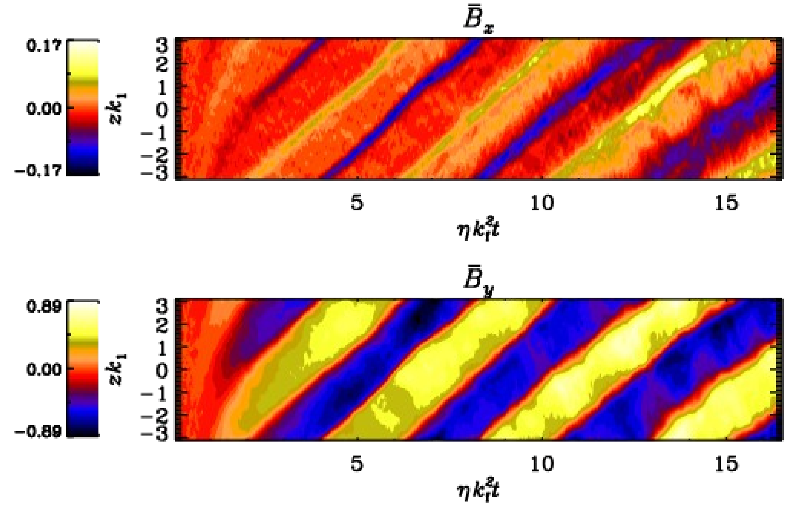
- It's not because of helicity (*cf*. nonhel dyn)
- Not because of 3-D: *cf*. 2-D decay problem
- It's really because $\alpha(x,y,z,t)$ and $\beta(x,y,z,t)$

Other good examples of dynamos

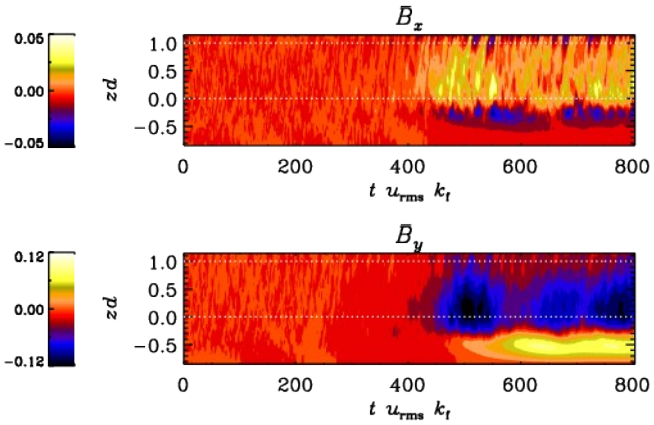
Helical turbulence (B_y)



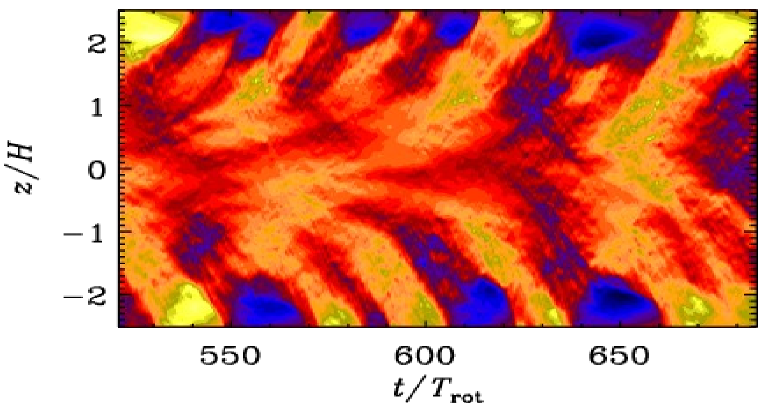
Helical shear flow turb.



