Fundamentals of magnetohydrodynamics

Part IV

Daniel Gómez 1,2

Group of Astrophysical Flows
Buenos Aires - Argentina

Email: dgomez@df.uba.ar Webpage: http://astro.df.uba.ar

(1) Instituto de Astronomia y Fisica del Espacio, CONICET, Argentina
(2) Departamento de Fisica, Universidad de Buenos Aires, Argentina

ICTP-SAIFR, Sao Paulo (Brazil), October 17, 2013

Hall-MHD equations

- The dimensionless version, for a length scale L_0, density n_0 and Alfven speed $v_A = B_0 / \sqrt{4\pi m_i n_0}$

$$\frac{dU}{dt} = \frac{1}{\varepsilon}(\vec{E} + \vec{U} \times \vec{B}) - \frac{\beta}{n}\vec{\nabla} p_i - \frac{\eta}{\varepsilon n}\vec{J} + \nu \nabla^2 \vec{U} \qquad \nu = \frac{\mu}{m_i n v_A L_0}$$

$$0 = -\frac{1}{\varepsilon}(\vec{E} + \vec{U}_e \times \vec{B}) - \frac{\beta}{n}\vec{\nabla} p_e + \frac{\eta}{\varepsilon n}\vec{J}$$

where $\vec{J} = \vec{\nabla} \times \vec{B} = \frac{n}{\varepsilon}(\vec{U} - \vec{U}_e)$

- We define the Hall parameter $\varepsilon = \frac{c}{\omega_{pi} L_0}$

as well as the plasma *beta* $\beta = \frac{p_0}{m_i n_0 v_A^2}$ and the electric resistivity $\eta = \frac{c^2 \nu_{ie}}{\omega_{pi}^2 L_0 v_A}$

- Adding these two equations yields:

$$n\frac{d\vec{U}}{dt} = (\vec{\nabla} \times \vec{B}) \times \vec{B} - \beta \vec{\nabla}(p_i + p_e) + \nu \nabla^2 \vec{U}$$

Hall-MHD equations

- On the other hand, using

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi, \qquad \vec{B} = \vec{\nabla} \times \vec{A} \quad \longrightarrow \quad \frac{\partial \vec{A}}{\partial t} = (\vec{U} - \frac{\varepsilon}{n}\vec{\nabla} \times \vec{B}) \times \vec{B} - \vec{\nabla}\phi + \frac{\varepsilon\beta}{n}\vec{\nabla} p_e - \frac{\eta}{n}\vec{\nabla} \times \vec{B}$$

Hall-MHD reconnection in 2.5D

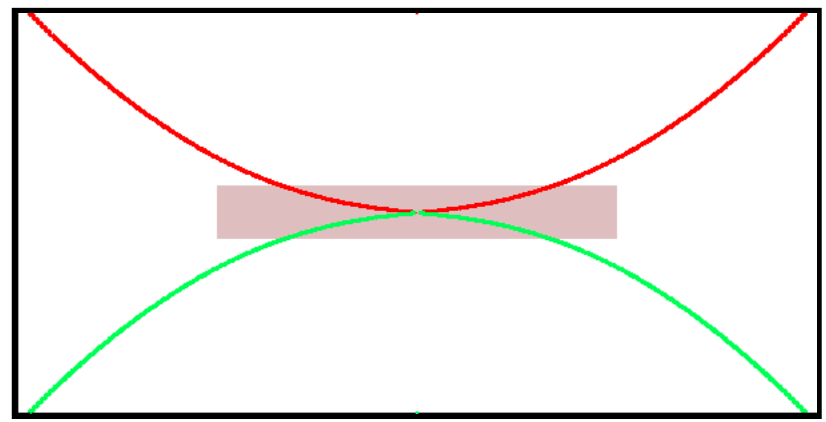
- Hall reconnection has extensively been studied for the Earth's magnetopause and also the magnetotail. The Hall effect is expected to increase the reconnection rate.

- The simplest geometrical setup is 2.5D, for which the velocity and magnetic field can be written in terms of four scalar fields (Gómez 2006, Space Sci. Rev. 122, 231; Gómez et al. 2006, Adv. Sp. Res. 37, 1287)

$$\vec{B}(x,y,t) = \vec{\nabla}\times\left[\hat{z}\,a(x,y,t)\right] \quad + \quad \hat{z}\,b(x,y,t)$$

$$\vec{U}(x,y,t) = \vec{\nabla}\times\left[\hat{z}\,\phi(x,y,t)\right] \quad + \quad \hat{z}\,u(x,y,t)$$

$$U_e = U - \varepsilon\,J = \nabla\times\left[\hat{z}\,(\phi-\varepsilon\,b)\right] \quad + \quad \hat{z}\,(u-\varepsilon\,j)$$



- The 2.5D Hall-MHD equations are

$$\partial_t a = \left[\phi-\varepsilon\,b, a\right] + \eta\nabla^2 a$$

$$\partial_t \omega = \left[\phi,\omega\right] + \left[j,a\right] + \nu\nabla^2\omega$$

$$\partial_t b = \left[\phi,b\right] + \left[u-\varepsilon\,j, a\right] + \eta\nabla^2 b$$

