On spectral geometry approach to Horava-Lifshitz gravity

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- What is wrong with Quantum Gravity?
- Lifshitz's idea
- Horava-Lifshitz Gravity
  - Projectable
  - Non-projectable
  - Detailed balance condition
  - Healthy (or natural) extension
  - Problems
- Spectral dimension
  - Random walk
  - Weyl's theorem
- Spectral geometry approach
  - Spectral Action
  - Geodesic motion?
- Conclusions

Problems with the quantization of gravity

$[\lambda] = \delta$ in momentum units

$D = d - (d/2 - 1)E - n\delta$

D - superficial degree of divergence

d - space-time dimension

E - numberf of the external legs

n - number of vertices

We can expect renormalizability only when $\delta \geq 0$

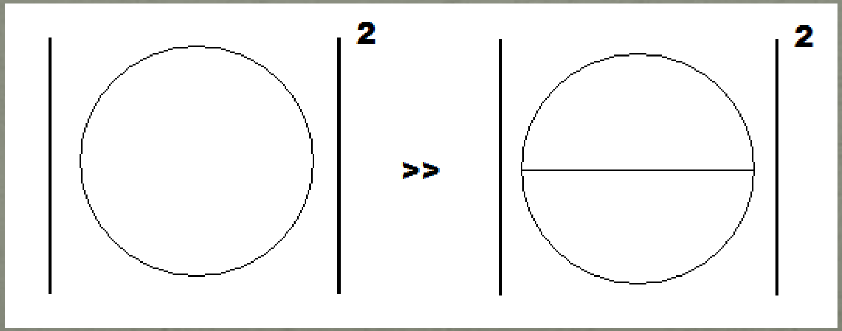
$$S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R \quad => \quad \delta \equiv [G] = 2-d$$

for $d = 4, \delta = -2 < 0$

As the result, the effective dimensionless constant is given by

$$GE^2 := \left(\frac{E}{M_P}\right)^2 \quad \text{where } M_P = \sqrt{\frac{\hbar c}{G}} = 1.22\times 10^{19} GeV$$

i.e. when $E << M_p$



Possible solutions

- (Super)string theory: contains a spin-2 massles mode => has to describe gravity. GR is recovered in long-wave regime. But, the predictive power is quite poor: the string theory landscape has 10^{500} vacua.
- Loop quantum gravity: one can perform non-perturbative quantization. Among problems, the difficulty of the recovery quasiclassical space.
- Some other approaches treat gravity as an emergent phenomenon (e.g., entropic gravity).

Horava's idea

- Lifshitz model (Lifshitz 1941)

$$S = \int dt d^n x \left(\dot{\phi}^2 - g(\Delta\phi)^2 + c^2 \phi \Delta \phi \right)$$

$$[x] = -1, [t] = -2, [\mathrm{c}] = 1$$

The propagator has the form :

$$G(\omega, \vec{k}) \propto \frac{1}{\omega^2 - c^2 \vec{k}^2 - g \vec{k}^4}$$

$$\mathrm{UV}: \frac{1}{\omega^2 - c^2\vec{k}^2 - g^2\vec{k}^4} = \frac{1}{\omega^2 - g^2\vec{k}^4} + \frac{1}{\omega^2 - g^2\vec{k}^4} c^2\vec{k}^2 \frac{1}{\omega^2 - g^2\vec{k}^4} + \ldots$$

$$\mathrm{IR}: \frac{1}{\omega^2 - c^2\vec{k}^2 - g^2\vec{k}^4} = \frac{1}{\omega^2 - c^2\vec{k}^2} + \frac{1}{\omega^2 - c^2\vec{k}^2} g^2\vec{k}^4 \frac{1}{\omega^2 - c^2\vec{k}^2} + \ldots$$

I.e. we have two fixed points: UV, which corresponds to z=2 and has significantly improved behavior and IR, in which by the time rescaling we can set c=1 and restore relativistic invariance, z=1

Why to break Lorentz invariance?