Convection with shear



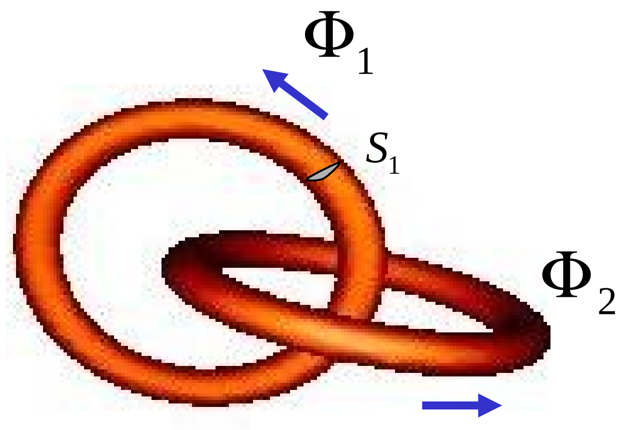
M



$$\omega = \eta_t k_1^2$$

$$c = \eta_t k_1$$

Magnetic helicity measures linkage of flux



$$H = \int_V \mathbf{A} \cdot \mathbf{B} \, \mathrm{d}V$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$H_1 = \int_{L_1} \mathbf{A} \cdot \mathrm{d}\ell \int_{S_1} \mathbf{B} \cdot \mathrm{d}\mathbf{S}$$