$$\partial_t u = \left[b,a\right] + \left[\phi,u\right] + \nu\nabla^2 u$$

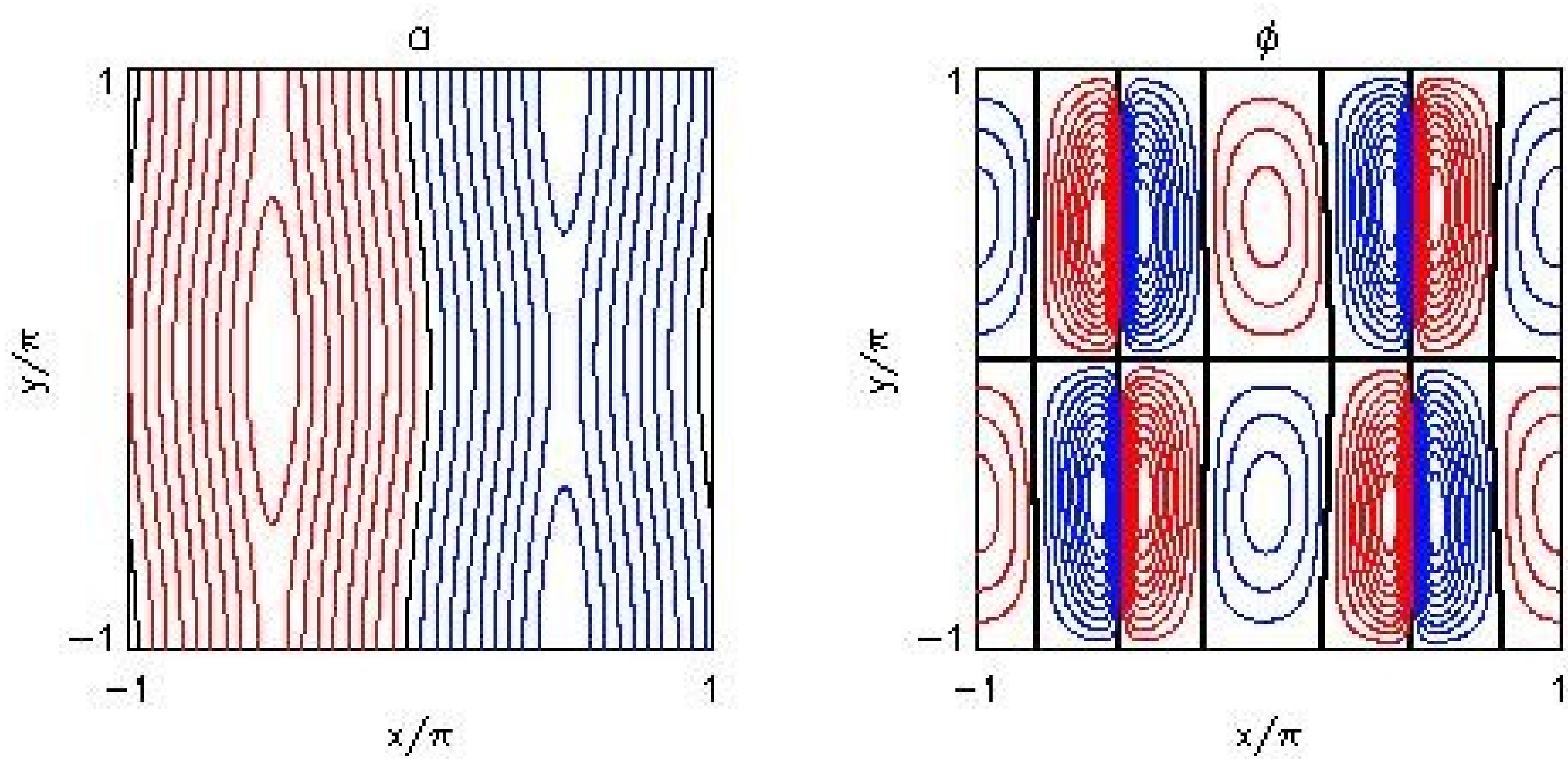
where

$$\omega = -\nabla^2\phi \quad , \quad j = -\nabla^2 a$$

- In the absence of Hall , the parallel components (u,b) have no influence on the perp. dynamics.

- When Hall is present, the parallel components will be turned on and couple to the perp. components.

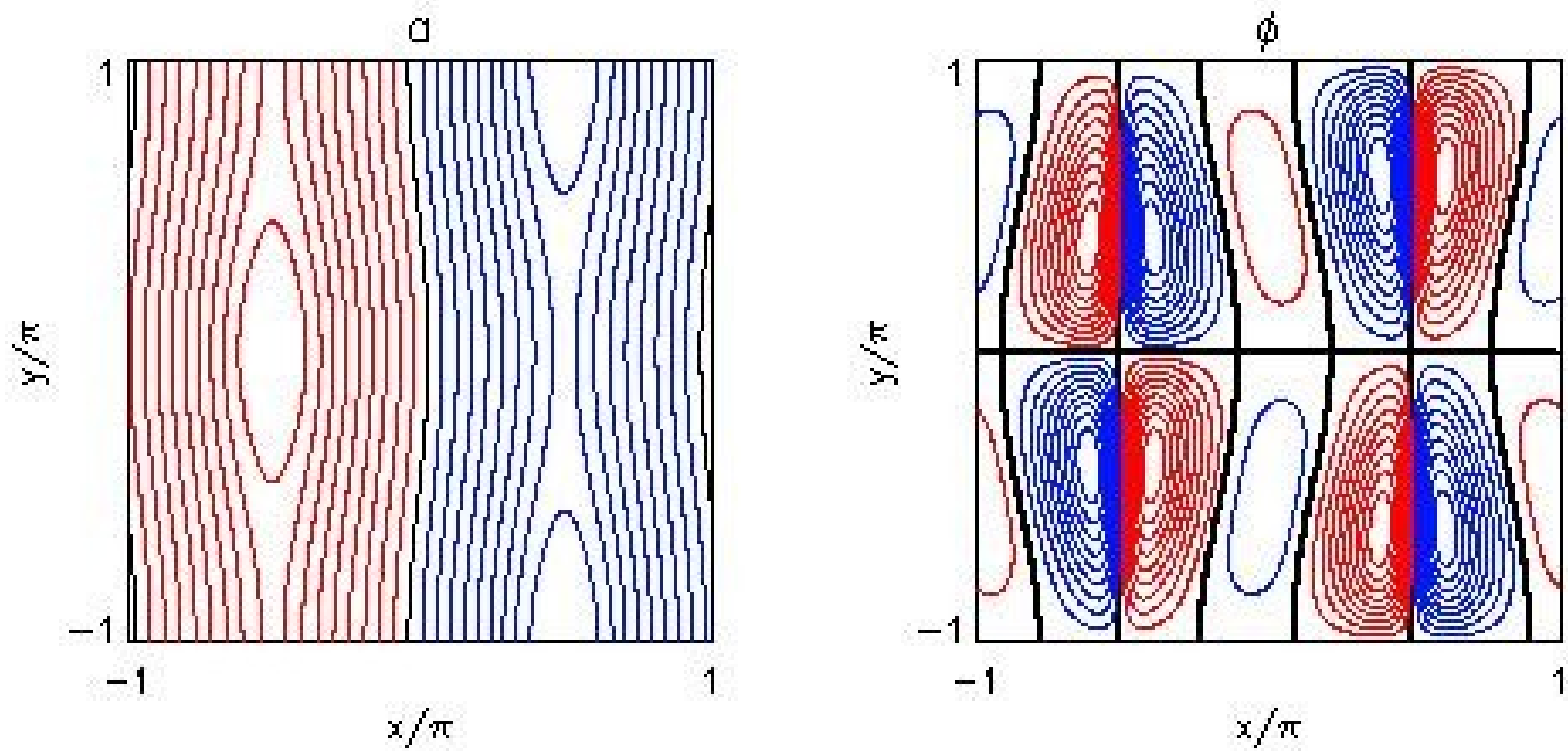
Hall-MHD reconnection in 2D



- When the Hall effect is neglected (pure MHD) **2D** reconnection is possible.

