Massive Kaluza-Klein Gravity

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1. **Massive and massless spin-2 fields.**

Early in the 70's, a debate emerged on the possibility that the massless gravitation described by the Einstein-Hilbert action could be considered as the zero mass limit $m \to 0$ of the Pauli-Fierz action.

However, it was soon noted by van Dam, Veltmann and Zakharov (vDVZ), that such limit would lead to a gravitational theory that contradicts by $1/4$ of the result of the light bending measurements of 1919.

In 1972 Vainshtein suggested that the vDVZ limit problem could be solved by considering a Fierz-Pauli theory with non-linear terms in the equation of motion. However, in that same year Boulware and Deser showed that such non-linear theory would be haunted by non-physical states with negative energy(ghosts).

More recently the subject emerged again as a possible explanation for the acceleration of the universe in a higher-dimensional program, mostly in connection with the DGP model of the brane-world program, with a possible intermediate massive spin-2 field.

The zero mass limit problem suggests a conflict between two variational principles for massless spin-2 fields: the Einstein-Hilbert and the Fierz-Pauli actions.

However, a theorem due to Soraj Gupta in 1960 and later reviewed by Stanley Deser an others in 1970, proved that *any massless spin-2 field defined by a symmetric rank-2 tensor is necessarily a solution of an Einstein-like equation*. The Gupta theorem essentially reverses the linear approximation of Einstein equations, applied to the Fierz-Pauli theory, in practice reconstructing the non-linear terms.

Therefore, the non-linearity of a massless spin-2 field would appear when the the Fierz-Pauli Lagrangian

$$\mathcal{L}= \frac{1}{4}\left[H_{,\mu}H^{,\mu} - H_{\nu\rho,\mu}H^{\nu\rho,\mu} - 2H_{\mu\nu}{}^{,\mu}H^{,\nu} + 2H_{\nu\rho,\mu}H^{\mu,\rho}\right]$$

is written in the geometric language as

$$\mathcal{L} = \mathcal{R}\sqrt{g}$$

This conclusion suggests an interesting corolary to Gupta's theorem:

The Einstein-Hilbert action is not a guess or an accident: The non-linear geometric expression for massless gravitation is derived from the Einstein-Hilbert principle, built with the Ricci scalar R itself, and not with an unknown function $F(\mathcal{R})$.

On the other hand, a massive spin-2 field does not lead to an Einstein-like equation and its physical properties are different when compared with massless gravitation. Indeed, such massive field may have a short range interaction with gauge fields at the Tev scale.

Such possibility emerged in 1971 when Isham, Salam and Stradthee, suggested the existence of a massive spin-2 field with mass, acting as an intermediate field between Einstein's gravitation and hadrons. They never found it because they were insisting on Gupta's theorem in General Relativity.

In the next sections we will show that ghost-free massive spin-2 field, appears in a modified massive Kaluza-Klein theory.

2 Kaluza-Klein

The non-Abelian Kaluza-Klein theory proposed in 1963 was defined in a higher-dimensional space with product topology $V_4 \times B_N$, $N > 1$, where B_N is a compact space with Planck size, and the $4+N$ geometry is defined by the Einstein-Hilbert action.

By writing the metric in the form (the Kaluza-Klein ansatz)

$$\mathcal{G}_{AB} = \begin{pmatrix} g_{\mu\nu} + g^{ab} A_{\mu a} A_{\nu b} & A_{\mu a} \\ A_{\nu b} & g_{ab} \end{pmatrix}$$

The action decomposes as

$$\mathcal{R}\sqrt{\mathcal{G}} = R\sqrt{-g} + \frac{1}{4} tr F_{\mu\nu} F^{\mu\nu} \sqrt{-g}$$

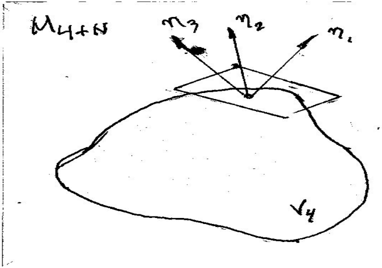
where $A_\mu = A_{\mu a}\hat{\eta}^a$, $F_{\mu\nu} = [D_\mu, D_\nu]$ and $D_\mu = \partial_\mu + A_\mu$

In 1984 it was noted that the predictions of the Kaluza-Klein theory at the electroweak limit did not agree with the observed chiral motion of fermions.