$$= \int_{S_2} \nabla \times \mathbf{A} \cdot \mathrm{d}\mathbf{S} = \Phi_2$$

$$= \Phi_1$$

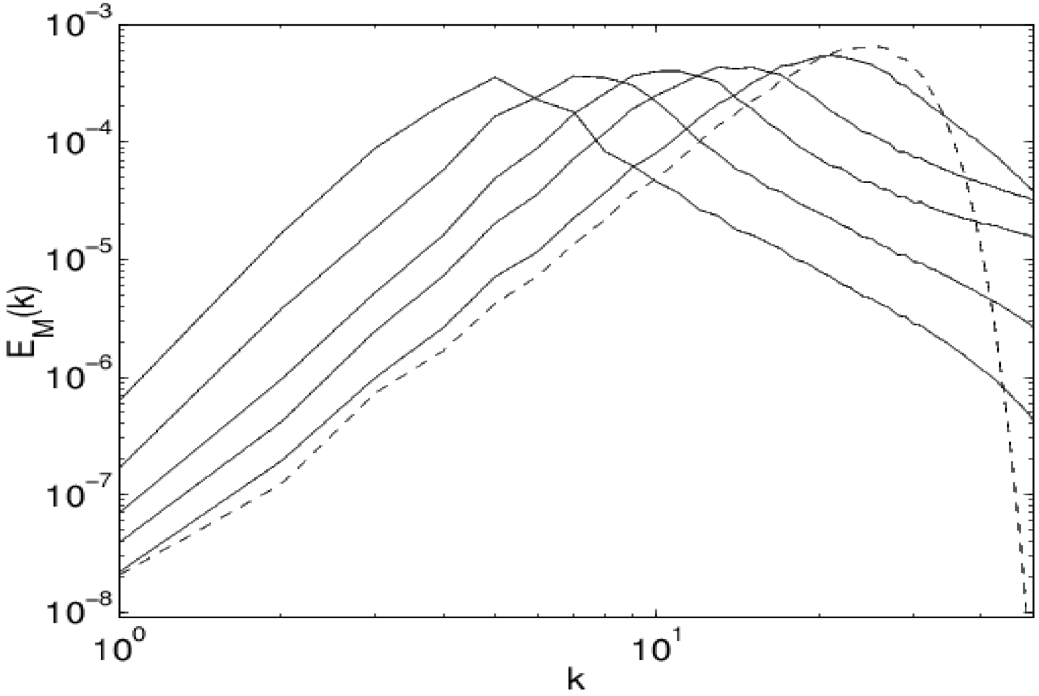
$$H = \pm 2 \Phi_1 \Phi_2$$

T

Decay of helical field: inverse cascade

Important applications to early Universe: EW & QCD phase transitions

- Inverse cascade on large scales
- Forward cascade on small scales



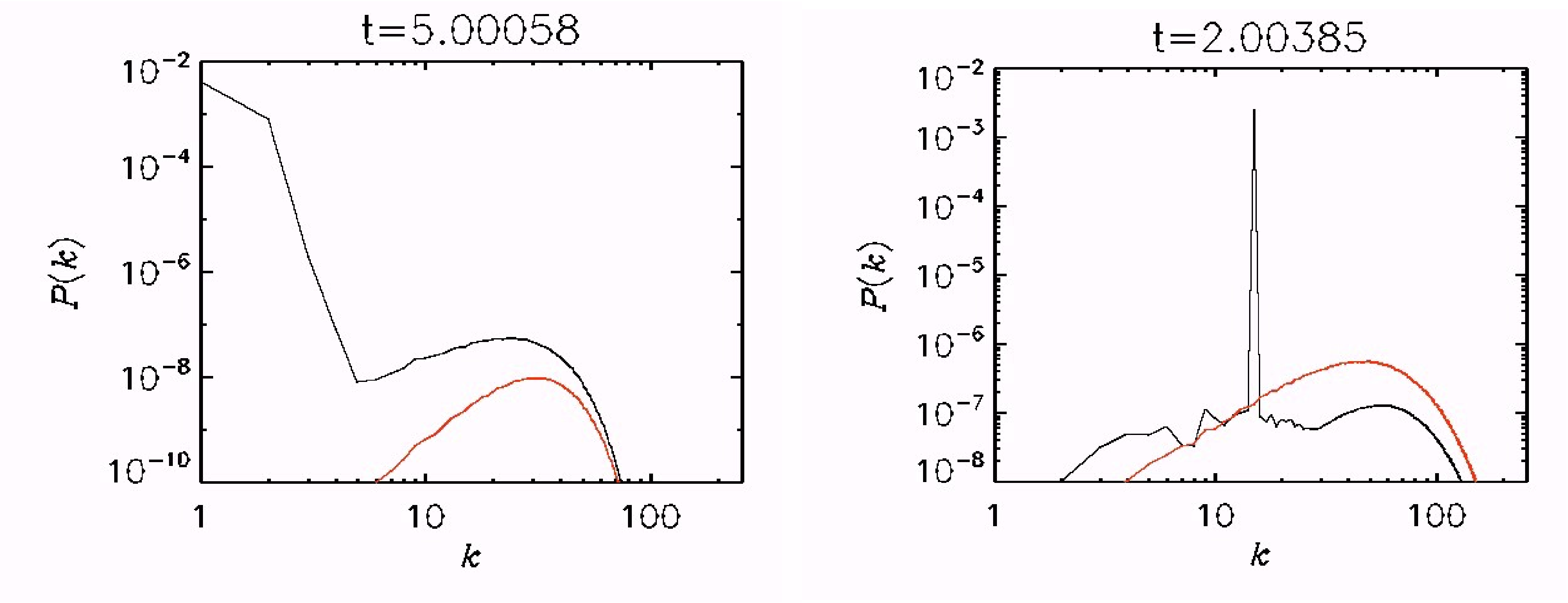
Initial slope
$E \sim k^4$

Christensson et al.
(2001, PRE 64, 056405)

