EFT Activity Day 5

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Today we will derive the HQET Lagrangian to $\mathcal{O}(1/m_Q)$ and the SCET Lagrangian to $\mathcal{O}(\lambda^0)$.

1 HQET Lagrangian

Exercise 1 Show that the Dirac propagator for a heavy quark with momentum $p = m_Q v + k$, where $k \ll m_Q$, is, to leading order in $\mathcal{O}(1/m_Q)$,

$$i\frac{1+\not{v}}{2v\cdot k + i\epsilon}\,. \tag{1}$$

Exercise 2 Write the Dirac Lagrangian for a heavy quark $Q(v)$ in terms of the projections $Q_v, \mathcal{Q}_v$, defined by:

$$Q(x) = e^{-im_Q v\cdot x}[Q_v(x) + \mathcal{Q}_v(x)]\,, \tag{2}$$

where

$$Q_v = e^{im_Q v\cdot x}\frac{1+\not{v}}{2}Q\,, \quad \mathcal{Q}_v = e^{im_Q v\cdot x}\frac{1-\not{v}}{2}Q\,. \tag{3}$$

The phase in Eq. (2) removes the large part of the momentum from the fields, and the fields $Q_v, \mathcal{Q}_v$ only have momentum fluctuations of order k. The two projections pick out the large and small components of a heavy quark field at low momentum.

Exercise 3 Look at the Lagrangian you got in Exercise 2. Why can you "integrate out" the field $\mathcal{Q}_v$? Find its classical equation of motion.

Exercise 4 Define the transverse component of an arbitrary vector X^μ by

$$X_\perp^\mu = X^\mu - X\cdot v\, X^\mu\,. \tag{4}$$

Using this notation, show that to $\mathcal{O}(1/m_Q)$,

$$\mathcal{L}_{\text{HQET}} = \bar{Q}_v\Big(iv\cdot D - \frac{1}{2m_Q}\not{D}_\perp\not{D}_\perp\Big)Q_v\,. \tag{5}$$

What is the Feynman rule for the quark-gluon interaction vertex at $\mathcal{O}(1/m_Q^0)$?

Exercise 5 Show that, in the HQET Lagrangian, you can use

$$\not{D}_\perp \not{D}_\perp = D_\perp^2 + \frac{g}{2}\sigma_{\mu\nu}G^{\mu\nu}\,, \tag{6}$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$. Why don't you need to put $\perp$ labels on the $G^{\mu\nu}$?

Exercise 6 HQET obeys **reparameterization invariance**, that is, under small changes in the label and residual momenta that leave the total momentum invariant:

$$\begin{aligned} v &\to v + \frac{\varepsilon}{m_Q} \\ k &\to k - \varepsilon\,. \end{aligned} \tag{7}$$

Find a constraint that must be satisfied by the 4-vector ε^μ. Work to linear order in ε.

Exercise 7 The heavy quark spinor field transforms under RPI as:

$$Q_v \to Q_v + \delta Q_v\,. \tag{8}$$

Find a constraint that must be satisfied by δQ_v (hint: consider what condition Q_v satisfies before the reparametrization Eq. (7)). Find an explicit form for δQ_v that satisfies this constraint.

Exercise 8 Show to $\mathcal{O}(1/m_Q)$ that $\mathcal{L}_{\text{HQET}}$ from is invariant under RPI.

2 SCET Lagrangian

In SCET we decompose the momenta in light-cone coordinates along a direction $n = (1,0,0,1)$ and its conjugate $\bar{n} = (1,0,0,-1)$ so that $n^2 = \bar{n}^2 = 0$ and $n\cdot\bar{n} = 2$. Namely,

$$p^\mu = \bar{n}\cdot p\frac{n^\mu}{2} + n\cdot p\frac{\bar{n}^\mu}{2} + p_\perp^\mu\,, \tag{9}$$

where

$$p_\perp^\mu = \Big(g_{\mu\nu} - \frac{n_\mu\bar{n}_\nu + n_\nu\bar{n}_\mu}{2}\Big)p^\nu\,. \tag{10}$$

For a collinear particle, we divide the momentum in a large label component and a small residual component, similar to HQET:

$$p_c^\mu = \tilde{p}_c^\mu + k^\mu\,, \tag{11}$$

where

$$\tilde{p}_c^\mu = \bar{n}\cdot\tilde{p}_c^\mu\frac{n^\mu}{2} + \tilde{p}_{c\perp}^\mu \tag{12}$$

contains the large $\mathcal{O}(Q)$ and $\mathcal{O}(Q\lambda)$ components of the collinear momentum, and k is the $\mathcal{O}(\lambda^2)$ residual momentum.

Exercise 9 Start from the QCD Dirac Lagrangian,

$$\mathcal{L} = \bar{q}(x) i \not{D}\, q(x) \,. \tag{13}$$

Factor out the large label momenta from the field:

$$q(x) = \sum_{\tilde{p}} e^{-i\tilde{p}\cdot x} q_{n,p}(x) \,, \tag{14}$$

with a sum over nonzero labels $\tilde{p}$. By convention we don't write the tilde in the field subscript. What is the QCD Lagrangian in terms of $q_{n,p}$ fields?