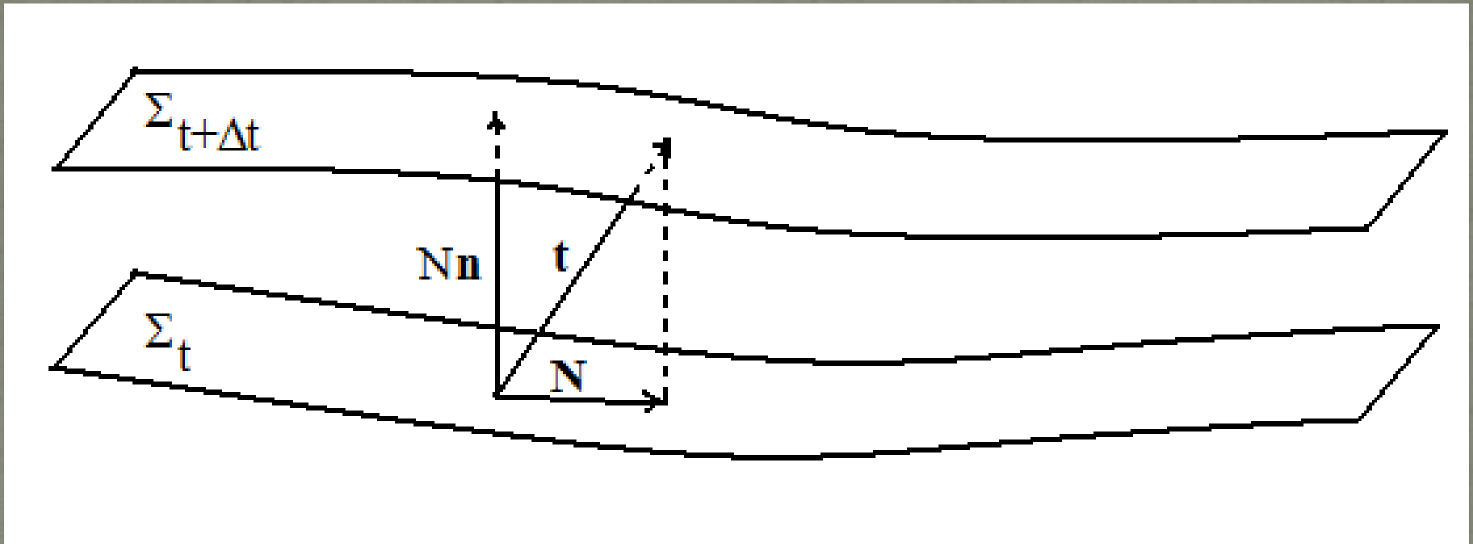
Let us consider the same type of the modification, but when the higher derivatives are added in the Lorentz invariant way.

$$S = \int d^4x \left(\partial_\mu \phi \partial^\mu \phi + g (\partial_\mu \phi \partial^\mu \phi)^2 \right)$$

The propagator takes the form:

$$G(\omega, \vec{k}) \propto \frac{1}{k^2 - gk^4} = \frac{1}{k^2(1 - gk^2)} = \frac{1}{k^2} - \frac{1}{k^2 - 1/g}$$

ADM



$$ds^2 = g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) - (Ncdt)^2$$

$$S_{EH} = \frac{1}{16\pi G}\int dt d^3x N\sqrt{g}\left(K_{ij}K^{ij} - K^2 + {}^3R\right)$$

where $K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right)$ - second fundamental form

We take ADM slicing as fundamental, i.e. instead of considering just a manifold, we endow it with the foliation structure:

$$\tilde{x}^i = \tilde{x}^i(\vec{x},t),\ \tilde{t} = \tilde{t}(t)$$

These are foliation - preserving diffeos or FDiffs

Also, we introduce anisotropic scaling between x and t:

$$\vec{x} \to \alpha\vec{x},\ t \to \alpha^z t \quad \text{or}\ [\vec{x}] = -1,\ [t] = -z$$

This is equivalent to prescribing the following dimensions :

$$[c] = z-1,\ [N] = [g_{ij}] = 0,\ [N_i] = z-1 \Rightarrow [G] = 3-z$$

- Projectable FDiff gravity (Horava 2009)

$$N = N(t), \quad N \to N\frac{\partial t}{\partial \tilde{t}}$$

$$S = \frac{M_P^2}{2}\int d^3x dt \sqrt{g} N\left(K_{ij}K^{ij} - \lambda K^2 - V_P\right)$$

$$V_P = 2\Lambda - \xi R + M_*^{-2}\left(A_1 R^2 + A_2 R_{ij}R^{ij}\right) +$$

$$+ M_*^{-4}\left(B_1 R \Delta R + B_2 R_{ij} R^{jk} R_k^i + B_3 \nabla_i R_{jk} \nabla^i R^{jk} + B_4 R R^{jk} R_{jk} + B_5 R^3\right)$$