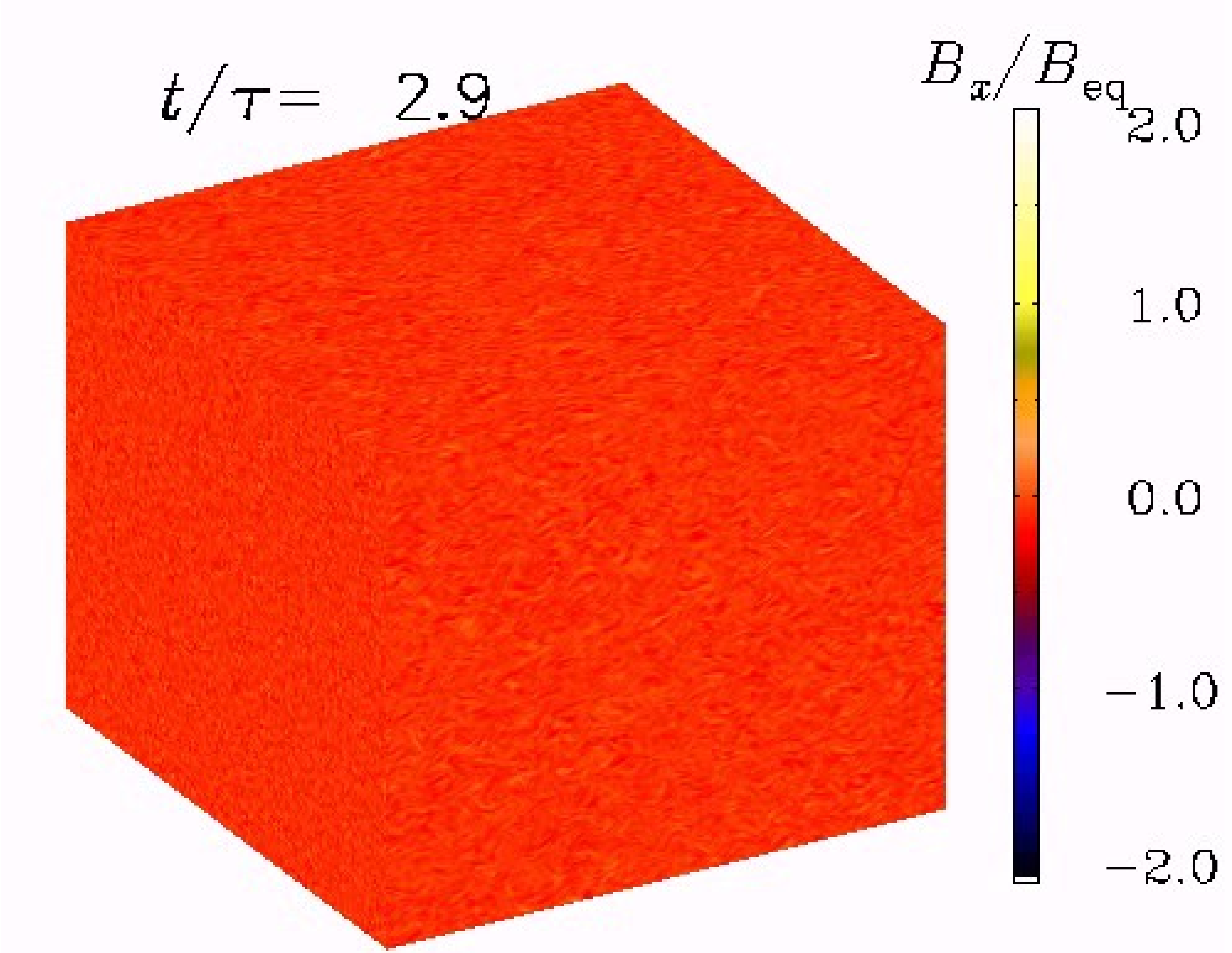
Nonhelical & helical turbulence

Dynamos in both cases: non-magnetic solutions do not exist

…when conductivity high enough



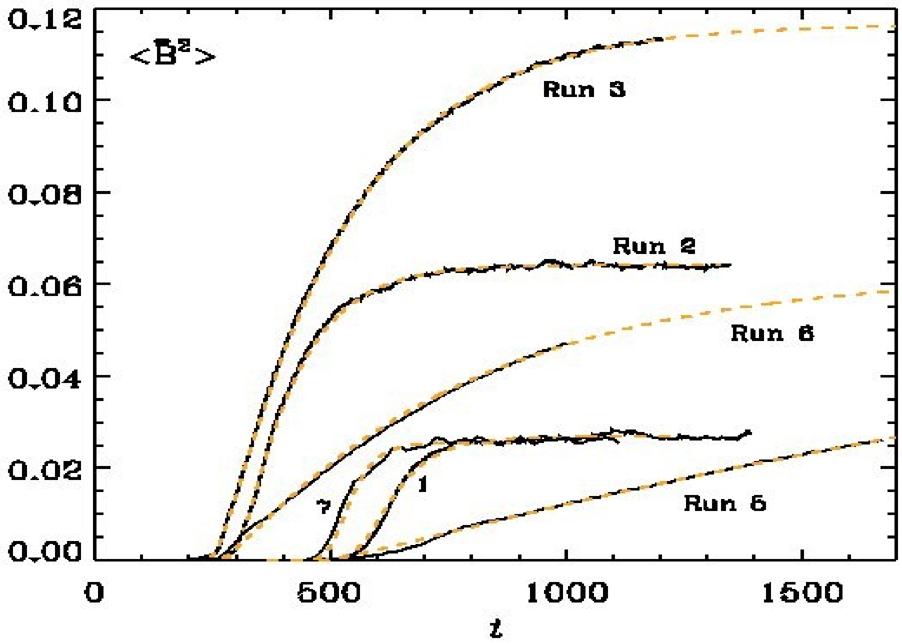
With helicity: gradual build-up of large-scale field