- Magnetic **fieldlines** (left) and flow **streamlines** (right) are shown at three succesive Alfven times. **Blue** contours are **positive** and the **red** ones are **negative**.

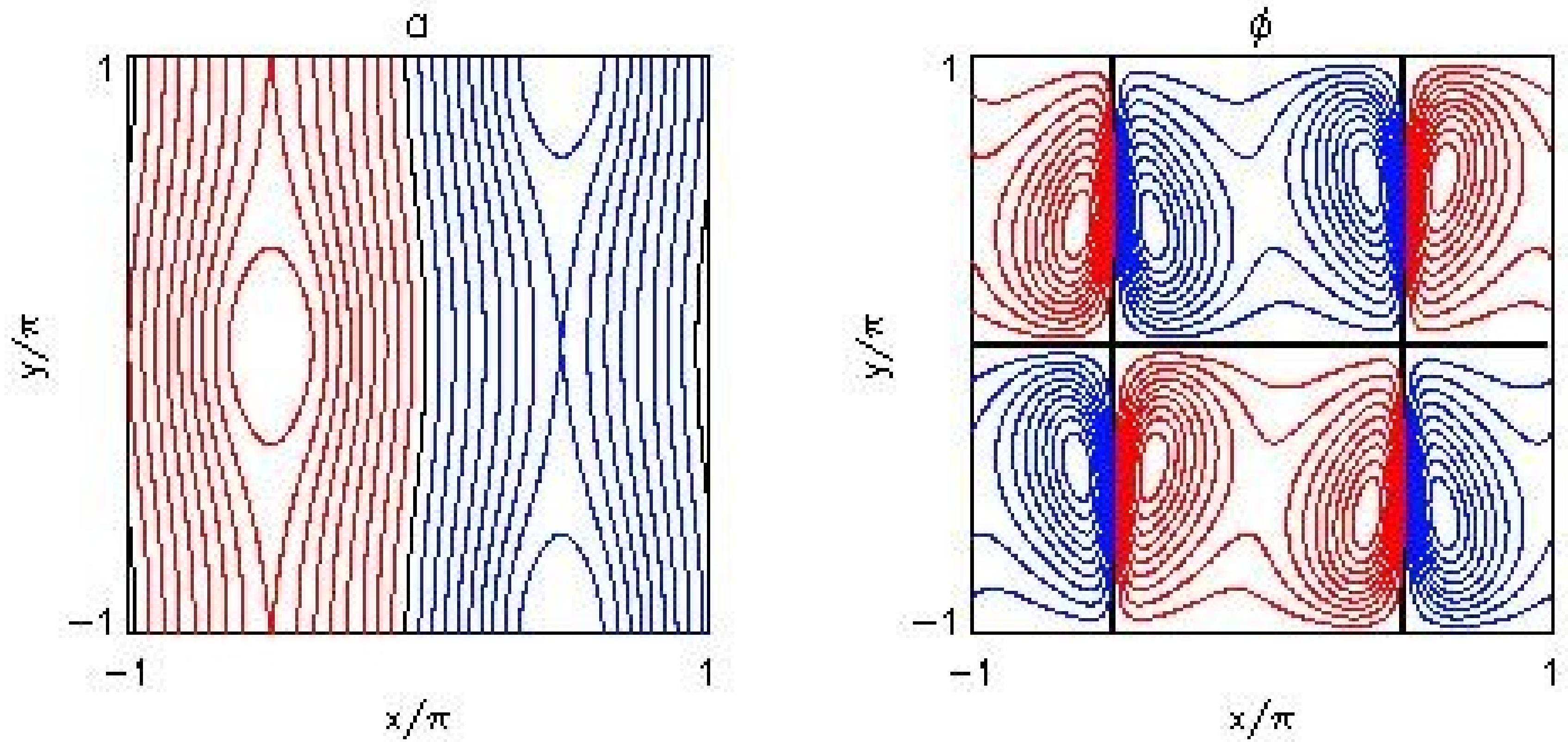
Hall-MHD reconnection in 2D



- When the Hall effect is neglected (pure MHD) **2D** reconnection is possible.

- Magnetic **fieldlines** (left) and flow **streamlines** (right) are shown at three succesive Alfven times. **Blue** contours are **positive** and the **red** ones are **negative**.