- Non-projectable FDiff gravity (Blas et al. 2010)

$$N = N(t, \vec{x}), \quad a_i := N^{-1}\nabla_i N$$

$$S = \frac{M_P^2}{2}\int d^3x dt \sqrt{g} N\left(K_{ij}K^{ij} - \lambda K^2 - V_{NP}\right)$$

$$V_{NP} = V_P - \alpha a_i a^i + M_*^{-2}\left(C_1 a_i \Delta a^i + C_2 (a_i a^i)^2 + C_3 a_i a_j R^{ij} \cdots\right) +$$

$$+ M_*^{-4}\left(D_1 a_i \Delta^2 a^i + D_2 (a_i a^i)^3 + D_3 a_k a^k a_i a_j R^{ij} \cdots\right)$$

- Horava's recipe to deal with the large number of terms: Detailed Balance condition:

$$S_V = \frac{\kappa^2}{8}\int dt d^3x \sqrt{g}\, N E^{ij} G_{ijkl} E^{kl}\ ,\ \text{where}$$

$$\sqrt{g}\, E^{ij} = \frac{\delta W[g]}{\delta g_{ij}}\ \text{and } W \text{ is some lower-dimensional action:}$$

$$W = \frac{1}{\omega^2}\int CS_3(\Gamma) + \mu\int d^3x\sqrt{g}\left(R - 2\Lambda_W\right)$$

$$G^{ijkl} = \frac{1}{2}\left(g^{ik}g^{jl} + g^{il}g^{jk}\right) - \lambda g^{ij}g^{kl}\quad \text{modified DeWitt metric}$$

- Even applied to the healthy extended model the detailed balance condition still requires serious fine tuning (Verniery&Sotiriou 2013)

Some properties

- Broken 4d diffeos => Lorentz violation
- Extra scalar mode in addition to two graviton polarizations
- In general the scalar mode does not decouple in IR, this can endanger the renormalizability
- The model with the detailed balance condition does not pass the Solar system tests
- The healthy extension (with a_i) has A LOT of free parameters and some of them still require fine tuning
- …

UV spectral dimension (Horava)

- CDT lattice calculations indicate that d=2 in UV (Ambjorn et al. 2005)
- Spectral dimension (Horava 2009)

$$d_s = -2\frac{d\log P(\sigma)}{d\log\sigma}$$

$P(\sigma)$ - everage return probability, σ - diffusion time

$$P(\sigma) := \rho(\tau,\vec{x};\tau,\vec{x};\sigma)$$

$$\frac{\partial}{\partial\sigma}\rho(\tau,\vec{x};\tau',\vec{x}';\sigma) = \left(\frac{\partial^2}{\partial\tau^2}+\Delta\right)\rho(\tau,\vec{x};\tau',\vec{x}';\sigma)$$

- Spectral dimension in Horava case

$$\frac{\partial}{\partial\sigma}\rho(\tau,\vec{x};\tau',\vec{x}';\sigma)=\left(\frac{\partial^2}{\partial\tau^2}+(-1)^{z+1}\Delta^z\right)\rho(\tau,\vec{x};\tau',\vec{x}';\sigma)$$

$$\rho(\tau,\vec{x};\tau,\vec{x};\sigma)=\frac{C}{\sigma^{(1+D/z)/2}}$$

$$d_s=1+\frac{D}{z}\ \text{ or, in the case of }\ D=3,\ z=3,\ d_s=2$$

- In IR, the diffusion equation will be dominated by z=1, leading to d_s=4

Spectral geometry

(Connes 1990 and up to now)

$T=(A,H,D)$ – spectral triple

A Riemannian manifold, M, is completely

Recovered from T. In this case

i. $A=C^{\infty}(M)$
ii. $H=L^{2}(M,S)$
iii. $D=\gamma^{\mu}(\partial_{\mu}+\omega_{\mu})$