The fermion chirality problem was soon understood to be a consequence of the small size of B_n, which contributed to a large fermionic mass proportional to the Planck mass.

In spite of the many efforts presented at that time to save the theory, it was literally abandoned.

In the 60's Joseph and Ne'eman conjectured that the internal (gauge) symmetry could be geometrically explained by the group of rotations of the space orthogonal to the space-time embedded in a flat space.



regarded as the ground state of the high-dimensional gravitational field, defined by the Einstein's equations

$$\mathcal{R}_{AB} - \frac{1}{2}\mathcal{R}\mathcal{G}_{AB} = \kappa_g^* T_{AB}^*$$

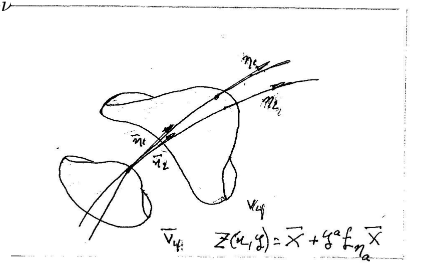
which are hyperbolic equations, compatible with the product topology $I\!R^4 \times I\!R^N$ in place of the $V_4 \times B_n$.

In 1985 we have checked the Joseph-Ne'eman conjecture, in a modified Kaluza-Klein theory, where the higher-dimensional space was the embedding space of the space-time, and found that the geometry of the embedded space-time is given by

$$\mathcal{G}_{AB} = \begin{pmatrix} \tilde{g}_{\mu\nu} + g^{ab}A_{\mu a}A_{\nu b} & A_{\mu a} \\ A_{\nu b} & g_{ab} \end{pmatrix}$$

where

$$\tilde{g}_{\mu\nu} = \bar{g}_{\mu\nu} - 2y^a\bar{k}_{\mu\nu a} + y^a y^b g^{\alpha\beta}\bar{k}_{\mu\alpha a}\bar{k}_{\nu \dots}$$
$$\tilde{k}_{\mu\nu a}(x,y) = \bar{k}_{\mu\nu a} - y^b\bar{g}^{\alpha\beta}\bar{k}_{\mu\alpha a}\bar{k}_{\nu\beta b}$$
$$A_{\mu a} = g_{\mu a}(x,y) = y^b\bar{A}_{\mu ab}$$



$$\tilde{k}_{\mu\nu a} = -\frac{1}{2}\frac{\partial\tilde{g}_{\mu\nu}}{\partial y^a}$$

Notice that $\mathcal{G}_{AB}$ looks like the Kaluza-Klein metric ansatz, with the difference that the space-time metric is replaced by

$$\tilde{g}_{\mu\nu} = \bar{g}_{\mu\nu} - \underbrace{2y^a \bar{k}_{\mu\nu a} + y^a y^b \bar{g}^{\alpha\beta} \bar{k}_{\mu\alpha a} \bar{k}_{\nu\beta b}}$$

so that the Kaluza-Klein Lagrangian decomposes as

$$\mathcal{R}\sqrt{\mathcal{G}} = \tilde{R}\sqrt{-\tilde{g}} + \frac{1}{4} tr F_{\mu\nu} F^{\mu\nu} \sqrt{-\tilde{g}}$$

where $F_{\mu\nu}$ is defined by the second fundamental form $A_{\mu ab}$, written in the Lie algebra of the rotations group:

$$D_{\mu a}{}^{b} = \tilde{g}_a^b + A_{\mu a}{}^{b}, \quad D_\mu = D_{\mu a}{}^{b} L_b^a, \quad F_{\mu\nu} = [D_\mu, D_\nu]$$

Since the internal space is non-compact, the Chiral fermions problem does not appear.

Rewriting the above Kaluza-Klein Lagrangian, but now replacing $\tilde{g}_{\mu\nu}$ in terms of the extrinsic curvature

$$\mathcal{L} = \mathcal{R}\sqrt{\mathcal{G}} = [R + (K^2 - h^2)]\sqrt{g} - \frac{1}{4} trF_{\mu\nu}F^{\mu\nu}\sqrt{g}$$

where $K^2 = k_{\mu\nu a}k^{\mu\nu a}$ and $h^2 = h_a h^a$, $h_a = g^{\mu\nu}k_{\mu\nu a}$
Showing that $k_{\mu\nu a}$ corresponds to a spin-2 field which intermediates between the gravitational field $g_{\mu\nu}$ and the gauge fields A_μ.
As we see the term $K^2 - h^2$ is proportional to the mass term of the the Fierz-Pauli action with mass (written for $k_{\mu\nu a}$):

$$\mathcal{L} = \frac{1}{4}\left[h_{a,\mu}h^{a,\mu} - k_{\rho\nu a;\mu}k^{\rho\nu a;\mu} - 2k_{\mu\nu a}{}^{;\mu}h_a{}^{,\nu} + 2k_{\nu\rho a;\mu}h^{a;\rho} - m^2(K^2 - h^2)\right]$$