Hall-MHD reconnection in 2D

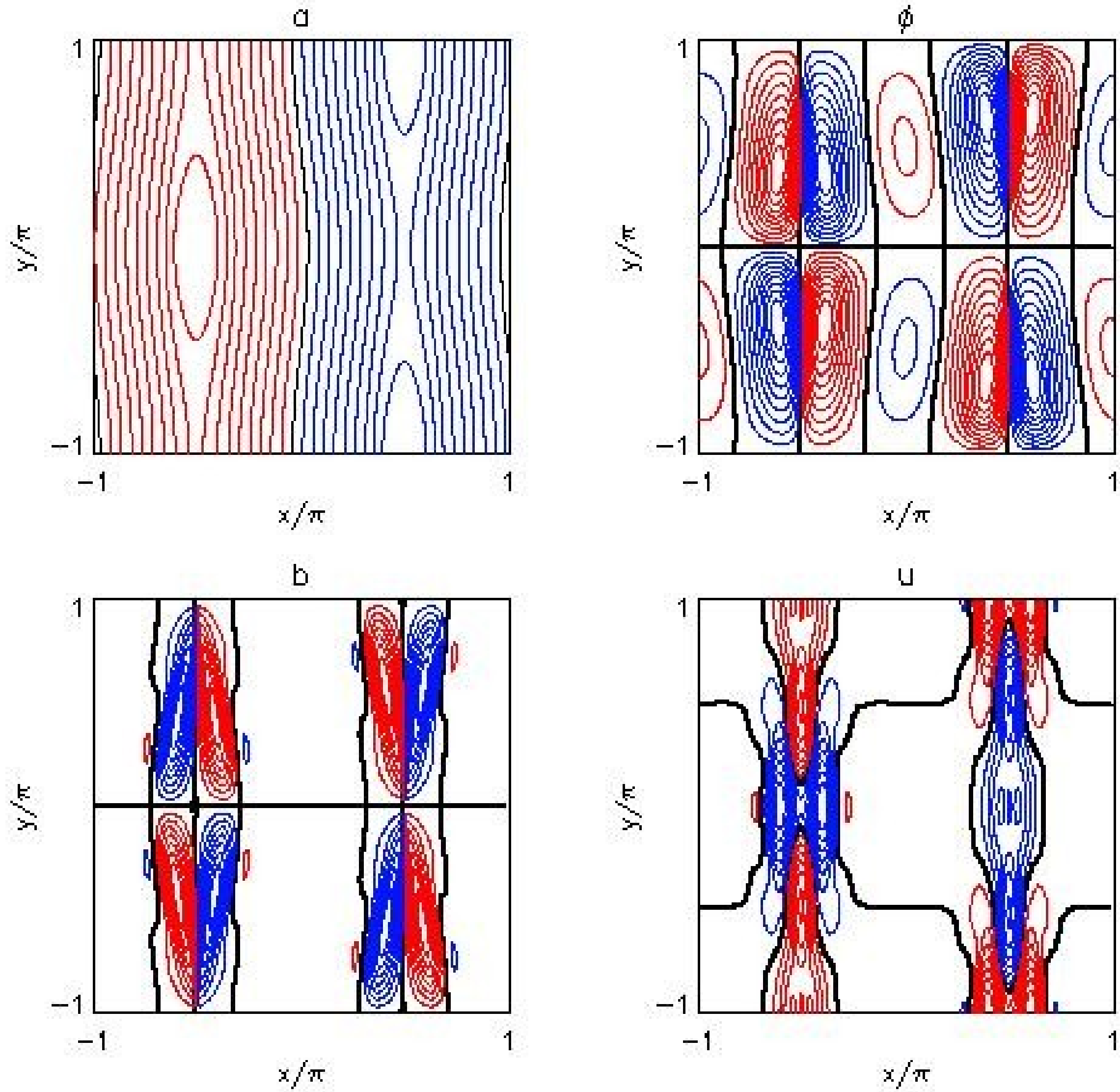


- When the Hall effect is neglected (pure MHD) **2D** reconnection is possible.

- Magnetic **fieldlines** (left) and flow **streamlines** (right) are shown at three succesive Alfven times. **Blue** contours are **positive** and the **red** ones are **negative**.

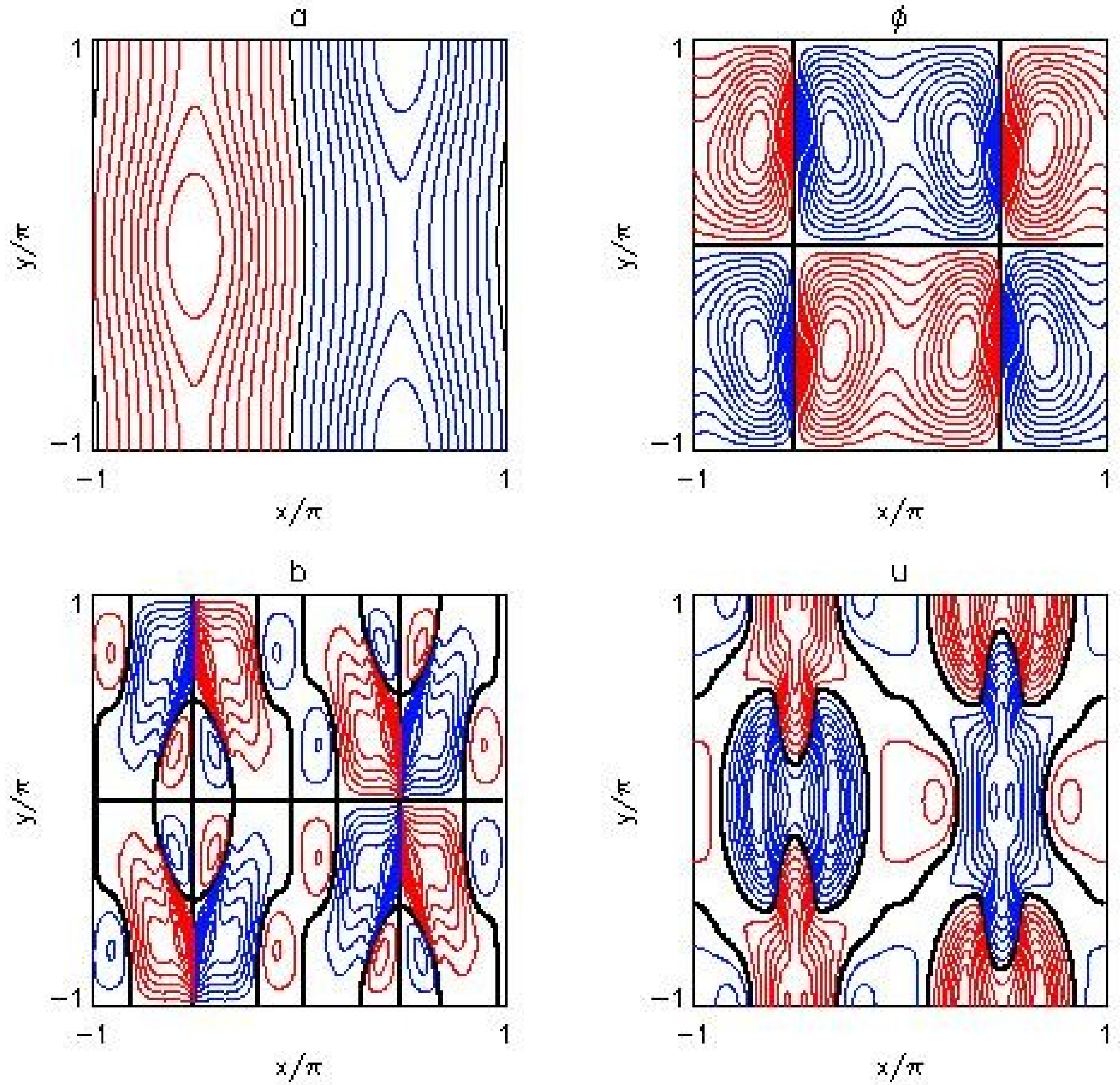
Hall-MHD reconnection in 2.5D

- When the Hall effect is considered, the out-of-plane fields are generated. They were initially set to zero.
- We show contour plots of the four scalar fields at three succesive Alfven times for $\varepsilon = 0.07$
- The out-of-plane magnetic field develops a quadrupolar pattern, while the velocity field develops a net flow at the reconnection region.
- **Blue** contours are **positive** and the **red** ones are **negative**.