$t/\tau = 2.9$
B_x/B_{eq}
2.0
1.0
0.0
−1.0
−2.0

One big flaw: slow saturation

(explained by magnetic helicity conservation)



$$\frac{d}{dt}\langle \mathbf{A}\cdot\mathbf{B}\rangle = -2\eta\langle \mathbf{J}\cdot\mathbf{B}\rangle$$

$$\langle \mathbf{J}\cdot\mathbf{B}\rangle = \langle \overline{\mathbf{J}}\cdot\overline{\mathbf{B}}\rangle + \langle \mathbf{j}\cdot\mathbf{b}\rangle$$

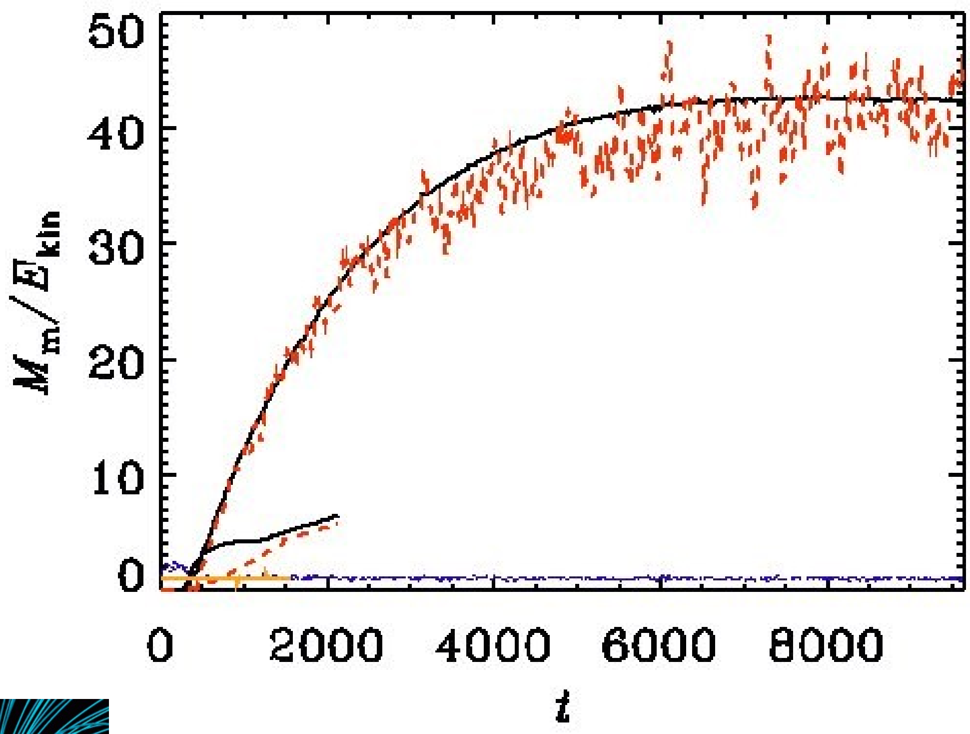
$$\langle \overline{\mathbf{J}}\cdot\overline{\mathbf{B}}\rangle \approx -k_1\langle \overline{\mathbf{B}}^2\rangle, \quad \langle \mathbf{j}\cdot\mathbf{b}\rangle \approx k_\mathrm{f}\langle \mathbf{b}^2\rangle$$

$$k_1^{-1}\frac{d}{dt}\langle \overline{\mathbf{B}}^2\rangle = -2\eta k_1\langle \overline{\mathbf{B}}^2\rangle + 2\eta k_\mathrm{f}\langle \mathbf{b}^2\rangle$$

molecular value!!

$$\langle \overline{\mathbf{B}}^2\rangle = \langle \mathbf{b}^2\rangle \frac{k_f}{k_1}\left[1 - e^{-2\eta k_1^2 (t - t_\mathrm{s})}\right]$$