Therefore, the redefined Kaluza-Klein in accordance with the Joseph-Ne'emann conjecture leads to a chiral fermion compatible theory at the electroweak level, including a massive, short range spin-2 term which corresponds to the second fundamental form.

The gauge symmetries are identified with the group of rotations of the normal vectors of the embedded space-time and the gauge fields are given by the third fundamental form.

3 Cosmology

In particular for N=1, the third fundamental form vanishes so that there is no unification. Einstein's equations simplifies to

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - Q_{\mu\nu} = \kappa_g^* G T_{\mu\nu}^*$$
$$k^\rho_{\mu;\rho} - h_{,\mu} = \alpha^* T^*_{\mu 5}$$
$$\mathcal{R}_{55} - \frac{1}{2} \mathcal{R} g_{55} = \alpha^*_{55} T^*_{55}$$

where

$$Q_{\mu\nu} = g^{55}(k^\rho_\mu k_{\rho\nu} - h k_{\mu\nu}) - \frac{1}{2}(K^2 - h^2) g_{\mu\nu},$$

$Q_{\mu\nu}$ is conserved in the sense of Noether and it reproduces the cosmological constant term in the particular case where $k_{\mu\nu} = \alpha g_{\mu\nu}$.

Applying to the FLRW metric

$$ds^2 = -dt^2 + a(t)^2[dr^2 + f^{(}r)(d\theta^2 + \sin^2(\theta)d\phi^2)] + \underbrace{\phi^2(r,t)dy^2}$$

where $f(r) = \sin r, r, \sinh r$ for k = 1, 0, -1 respectively and $\phi = g_{55}$. The gravitational equations are

$$\begin{aligned}
&\dot{a}^2 + k - \frac{1}{\phi^2}\frac{k_{11}^2}{a^2} = \frac{8\pi G}{3}\rho a^2 \\
&k^{\rho}_{\mu;\rho} - h_{,\mu} = 0 \\
&\phi^2\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right) = 0
\end{aligned}$$

Note that the term $-\frac{1}{\phi^2}\frac{k_{11}^2}{a^2}$ corresponds to $Q_{\mu\nu}$. k_{11} is determined from the the second equation, with arbitrary constants. The tests based on the WMAP say that it is a better fit than the ΛCDM model for the observed the acceleration of the universe.

4 Induced Quantum Gravity

Consider again the Kaluza-Klein metric in an embedding space

$$\mathcal{G}_{AB} = \begin{pmatrix} \tilde{g}_{\mu\nu} + g^{ab} A_{\mu a} A_{\nu b} & A_{\mu a} \\ A_{\nu b} & g_{ab} \end{pmatrix}$$

leading to the Kaluza-Klein Lagrangian

$$\mathcal{L} = \mathcal{R}\sqrt{\mathcal{G}} = \tilde{R}\sqrt{-\tilde{g}} + \frac{1}{4} trF_{\mu\nu}F^{\mu\nu}\sqrt{-\tilde{g}}$$

or, in terms of $k_{\mu\nu a}$

$$\mathcal{L} = \mathcal{R}\sqrt{\mathcal{G}} = [R + (K^2 - h^2)]\sqrt{g} - \frac{1}{4} trF_{\mu\nu}F^{\mu\nu}\sqrt{g}$$

Since A_μ is a gauge field, it is perturbatively quantizable, then it will induce quantum fluctuations in the remaining components of the metric.

This is the same principle introduced by Ashtekar, with the difference that now we do not need to introduce a gauge field via the group of holonomy of the triads.

Therefore, the four-dimensional space-time has a quantum geometry induced by the quantization of its gauge field components, meaning that the metric $g_{\mu\nu}$ the extrinsic curvature $k_{\mu\nu a}$ and the third fundamental form $A_{\mu ab}$ are subject to quantum fluctuations.

This induced quantization in the sense of Asktekar suggests that there would be also a canonical quantization in the same sense of the ADM program, adapted to the 4+N topology in the embedded Kaluza-Klein just described.

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