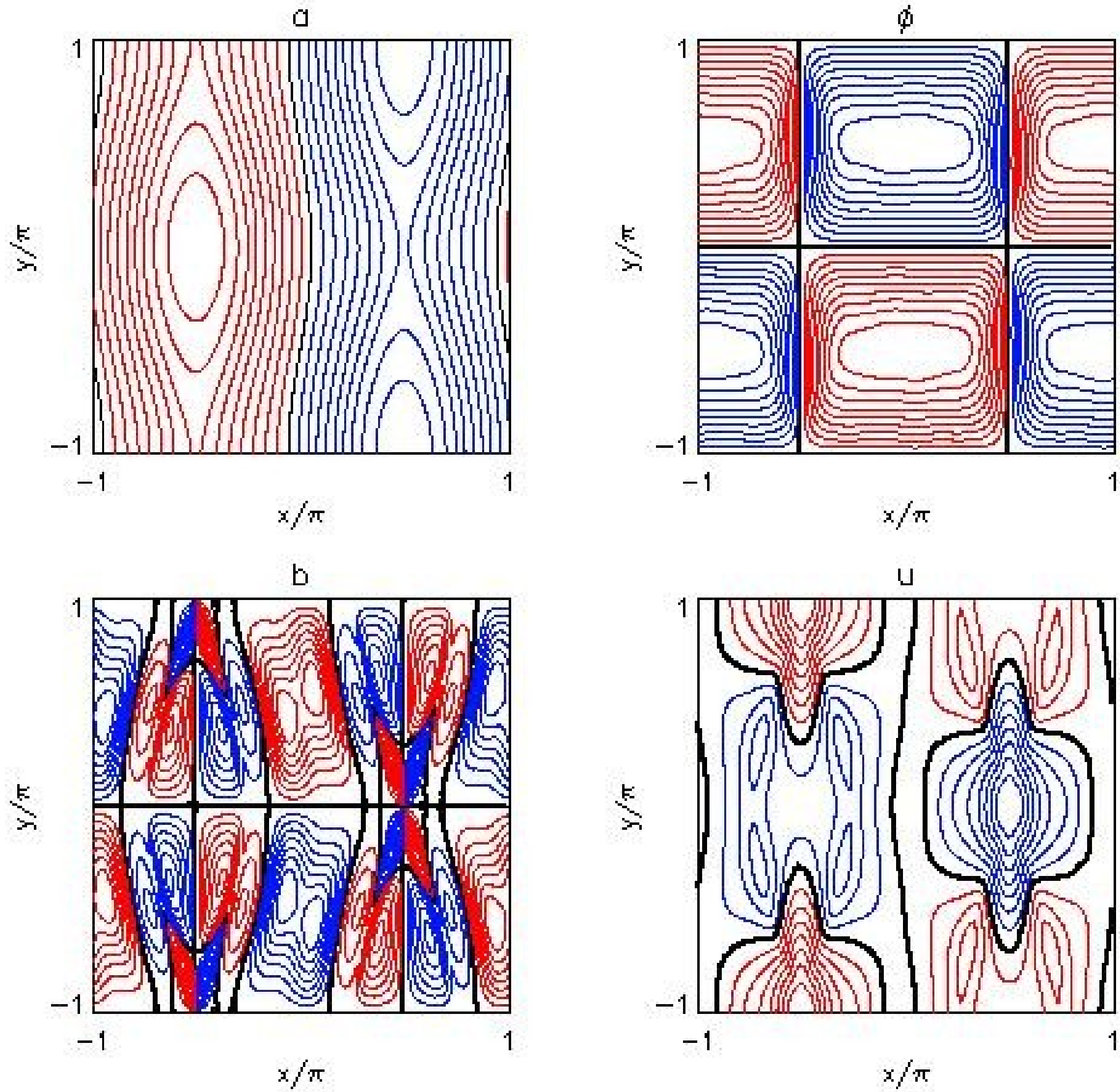
Hall-MHD reconnection in 2.5D

- When the Hall effect is considered, the out-of-plane fields are generated. They were initially set to zero.
- We show contour plots of the four scalar fields at three succesive Alfven times for $\varepsilon = 0.07$
- The out-of-plane magnetic field develops a quadrupolar pattern, while the velocity field develops a net flow at the reconnection region.
- **Blue** contours are **positive** and the **red** ones are **negative**.

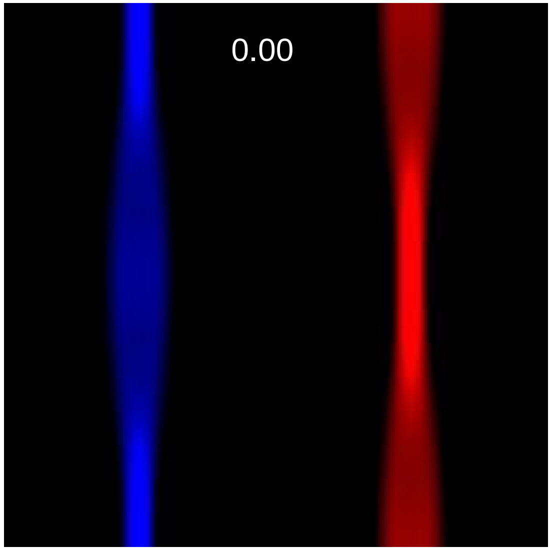


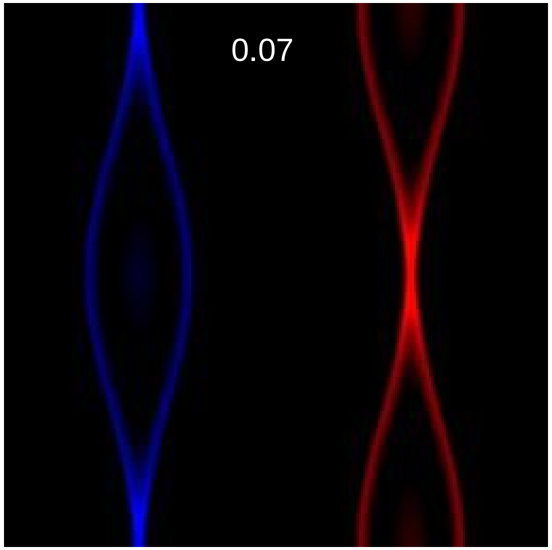
Hall-MHD reconnection in 2.5D

- When the Hall effect is considered, the out-of-plane fields are generated. They were initially set to zero.
- We show contour plots of the four scalar fields at three succesive Alfven times for $\varepsilon = 0.07$
- The out-of-plane magnetic field develops a quadrupolar pattern, while the velocity field develops a net flow at the reconnection region.
- **Blue** contours are **positive** and the **red** ones are **negative**.