- $d(x,y) = \sup\left\{ | f(x)\text{-}f(y) | : f \in C(M), \| [D, f] \| \leq 1 \right\}$
- $N_{|D|}(\lambda) \underset{\lambda\to\infty}{\longrightarrow} \frac{2^m \Omega_n}{n(2\pi)^n} Vol(M) \lambda^n$
- $Tr^+(f\,|D|^{-n}) = \frac{2^m \Omega_n}{n(2\pi)^n} \int_M f \nu_g$, where $Tr^+ A = \lim_{N\to\infty} \frac{\sigma_N(A)}{\log N}$
- $Tr\chi\left(\frac{D^2}{m_0^2}\right) = \frac{N}{48\pi^2}\Big[12 m_0^4 f_0 \int d^4x \sqrt{g} + m_0^2 f_2 \int d^4x \sqrt{g} R +$

$$+ f_2 \int d^4x \sqrt{g}\left(-\frac{3}{20} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \frac{1}{10} R_{;\mu}{}^{\mu} + \frac{11}{20} R^* R^* \right) + O\left(\frac{1}{m_0^2}\right)\Big]$$

Spectral dimension (NC geometry)

(AP 2010)

- The choice of the Dirac operator in the form $D=\gamma^\mu(\partial_\mu+\omega_\mu)$ is not natural anymore
- The foliation structure dictates the following (schematic) form for D (for z=3)
  $D= \partial_t+\sigma^\mu\partial_\mu\Delta+M_*\Delta+M_*^2\sigma^\mu\partial_\mu$
- This D should be used to obtain "physical" geometry instead of auxiliary 3+1 dimensional space. (AP 2010, Gregory & AP 2012)

Model calculation

- $M=S^1\times T^3$, $D^2=\partial_t^{\,2}+\Delta^3+M_*^{\,2}\Delta^2+M_*^{\,4}\,\Delta$
- $sp(D^2)=\{n^2+(n_1^{\,2}+n_2^{\,2}+n_3^{\,2})^{3}+M_*^{\,2}(n_1^{\,2}+n_2^{\,2}+n_3^{\,2})^2$ $+M_*^{\,4}(n_1^{\,2}+n_2^{\,2}+n_3^{\,2})\ ,\ n_i\in Z\}$
- $N_{|D|}(\lambda)=\{\#\text{ eigenvalues} < \lambda\}$
- when $\lambda<<M_*^{\,6}$ the last term dominates:

$$N_{|D|}(\lambda)\cong\int_0^{\lambda}dn\int_0^{(\lambda^2-n^2)^{1/2}}4\pi\rho^2d\rho\propto\lambda^4\quad\Rightarrow\quad d=4$$

when $\lambda>>M_*^{\,6}$ the first term dominates:

$$N_{|D|}(\lambda)\cong\int_0^{\lambda}dn\int_0^{(\lambda^2-n^2)^{1/6}}4\pi\rho^2d\rho\propto\lambda^2\quad\Rightarrow\quad d=2$$

One can do better and go beyond the flat case.

- Define a generalized ζ-function

$$\zeta_\Delta(s) := \mathrm{Tr}(\Delta^{-s})$$

- Now Δ can be any generalized elliptic operator.
- ζ-function can be extended to a meromorphic function on the whole complex plane with the only poles given by

$$\frac{n-p+zp}{2z}, \frac{n-p+zp-1}{2z}, \ldots, \frac{n-p+zp-k}{2z}, \ldots$$

- The first pole is related to the analytic dimension

$$\frac{n-p+zp}{2z} = \frac{n_a}{2}$$

- n=D+1, p=1 we have $\quad n_a = 1 + \frac{D}{z}$

Spectral Action (the speculative part)

Part I $\quad Tr\chi\left(\frac{D^2}{m_0^2}\right) = \text{Horava - Lifshitz gravity?}$

- Dirac operator is very complicated:

$$D^2 = \Delta_\tau + f(\Delta_x),$$

$$\text{where } \Delta_\tau = -\frac{1}{N\sqrt{g}}\partial_\tau\left(\frac{\sqrt{g}}{N}\partial_\tau\right) \text{ and } \Delta_x = \frac{1}{N\sqrt{g}}\partial_i\left(N\sqrt{g}\,g^{ij}\partial_j\right)$$

- To calculate the trace of this operator one has to find the heat kernel

$$\begin{cases} (\partial_s + D^2)K(x,x';s) = 0 \\ K(x,x';+0) = \delta(x,x') \end{cases}$$