Exercise 10 Project out the large and small components of the fields:

$$\xi_{n,p} = \frac{\not{n}\bar{\not{n}}}{4} q_{n,p} \,, \quad \Xi_{n,p} = \frac{\bar{\not{n}}\not{n}}{4} q_{n,p} \,. \tag{15}$$

Write the Lagrangian in terms of $\xi_{n,p}$ and $\Xi_{n,p}$ fields.

Exercise 11 Argue why you can integrate out the $\Xi_{n,p}$ fields. What is its classical equation of motion? Use it to obtain the form of the Lagrangian,

$$\mathcal{L} = \sum_{\tilde{p},\tilde{p}'} e^{-i(\tilde{p}-\tilde{p}')\cdot x} \bar{\xi}_{n,\tilde{p}'} \left[in\cdot D + (\tilde{\not{p}}_\perp + i\not{D}_\perp) \frac{1}{\bar{n}\cdot\tilde{p} + i\bar{n}\cdot D} (\tilde{\not{p}}_\perp + i\not{D}_\perp) \right] \frac{\bar{\not{n}}}{2} \xi_{n,p}(x) \,. \tag{16}$$

Exercise 12 The gluon field in the covariant derivatives above are also divided into collinear and soft fields:

$$A \to A_n + A_s \,, \tag{17}$$

where the collinear and soft gluon fields scale just like the corresponding momenta:

$$A_n \sim Q(1, \lambda^2, \lambda) \,, \quad A_s \sim Q\lambda^2(1,1,1) \,, \tag{18}$$

in light-cone components $(\bar{n}\cdot A, n\cdot A, A_\perp)$. Drop the soft gluons everywhere you can in the Lagrangian Eq. (16) to leading order in λ. You should be left with the Lagrangian,

$$\mathcal{L}_{\text{SCET}} = \sum_{\tilde{p},\tilde{p}'} e^{-i(\tilde{p}-\tilde{p}')\cdot x} \bar{\xi}_{n,\tilde{p}'} \left[in\cdot D + (\tilde{\not{p}}_\perp + i\not{D}^c_\perp) \frac{1}{\bar{n}\cdot\tilde{p} + i\bar{n}\cdot D^c} (\tilde{\not{p}}_\perp + i\not{D}^c_\perp) \right] \frac{\bar{\not{n}}}{2} \xi_{n,p}(x) \,, \tag{19}$$

where $D^c = \partial + igA_n$ is the purely collinear covariant derivative. This is one form of the SCET Lagrangian, from which you can derive the Feynman rules shown in A. Manohar's lecture this morning.

Exercise 13 Define the *label operators* acting on collinear fields as:

$$\mathcal{P}^\mu \phi_{n,p} = \tilde{p}^\mu \phi_{n,p} \,, \tag{20}$$

where ϕ can be ξ or A. Using these operators, show that the SCET Lagrangian can be written in the form

$$\mathcal{L}_{\rm SCET} = \bar{\xi}_n(x)\Big[in\cdot D + i\not{\mathcal{D}}^c_\perp \frac{1}{i\bar{n}\cdot \not{\mathcal{D}}^c} i\not{\mathcal{D}}^c_\perp\Big]\frac{\not{\bar{n}}}{2}\xi_n(x)\,, \tag{21}$$

where now

$$\mathcal{D}^c_\mu = \mathcal{P}_\mu + igA^\mu_n\,, \tag{22}$$

and

$$\xi_n(x) = \sum_{\tilde{p}} e^{-i\tilde{p}\cdot x}\xi_{n,p}(x)\,. \tag{23}$$

Note Eq. (21) contains arbitrarily many $\bar{n}\cdot A_n$ gluons coupling to the collinear quarks because the component $\bar{n}\cdot A_n$ is not suppressed by any powers of λ and thus can appear arbitrarily many times in the $\mathcal{O}(\lambda^0)$ Lagrangian.

Exercise 14 Using the collinear Wilson lines

$$W_n(x) = P\exp\Big[-ig\int_{-\infty}^{x} ds\,\bar{n}\cdot A_n(x)\Big]\,, \tag{24}$$

show that

$$W_n^\dagger(x)\bar{n}\cdot\mathcal{P}W_n(x) = \bar{n}\cdot\mathcal{D}^c\,. \tag{25}$$

Show that the SCET Lagrangian can then be written:

$$\mathcal{L}_{\rm SCET} = \bar{\xi}_n(x)\Big[in\cdot D + i\not{\mathcal{D}}^c_\perp W_n^\dagger(x)\frac{1}{i\bar{n}\cdot\mathcal{P}}W_n(x)i\not{\mathcal{D}}^c_\perp\Big]\frac{\not{\bar{n}}}{2}\xi_n(x)\,, \tag{26}$$

which now has no derivatives or gluons in the denominator of the second term.

Exercise 15 Show the gauge transformation properties of Wilson lines given in A. Manohar's lecture this morning.

Now you should be able to read papers in HQET and SCET! *Muito obrigado* for your hard work and attentive interest in the EFT activities this week. If you have any questions about them after we part ways from São Paulo feel free to contact me. Best of luck in your future research!