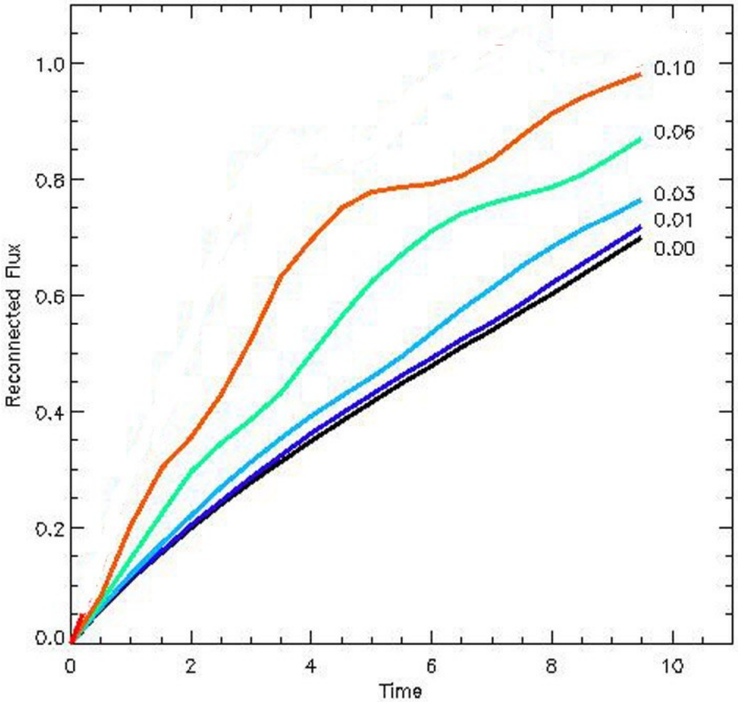


Hall-MHD reconnection rates





- The out-of-plane **current density** is shown for the cases $\varepsilon = 0.00 \; and \; 0.07$
- The current sheets becomes narrower and smaller as the Hall parameter is increased.
- The **reconnected flux** also increases with the Hall parameter, confirming previous results from collisionless and also Hall-MHD simulations.
- The plot shows reconnected flux vs. time for different values of the Hall parameter.



… back to multi-species plasmas

- For each species s we have (Goldston & Rutherford 1995):
  - Mass conservation
  $$\frac{\partial n_s}{\partial t}+\vec{\nabla}\cdot(n_s\vec{U}_s)=0$$
  - Equation of motion
  $$m_s n_s \frac{dU_s}{dt}=q_s n_s\left(\vec{E}+\frac{1}{c}\vec{U}_s\times\vec{B}\right)-\vec{\nabla}p_s+\vec{\nabla}\bullet\vec{\sigma}_s+\sum_{s'}\vec{R}_{ss'}$$
  - Momentum exchange rate
  $$R_{ss'}=-m_s n_s \upsilon_{ss'}(U_s-U_{s'})$$
- These moving charges act as sources for electric and magnetic fields:
  - Charge density
  $$\rho_c=\sum_s q_s n_s\approx 0$$
  - Electric current density
  $$\vec{J}=\frac{c}{4\pi}\vec{\nabla}\times\vec{B}=\sum_s q_s n_s\vec{U}_s$$

Two-fluid MHD equations

- Let us now retain electron inertia (i.e. $0 < m_e << m_i$):

  - Mass conservation: $$0 = \frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\vec{U}) \quad , \qquad n_e \cong n_i \cong n$$
  - Ions: $$m_i n \frac{d\vec{U}}{dt} = en(\vec{E} + \frac{1}{c}\vec{U} \times \vec{B}) - \vec{\nabla} p_i + \vec{R}$$
  - Electrons: $$m_e n \frac{d\vec{U}_e}{dt} = -en(\vec{E} + \frac{1}{c}\vec{U}_e \times \vec{B}) - \vec{\nabla} p_e - \vec{R}$$
  - Friction force: $$\vec{R} = -m_i n \nu_{ie} (\vec{U} - \vec{U}_e)$$
  - Ampere's law: $$\vec{J} = \frac{c}{4\pi} \vec{\nabla} \times \vec{B} = en(\vec{U} - \vec{U}_e) \quad \Rightarrow \quad \vec{R} = -\frac{m \upsilon_{ie}}{e} \vec{J}$$
  - Polytropic laws: $$p_i \propto n^\gamma \quad , \qquad p_e \propto n^\gamma$$

Retaining electron inertia: EIHMHD equations

- The dimensionless version, for a length scale L_0, density n_0 and Alfven speed $v_A = B_0 / \sqrt{4\pi m_i n_0}$

$$\frac{d\vec{U}_i}{dt} = \frac{1}{\varepsilon}(\vec{E} + \vec{U}_i \times \vec{B}) - \frac{\beta}{n}\vec{\nabla} p_i - \frac{\eta}{\varepsilon n}\vec{J}$$

$$\frac{m_e}{m_i}\frac{d\vec{U}_e}{dt} = -\frac{1}{\varepsilon}(\vec{E} + \vec{U}_e \times \vec{B}) - \frac{\beta}{n}\vec{\nabla} p_e + \frac{\eta}{\varepsilon n}\vec{J}$$

where $\vec{J} = \vec{\nabla} \times \vec{B} = \frac{n}{\varepsilon}(\vec{U}_i - \vec{U}_e)$

- We defined the Hall parameter $\varepsilon = \frac{c}{\omega_{pi} L_0}$

as well as the plasma *beta* $\beta = \frac{p_0}{m_i n_0 v_A^2}$ and the electric resistivity $\eta = \frac{c^2 \nu_{ie}}{\omega_{pi}^2 L_0 v_A}$

- Adding these two equations yields:

$$\frac{dU}{dt} = (\vec{\nabla} \times \vec{B}) \times (\vec{B} + \varepsilon_e^2 \vec{\nabla} \times \vec{J}) - \vec{\nabla} p$$

where $\vec{U} = \frac{m_i U_i + m_e U_e}{m_i + m_e}$

and $p = p_i + p_e$

Retaining electron inertia: EIHMHD equations

- In the equation for electrons (assuming incompressibility)

$$\frac{m_e}{m_i}\frac{d\vec{U}_e}{dt} = -\frac{1}{\varepsilon}(\vec{E} + \vec{U}_e \times \vec{B}) - \beta_e \vec{\nabla} p_e + \frac{\eta}{\varepsilon}\vec{J} \qquad \vec{J} = \vec{\nabla} \times \vec{B} = \frac{1}{\varepsilon}(\vec{U}_i - \vec{U}_e)$$

we replace $\vec{E} = -\frac{1}{c}\frac{\partial A}{\partial t} - \vec{\nabla}\phi \quad and \quad \vec{B} = \vec{\nabla} \times \vec{A}$

to obtain

$$\frac{\partial}{\partial t}(\vec{A} - \varepsilon_e^2 \nabla^2 \vec{A} - \frac{\varepsilon_e^2}{\varepsilon}\vec{U}) = (\vec{U} - \varepsilon \vec{J}) \times (\vec{B} - \varepsilon_e^2 \nabla^2 \vec{B} - \frac{\varepsilon_e^2}{\varepsilon}\vec{w}) - \vec{\nabla}(\phi - \varepsilon \beta_e p_e - \frac{\varepsilon_e^2}{\varepsilon}\frac{U_e^2}{2}) + \eta \nabla^2 \vec{A}$$