- Even the flat case is not trivial (Mamiya & AP 2013)

$$K(x-x';\tau) = \frac{1}{z(4\pi)^2\tau^{\frac{1}{2}(1+3/z)}} \mathrm{e}^{-\frac{(t-t')^2}{4\tau}} \sum_{\{j_k\}=0}^{\infty} \left(\prod_{k=0}^{z-1} \frac{(-\tau\gamma_k)^{j_k}}{j_k!} \right) (\tau\gamma_z)^{-\sum_k k j_k/z} \times$$
$$\times {}_1\Psi_1 \left[((3/2 + \textstyle\sum_k k j_k)/z, 1/z); (3/2, 1); -\frac{|\vec{x}-\vec{x}'|^2}{4(\tau\gamma_z)^{1/z}} \right] .$$

- This allows to perform a completely analytical study of the spectral dimension flow:

$$d_S = 1 + \frac{3}{z} + 2\gamma\gamma_z^{-\frac{k}{z}}\tau^{1-\frac{k}{z}} \left(1 - \frac{k}{z}\right) \frac{{}_1\Psi_0\left[\left(\frac{3+2k}{2z}, \frac{k}{z}\right); -\gamma\gamma_z^{-\frac{k}{z}}\tau^{1-\frac{k}{z}}\right]}{{}_1\Psi_0\left[\left(\frac{3}{2z}, \frac{k}{z}\right); -\gamma\gamma_z^{-\frac{k}{z}}\tau^{1-\frac{k}{z}}\right]}$$

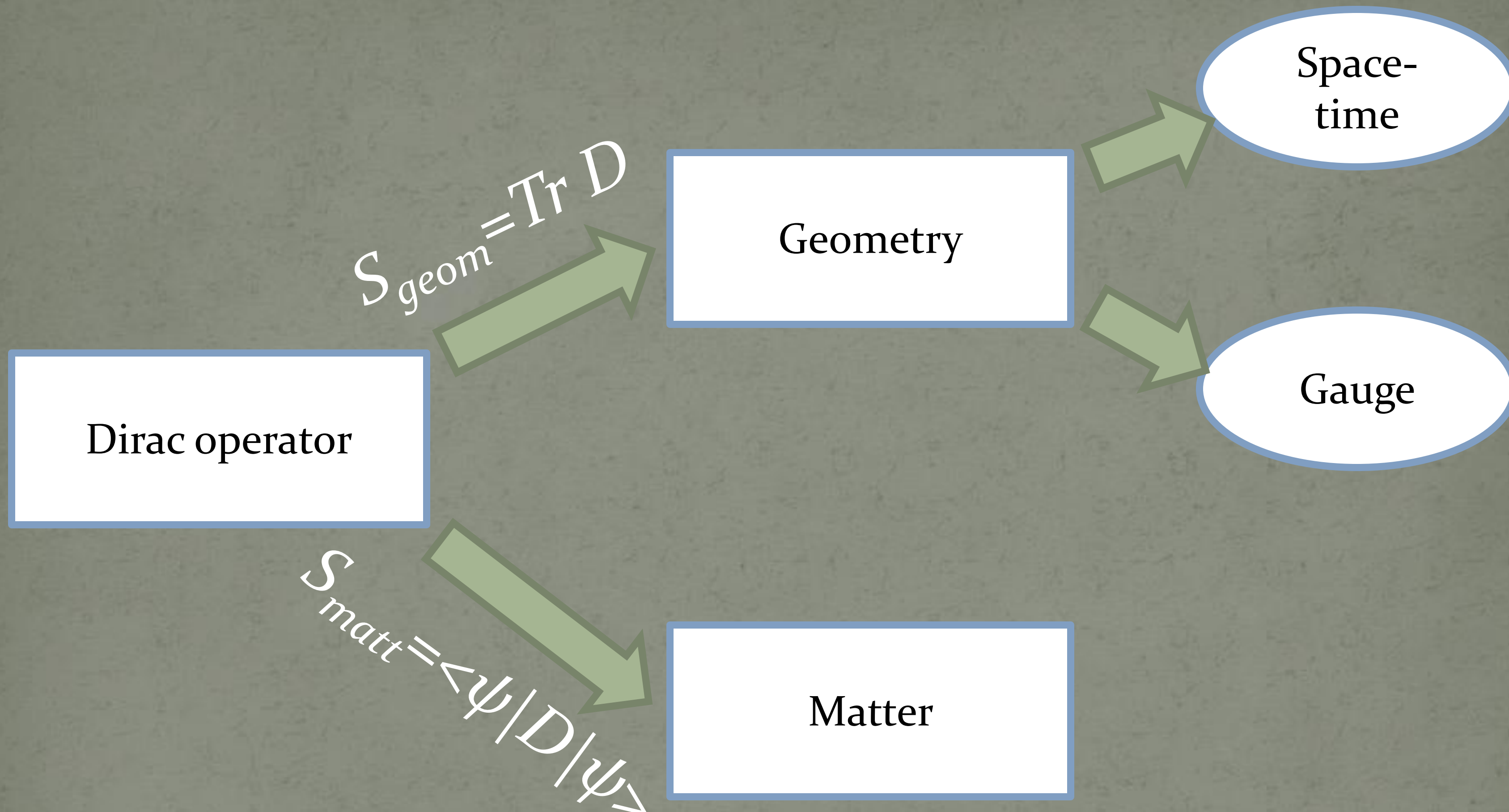
- Using the approach of Nesterov&Solodukhin 2010 we hope to show how to recover the HL action as the spectral action.