Helical dynamo saturation with hyperdiffusivity



PRL 88, 055003

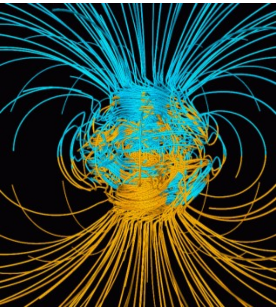
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{A}\cdot\mathbf{B}\rangle = -2\eta\langle \mathbf{J}\cdot\mathbf{B}\rangle$$

$$k_1^3\left\langle \overline{\mathbf{B}}^2\right\rangle = k_f^3\left\langle \mathbf{b}^2\right\rangle$$

for ordinary hyperdiffusion $\propto \eta_2 k^4$

$$k_1\left\langle \overline{\mathbf{B}}^2\right\rangle = k_f\left\langle \mathbf{b}^2\right\rangle$$

ratio 5^3=125 instead of 5



MTA – the Minimal Tau Approximation

1st aspect: replace triple correlation by quadradatic

$$\overline{uu\partial b} \approx \frac{\overline{ub}}{\tau}$$

Similar in spirit to tau approx in EDQNM → $\overline{uuu\partial b} \approx \frac{\overline{uub}}{\tau}$

2nd aspect: do not neglect triple correlation

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times \left(\overline{\mathbf{U}} \times \mathbf{b} + \mathbf{u} \times \overline{\mathbf{B}} + \underbrace{\mathbf{u} \times \mathbf{b}}_{\text{not neglected!}} - \overline{\mathbf{u} \times \mathbf{b}} \right)$$

3rd aspect: calculate $\partial \overline{\mathsf{E}} / \partial t = \overline{\mathbf{u} \times \dot{\mathbf{b}}} + \overline{\dot{\mathbf{u}} \times \mathbf{b}}$

rather than $\overline{\mathsf{E}} = \overline{\mathbf{u} \times \int \dot{\mathbf{b}}(t')\mathrm{d}t'}$

(Kleeorin, Mond, & Rogachevskii 1996, Blackman & Field 2002,
Rädler, Kleeorin, & Rogachevskii 2003)

Implications of MTA

1. MTA does not *a priori* break down at large R_m.

   (Strong fluctuations of **b** are possible!)

1. Extra time derivative of emf $\frac{\partial \overline{\mathcal{E}}}{\partial t} = \tilde{\alpha}\,\overline{\mathbf{B}} - \tilde{\beta}\,\overline{\mathbf{J}} - \frac{\overline{\mathcal{E}}}{\tau}$

$$\overline{\mathcal{E}} = \alpha\overline{\mathbf{B}} - \beta\overline{\mathbf{J}} - \underbrace{\tau\frac{\partial \overline{\mathcal{E}}}{\partial t}}_{\text{new}}$$

with

$$\alpha = \tau\tilde{\alpha}, \quad \tilde{\alpha} = -\tfrac{1}{3}\overline{\boldsymbol{\omega}\cdot\mathbf{u}} + \tfrac{1}{3}\overline{\mathbf{j}\cdot\mathbf{b}}$$

$$\beta = \tau\tilde{\beta}, \quad \tilde{\beta} = \tfrac{1}{3}\overline{\mathbf{u}^2}$$

→ hyperbolic eqn, oscillatory behavior possible!

3. τ is not correlation time, but *relaxation* time

Revised nonlinear dynamo theory
(originally due to Kleeorin & Ruzmaikin 1982)

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{A}\cdot\mathbf{B}\rangle = -2\eta\langle \mathbf{J}\cdot\mathbf{B}\rangle$$

Two-scale assumption

$$\overline{\mathcal{E}} = \alpha\overline{\mathbf{B}} - \eta_\mathrm{t}\overline{\mathbf{J}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \overline{\mathbf{A}}\cdot\overline{\mathbf{B}}\rangle = +2\langle \overline{\mathcal{E}}\cdot\overline{\mathbf{B}}\rangle - 2\eta\langle \overline{\mathbf{J}}\cdot\overline{\mathbf{B}}\rangle$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{a}\cdot\mathbf{b}\rangle = -2\langle \overline{\mathcal{E}}\cdot\overline{\mathbf{B}}\rangle - 2\eta\langle \mathbf{j}\cdot\mathbf{b}\rangle$$

Production of large scale helicity comes at the price of producing also small scale magnetic helicity