- Electron inertia is quantified by the dimensionless parameter $\varepsilon_e = \sqrt{\frac{m_e}{m_i}}\varepsilon = \frac{c}{\omega_{pe} L_0}$

- Just as the Hall effect introduces the new spatial scale $k_H = \frac{1}{\varepsilon}$ (the ion skin depth), electron inertia introduces the electron skin depth $k_e = \frac{1}{\varepsilon_e}$ which satisfies

$$k_e = \sqrt{\frac{m_i}{m_e}}\, k_H >> k_H$$

EIHMHD in 2.5D

- We now express the EIHMHD equations in 2.5D geometry. I.e. for simplicity we assume $\partial_z = 0$ and therefore

$$\vec{B} = \vec{\nabla} \times \left[\hat{z}\, a(x,y,t)\right] + \hat{z}\, b(x,y,t)$$
$$U = \nabla \times \left[\hat{z}\, \varphi(x,y,t)\right] + \hat{z}\, u(x,y,t)$$

- The equations for these four scalar fields are

$$\partial_t a' = [\varphi - \varepsilon b, a'] \qquad + \eta \nabla_\perp^2 a$$
$$\partial_t \omega = [\varphi, \omega] - [a, j] \qquad + \nu \nabla_\perp^2 \omega$$
$$\partial_t b' = [\varphi - \varepsilon b, b'] + [u - \varepsilon j, a'] \quad + \eta \nabla_\perp^2 b$$
$$\partial_t u = [\varphi, u] - [a, b] \qquad + \nu \nabla_\perp^2 u$$

where

$$a' = (1 - \varepsilon_e^2 \nabla_\perp^2) a - \frac{\varepsilon_e^2}{\varepsilon} u \qquad \textit{and} \qquad b' = (1 - \varepsilon_e^2 \nabla_\perp^2) b - \frac{\varepsilon_e^2}{\varepsilon} w$$

Normal modes in EIHMHD

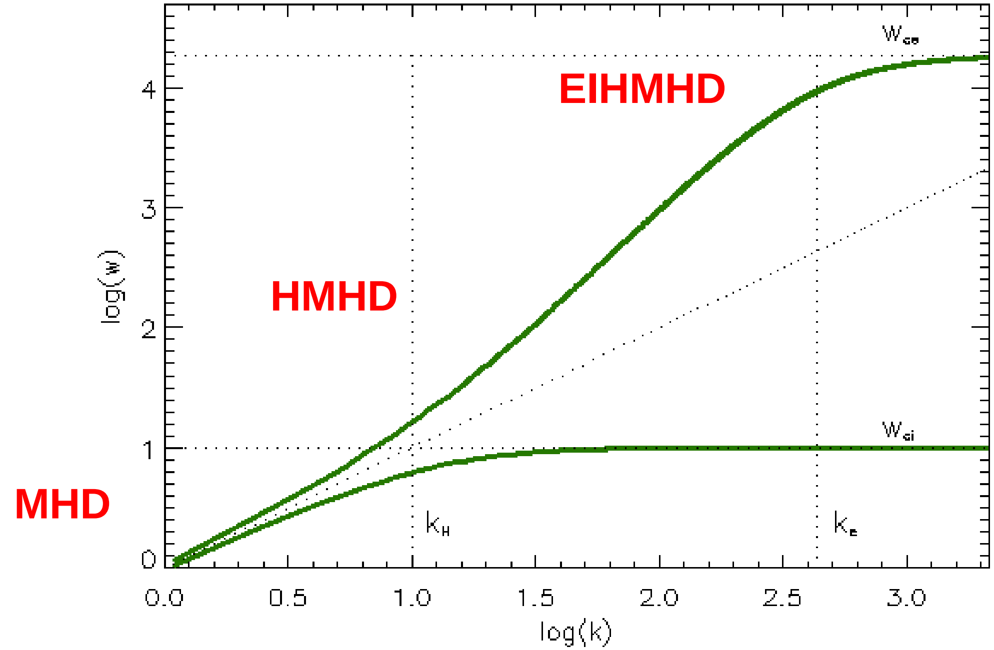
- If we linearize our equations around an equilibrium characterized by a uniform magnetic field, we obtain the following dispersion relation:

$$\left(\frac{\omega}{\vec{k}\bullet\vec{B}_0}\right)^2 \pm \frac{k\varepsilon}{1+\varepsilon_e^2 k^2}\left(\frac{\omega}{\vec{k}\bullet\vec{B}_0}\right) - \frac{1}{1+\varepsilon_e^2 k^2} = 0$$

- Asymptotically, at very large k, we have two branches

$$\omega \xrightarrow[k\to\infty]{} \omega_{ce}\cos\theta$$

$$\omega \xrightarrow[k\to\infty]{} \omega_{ci}\cos\theta$$



while for very small k, both branches simply become Alfven modes, i.e.

$$\omega \xrightarrow[k\to 0]{} k\cos\theta$$

- Different approximations, just as one-fluid MHD, Hall-MHD and electron-inertia HMHD can clearly be identified in this diagram.

Ideal invariants in EIHMHD

- For each species s in the incompressible and ideal limit

$$m_s n_s\left(\partial_t \vec{U}_s - \vec{U}_s \times \vec{W}_s\right) = q_s n_s\left(\vec{E} + \frac{1}{c}\vec{U}_s \times \vec{B}\right) - \nabla\left(p_s + m_s n_s \frac{U_s^2}{2}\right)$$