Part II Matter

- The matter coupling to geometry is restricted only by FPDiff.
- This permits inclusion of the higher spatial derivatives in S_{matter} .
- There is no guiding principle on how to proceed except the control over the amount of Lorentz violation (Pospelov&Shang 2010, Kimpton&Padilla 2013)
- The spectral action approach has the second part (Chamsedinne&Connes 1996)

$$S_{matter} \propto \langle \psi | D | \psi \rangle$$

- The operator D is the same that was used for the gravity part!


Dirac operator
S_{geom}=Tr D
Geometry
Space-
time
Gauge
$S_{matt}=<\psi|D|\psi>$
Matter

- What happens to the geodesic motion?

$$\nabla_\mu T^{\mu\nu} = 0 \Rightarrow \text{geodesic motion}$$

(Dixon 1970, Hawking&Ellis 1973)

- Now we DO NOT have $\nabla_\mu T^{\mu\nu} = 0$

  Instead we do have $h_{\lambda\nu} \nabla_\mu T^{\mu\nu} = 0$, where $T^{\mu\nu} \propto \frac{\delta S_{matt}}{\delta g_{\mu\nu}}$

- Alternative way to get geodesics:
  - Write a field theory
  - Find field equations
  - Restrict to the 1-particle sector
  - Do quasi-classical analysis
  - Hamilton-Jacobi => geodesic motion

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}\left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + \frac{m^2c^2}{\hbar^2}\phi^2\right)$$

$$\Box\phi - \frac{m^2c^2}{\hbar^2}\phi = 0$$

$$\phi = A\mathrm{e}^{\frac{i}{\hbar}S}$$

$$\begin{cases} 2\nabla_\mu A\nabla^\mu S + A\Box S = 0 \;\Rightarrow\; \nabla_\mu(A^2\nabla^\mu S) = 0 \\ \nabla_\mu S\nabla^\mu S + m^2c^2 = \hbar^2\frac{\Box A}{A} \end{cases}$$

$$H = g^{\mu\nu}p_\mu p_\nu + m^2c^2$$

$$\begin{cases} g^{\mu\nu}p_\mu p_\nu + m^2c^2 = 0 \\ \dot{x}^\mu = 2N(\tau)g^{\mu\nu}p_\nu \\ \dot{p}_\mu = -N(\tau)\frac{\partial g^{\nu\lambda}}{\partial x^\mu}p_\nu p_\lambda \end{cases}$$

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0\ ,\ \tau \text{ is a proper time}$$

- Immediate result is that “geodesics” change

Conclusions/Discussions

- Horava-Lifshitz could provide a UV completion of GR
- For this the original proposal should be modified ("healthy" extension?)
- It would be good to have a more geometrical approach to construct the theory
- What is the choice of the coupling to matter?
- What is the correct physical motion of a test particle? Geodesics?

- What is the underlying geometry? E.g. can one get the physical motion of point particle as geodesical motion in this geometry?
- Gauge sector, matter content (it is more natural now to have fields in reps of SO(3))
- Methods of spectral geometry plus spectra action principle might prove useful.

Science is the best way to satisfy you own curiosity at the government's expense.

L.A.Artsimovich