Express in terms of α

$$\alpha_M = \tfrac{1}{3}\tau\langle \mathbf{j}\cdot\mathbf{b}\rangle/\rho_0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{a}\cdot\mathbf{b}\rangle = -2\langle \overline{\mathcal{E}}\cdot\overline{\mathbf{B}}\rangle - 2\eta\langle \mathbf{j}\cdot\mathbf{b}\rangle$$

$$k_\mathrm{f}^2\langle \mathbf{a}\cdot\mathbf{b}\rangle = \langle \mathbf{j}\cdot\mathbf{b}\rangle$$

→ Dynamical α-quenching (Kleeorin & Ruzmaikin 1982)

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha_M = -2\eta k_f^2\left(R_m \frac{\langle \overline{\mathcal{E}}\cdot\overline{\mathbf{B}}\rangle}{B_{eq}^2} + \alpha_M \right)$$

Also: Schmalz & Stix (1991)

no additional free parameters

Steady limit:
consistent with Vainshtein & Cattaneo (1992)

(algebraic quenching)

$$\alpha = \frac{\alpha_0 + \eta_t R_m \overline{\mathbf{J}}\cdot\overline{\mathbf{B}}/B_{eq}^2}{1+R_m\overline{\mathbf{B}}^2/B_{eq}^2}$$

$$\alpha = \frac{\alpha_0}{1+R_m\overline{\mathbf{B}}^2/B_{eq}^2}$$

$$\alpha = \eta_t k_\mathrm{m}$$

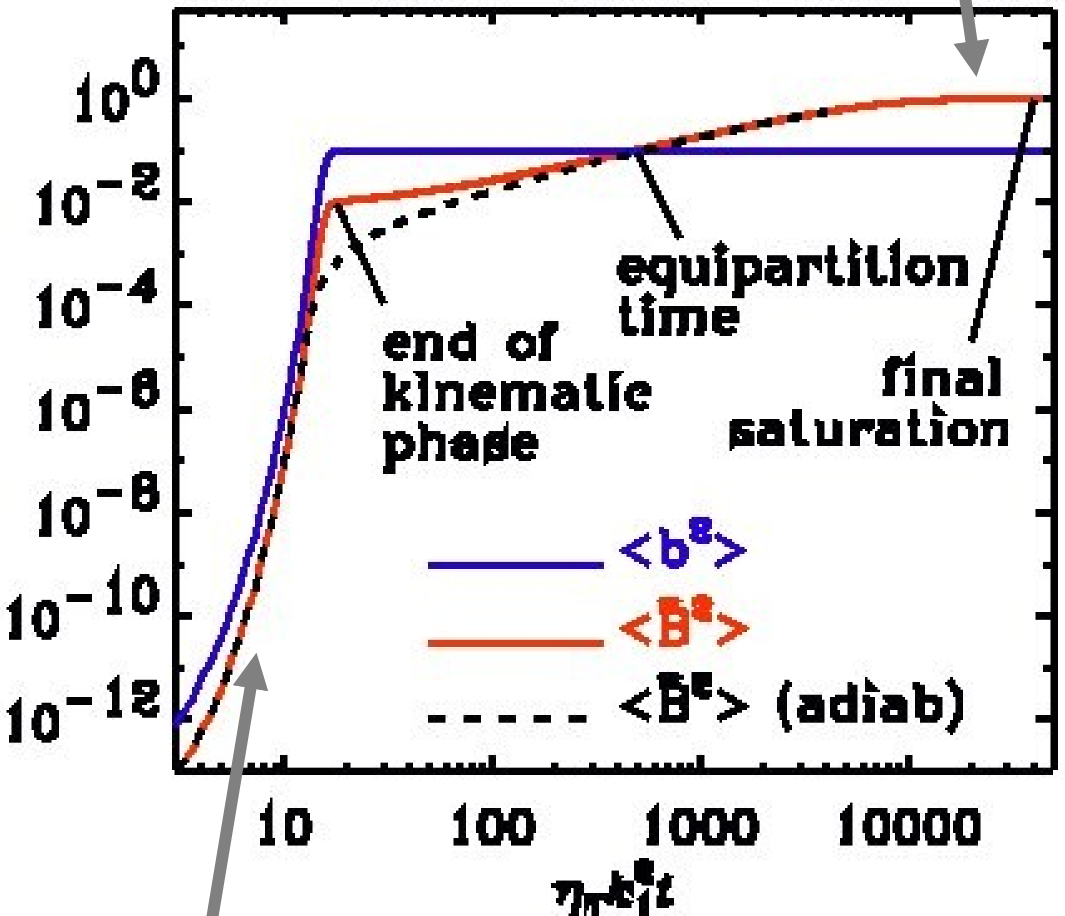
$$k_\mathrm{m} = \overline{\mathbf{J}}\cdot\overline{\mathbf{B}}/\overline{\mathbf{B}}^2$$