- Using that $\vec{J} = \frac{c}{4\pi}\nabla\times\vec{B} = \sum_s q_s n_s \vec{U}_s$ and $\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \nabla\phi$

  we can readily show that energy is an ideal invariant, where

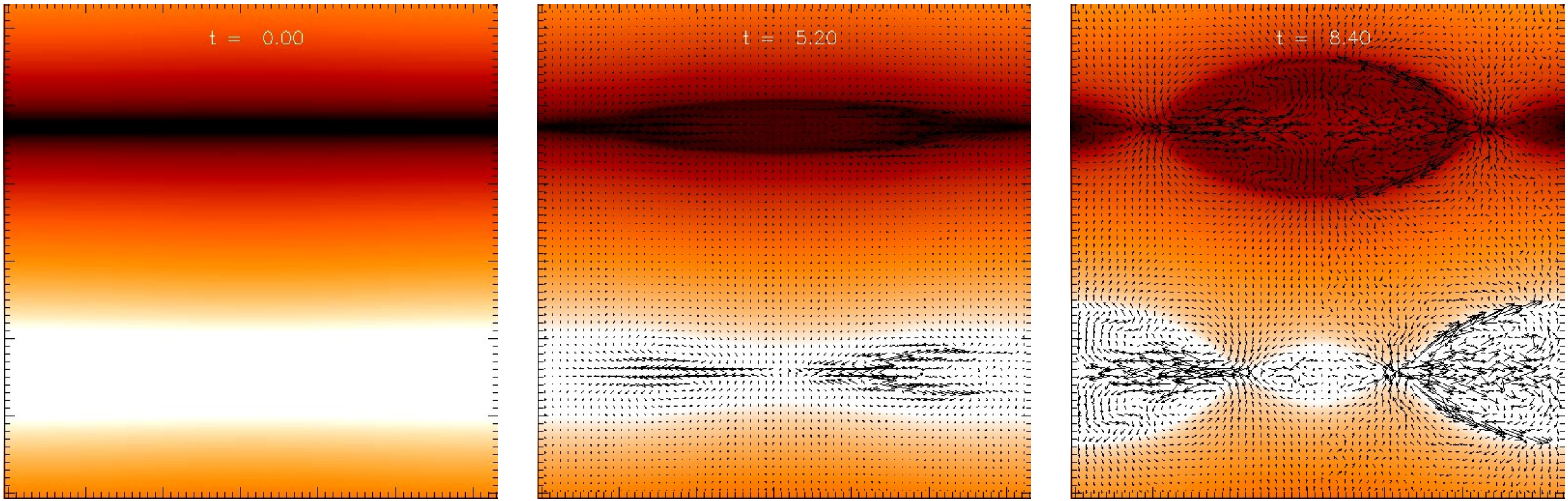
$$E = \int d^3r \left(\sum_s m_s n_s \frac{U_s^2}{2} + \frac{B^2}{8\Pi}\right)$$

- We also have a helicity per species which is conserved, where

$$H_s = \int d^3r \left(\vec{A} + \frac{c m_s}{q_s}\vec{U}_s\right) \bullet \left(\vec{B} + \frac{c m_s}{q_s}\vec{W}_s\right)$$

EIHMHD simulations

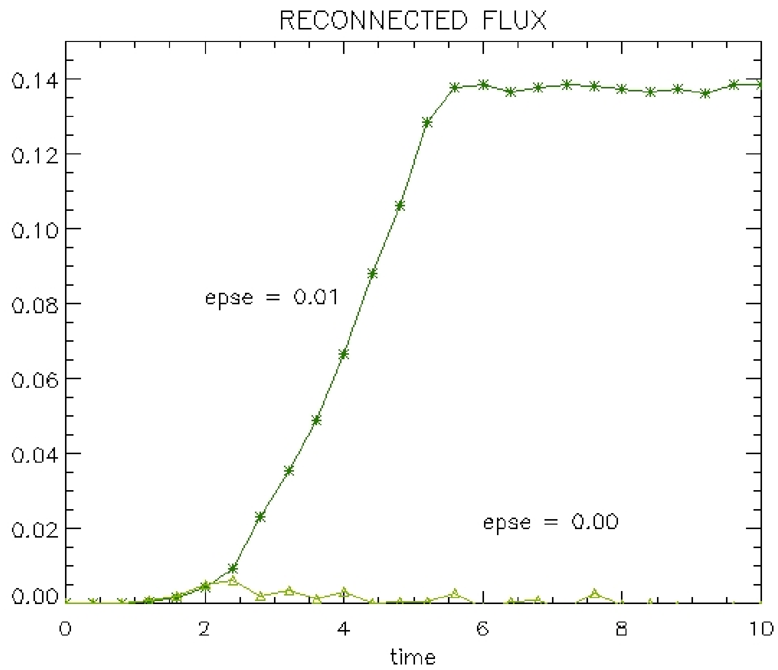
- We perform 512x512 simulations of the EIHMHD equations in 2.5D geometry to study magnetic reconnection.
- We force an external field with a double hyperbolic tangent profile to drive reconnection at two X points.
- At three succesive times we show the current density in the background, the proton flow in the left half of each frame, and the electron flow on the right half.
- Although at large scales both flows look quite similar, in the vecinity of the X points, electrons tend to move much faster, close to the Alfven velocity.

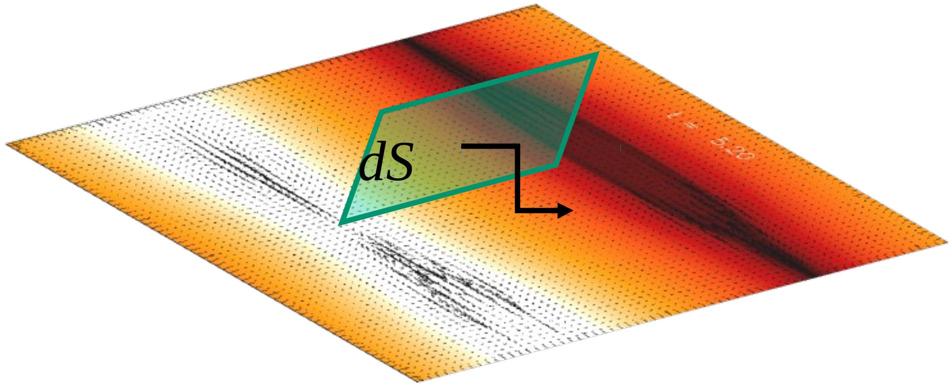


Reconnected flux in EIHMHD

- The total reconnected flux at the X-point is the magnetic flux through the perpendicular surface that extends from the O-point to the X-point.

- We compare the total reconnected flux between a run that includes electron inertia and another one that does not.





- The reconnection rate is the time derivative of these two curves.

- The apparent saturation is just a spurious effect stemming from the dynamical destruction of the X-point.

Conclusions

- In this presentation, we integrated the Hall-MHD equations numerically, to study magnetic reconnection. Even though the Hall effect does not produce reconnection, its role is to enhance the Ohmic reconnection rate.

- The existence of parallel electric fields can provide particle acceleration.

- We extended the Hall-MHD equations to include electron inertia, leading to what we call the EIHMHD equations.

- Integrating the EIHMHD equations in a 2.5D setup, we show that electron inertia leads to efficient magnetic reconnection, even in the absence of magnetic resistivity.

- The ideal invariants of a multi-species plasma are the total energy and also one helicity per species.