1. Discuss non-planarity for baryon diagrams: Work out the $N_c$ scaling of a two-gluon exchange between two quarks in a baryon, in the box and crossed box case.

2. The 4-meson vertex scales like $1/N_c$ and the baryon mass as $N_c$. You can obtain some insight into the motivation of considering the baryon as a soliton in large $N_c$ from the following simple(st) example:

   (a) Consider $\lambda \phi^4$ in the broken phase ($\mu^2 > 0$)

   $$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \mu^2 \phi^2 - \frac{\lambda}{4} \phi^4$$

   Find the lowest energy state.

   (b) Show that

   $$\phi(x) = v \tanh \left( \frac{\mu}{\sqrt{2}} x \right)$$

   with $v = \sqrt{\mu^2/\lambda}$, is a static solution that interpolates between the two vacuum states. This solution is a finite energy solution, known as "the kink".

   (c) Compute the energy of this solution. You should get

   $$E_0 = \frac{4}{3} \frac{\mu^3}{\lambda \sqrt{2}}$$

   Notice that it scales as the inverse of the $\phi^4$ coupling constant.

   (d) Consider the time dependent translation $\phi(x,t) = \phi(x - a(t))$ and compute the energy again ($a(t)$ is called a collective coordinate). You will obtain

   $$E = E_0 + \frac{1}{2} E_0 \left( \frac{da}{dt} \right)^2$$

   which you can compare with the motion of a free particle and identify $E_0$ as the mass of the particle.

   (e) A model that realizes this picture for baryons is the Skyrme model, in particular you can also compute the mass splitting of the $N$ and $\Delta$ states in this model and see that it matches the general large $N$ expression (see next problem for the quark model calculation).

3. To order $1/N_c$ and in flavor $SU(2)$ the mass operator is

$$ M = N_c c_0 + \frac{1}{N_c} c_1 J^2 $$

Compute $c_1$ in the quark model. Use $H = H_0 + H_{hyp}$ where

$$ H_0 = \sum_{i=3}^3 \frac{p_i^2}{2m_i} + \frac{K}{2} \sum_{i<j} (r_i - r_j)^2 $$

$$ H_{hyp} = A \sum_{i<j} s_i \cdot s_j \delta(3)(r_{ij}) $$

Eliminating the center of mass motion using the coordinates

$$ \rho = (r_1 - r_2)/\sqrt{2} $$

$$ \lambda = (r_1 + r_2 - 2r_3)/\sqrt{6} $$

you obtain a harmonic oscillator with ground state

$$ \Psi_{00} = \alpha^3 \frac{\pi^{3/2}}{\sqrt{2}} \exp\left(-\frac{\alpha^2}{2}(\rho^2 + \lambda^2)\right) $$

where $\alpha = (3Km)^{1/4}$. Compute $m_\Delta - m_N$. You should obtain

$$ m_\Delta - m_N = \frac{3\alpha^3}{4\pi\sqrt{2\pi}} A $$

Solve for $c_1$.

4. (a) Derive the consistency relations for pion-nucleon scattering

$$ [X^{ia}, X^{jb}] = 0 $$

(b) Solve the consistency relations. Using the Wigner-Eckart theorem you can write

$$ \langle J' m' | X^{ia} | J m \rangle = X(J, J') \sqrt{\frac{2J + 1}{2J' + 1}} \left( \begin{array}{c} J \ 1 \ 1 \ J' \\
 m \ i \ \alpha \ a \ m' \ \alpha' \ a' \end{array} \right) \right) $$

Introduce a complete set of intermediate states and multiply by

$$ (-1)^{J+b} \left( \begin{array}{c} J \ 1 \ H \ i \\
 m \ \alpha \ a \ h \end{array} \right) \left( \begin{array}{c} J' \ 1 \ K \ \eta \\
 m' \ -j \ m' \ \alpha' \ -b \ \eta' \end{array} \right) $$

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After summing over projections you can use the following useful expression
\[
\sum_{m_1,j,m',j} \begin{pmatrix} J' & 1 & H' \\ m' & j & h' \end{pmatrix} \begin{pmatrix} J_1 & 1 & J' \\ m_1 & i & m' \end{pmatrix} \begin{pmatrix} J_1 & 1 & J \\ m_1 & j & m \end{pmatrix} \begin{pmatrix} J & 1 & H \\ m & i & h \end{pmatrix} = (-1)^{J+J'} \begin{pmatrix} J & 1 & H \\ J' & 1 & J_1 \end{pmatrix} \sqrt{(2J'+1)(2J+1)} \delta_{HH'} \delta_{hh'}
\] (15)

and the orthogonality of the Clebsch’s.

(c) Finally, using the identity
\[
(2H+1) \sum_{J_1} (2J_1+1) \begin{pmatrix} J & 1 & H \\ J' & 1 & J_1 \end{pmatrix} \begin{pmatrix} J & 1 & K \\ J' & 1 & J_1 \end{pmatrix} = \delta_{KK}
\] (16)

you can see that \( X(J,J') = 1 \) is a solution. This has implications, like the one of the next exercise.


5. Show that the solution of the consistency equations obtained in the previous exercise predicts \( g_{\pi NN}/g_{\pi N\Delta} = \frac{3}{2} \). This is a result also obtained in the quark model or the Skyrme model in the large \( N_c \) limit.