Is η_t quenched? → can be checked in models with shear

Full time evolution

(η_t quenched constant)

$$\langle \overline{\mathbf{J}} \cdot \overline{\mathbf{B}} \rangle + \langle \mathbf{j} \cdot \mathbf{b} \rangle = 0$$



$$\langle \overline{\mathbf{A}} \cdot \overline{\mathbf{B}} \rangle + \langle \mathbf{a} \cdot \mathbf{b} \rangle = 0$$

Significant field already after kinematic growth phase

followed by slow resistive adjustment

Magnetic helicity flux

Advantage over magnetic helicity
1) <**j.b**> is what enters α effect
2) Can define helicity density

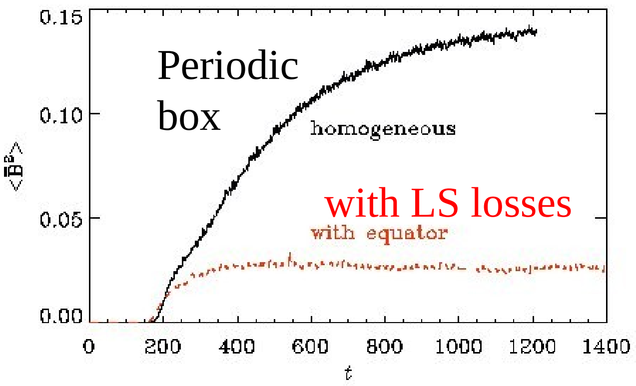
$$\overline{\mathsf{F}}_M^{SS} = \overline{\mathbf{e}\times\mathbf{a}}$$

$$\frac{\partial}{\partial t}\overline{\mathbf{a}\cdot\mathbf{b}} = -2\overline{\mathbf{e}\cdot\mathbf{b}} - \nabla\cdot\overline{\mathsf{F}}_M^{SS}$$

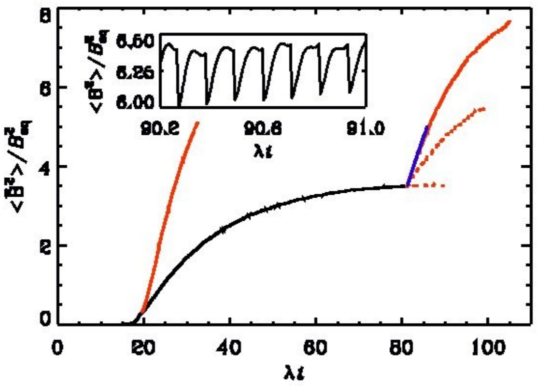
R_m also in the numerator

$$\alpha = \frac{\alpha_K + R_m\left[\left(\overline{\mathbf{J}}\cdot\overline{\mathbf{B}} - \tfrac{1}{2}k_\mathrm{f}^2\nabla\cdot\overline{\mathsf{F}}_\mathrm{M}^\mathrm{SS}\right)/B_{eq}^2 - \dfrac{\partial\alpha/\partial t}{2\eta_\mathrm{t}k_\mathrm{f}^2}\right]}{1+R_m\overline{\mathbf{B}}^2/B_{eq}^2}$$

Large scale vs small scale losses



Diffusive large scale losses:
→ lower saturation level
(Brandenburg & Dobler 2001)



Small scale losses (artificial)
→ higher saturation level
→ still slow time scale

Numerical experiment:
remove field for k>4
every 1-3 turnover times
(Brandenburg et al. 2002)

Where do we stand after 30 years

- Mean-field theory qualitatively confirmed!
  - Convection (e.g. Ossendrijver), forced turbulence
  - Alternatives (e.g. **ΩxJ** and **SJ** effects) to be explored
- Homogeneous dynamos saturate resistively
  - Entirely magnetic helicity controlled
- Inhomogeneous dynamo
  - Open surface, equator
  - Current helicity flux important
    - Finite if there is shear
  - Avoid magnetic helicity, use current helicity