Large N Phase Transitions in massive $\mathcal{N} = 2$ Gauge Theories

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Plan

1. Large N Gauge Theories and Quantum Gravity
2. $\mathcal{N} = 4$ SU(N) super Yang-Mills theory (SYM) at large N
3. Mass deformed $\mathcal{N} = 4$ SU(N) SYM at large N *- phase transitions at critical couplings !*
4. Mass deformed $\mathcal{N} = 2$ SCF SYM *- smooth weak-strong interpolation*
5. $\mathcal{N} = 2$ SQCD *- phase transition at a critical quark mass!*

1. Quantum Gravity and large N Gauge Theories

Is string theory a consistent Quantum Gravity theory?

BEST HINT: **HOLOGRAPHY**

Conjectured equivalence with supersymmetric gauge theories.

Most studied example:

Closed strings on $AdS_5 \times S^5$ =	$\mathcal{N} = 4$ $SU(N)$ SYM
Gravity theory on anti de Sitter space =	superconformal gauge theory

This is a quantum equivalence: the well-established consistency of quantum gauge theory implies the consistence of the dual gravity (superstring) theory.

The quantum theories in both sides of the duality are severely constrained by the high amount of symmetry.
In particular, these theories have maximal supersymmetry and conformal invariance.

It is important to establish if holography is an exact duality connecting gravity and gauge theories in less symmetric, non-conformal settings.

$\mathcal{N} = 2$ supersymmetric SU(N) gauge theories

Some $\mathcal{N} = 2$ theories exhibit physical features similar to QCD, like asymptotic freedom.

But many features of the deep quantum regime of general $\mathcal{N} = 2$ theories are still unknown.

Representations of $\mathcal{N} = 2$ supersymmetry

"Vector multiplet": 1 vector boson, 2 real scalars and two Weyl fermions

"Hypermultiplet": 4 real scalars and two Weyl fermions

Simple $\mathcal{N} = 2$ theory:
unique mass deformation of $\mathcal{N} = 4$ SYM that preserves $\mathcal{N} = 2$ supersymmetry

$\mathcal{N} = 4$ theory contains **1** vector boson, **4** Weyl fermions and **6** real scalars in the adjoint rep.

In terms of the $\mathcal{N} = 2$ subalgebra: 1 vector multiplet and 1 hypermultiplet.

Now add a mass term for the hypermultiplet.
That is, the same mass term for **4** scalars and **2** fermions.

The resulting theory is called **$\mathcal{N} = 2^*$ SYM theory**.

In which regime of gauge theory parameters we can compare with string theory ?

The SU(N) gauge theory parameters: ***N*, *g*, masses**.

In holography, quantum gravity corrections are controlled by the coupling **1/*N***
In order to make contact with classical gravity, **we need to study SU(*N*) gauge theories at large *N***.

$$N \to \infty\,, \quad g \to 0 \quad with \quad g^2 N \equiv \lambda = fixed$$

Gauge theory perturbation theory: $\lambda << 1$, *planar loop diagrams.*

*However, w**e need exact results in the gauge theory** that can be extrapolated to strong* λ ***>> 1*** *coupling.*
This is the regime where string theory calculations can be carried out in a controlled way.

Fortunately, for $\mathcal{N} = 2$ theories, exact results exist for certain supersymmetric observables. *These are obtained by "localization".*

Localization

Theories with a fermionic symmetry: the exact functional integral may localize in a subset of field configurations obeying classical field equations plus a one-loop contribution.

Exact partition function for general $\mathcal{N} = 2$ supersymmetric YM theory on S^4, with arbitrary matter content !! [Pestun, 0712.2824]

$$Z = \int d^{N-1}a \prod_{i<j} (a_i - a_j)^2 \, e^{-S_{cl}(a)} \, z_{1-loop}(a) \, \left| z_{inst}(a; g^2) \right|^2$$

known factors that depend on field content

$$S_{cl} = \frac{1}{4g^2} \int_{S^4} d^4x \sqrt{g} \, \mathrm{R} \, \mathrm{tr}\Phi^2 = \frac{8\pi^2}{g^2} \sum_i a_i^2$$

$$\langle \Phi \rangle = diag(a_1, \ldots, a_N)$$

VEV of scalar of vector multiplet

Integral over Couloumb moduli.

The one-loop factor

It is expressed in terms of a single function

$$H(x) \equiv \prod_{n=1}^{\infty}\left(1+\frac{x^2}{n^2}\right)^n \mathrm{e}^{-\frac{x^2}{n}}$$

The different multiplets contribute as follows:

Vector multiplet $\prod_{i<j} H^2(a_i - a_j)$

Adjoint hypermultiplet $\prod_{i<j} \frac{1}{H(a_i - a_j - M)H(a_i - a_j + M)}$

Fundamental hypermultiplet $\prod_{i} \frac{1}{H(a_i + M)}$

Observables

•Free energy $F = -\ln Z$

n-derivatives of free energy gives correlators of integrated operators.

For example:

- $d^nF/d\lambda^n$ gives correlators of the integrated action density.
- d^nF/dM^n gives correlators of integrated mass operator.

•Supersymmetric Wilson loops

Circular Wilson loop:

$$W(C) = \left\langle \mathrm{tr}\, P \exp\left[\int_C ds\, (A_\mu(x)\dot{x}^\mu + i\Phi\, |\dot{x}|) \right] \right\rangle$$

It localizes to

$$W(C) = \left\langle \sum_{j=1}^{N} e^{2\pi a_j} \right\rangle$$

We are interested in computing these observables in the large N limit.

$\mathcal{N}$= 4 Super Yang-Mills theory on S^4

•Instantons do not contribute.

•1-loop corrections cancel

Gaussian matrix model:

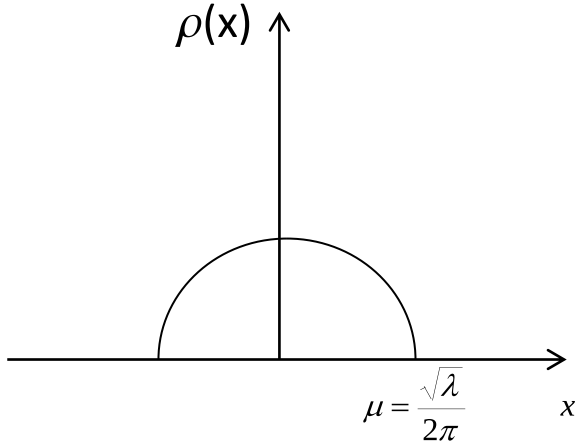
$$Z = \int d^{N-1}a \prod_{i<j} (a_i - a_j)^2 \, \mathrm{e}^{-\frac{8\pi^2}{g^2}\sum_i a_i^2}$$

At large N the integral is dominated by a saddle-point.
Minimize

$$S\,[a] = \frac{8\pi^2 N}{\lambda}\sum_{i=1}^{N} a_i^2 - \frac{1}{2}\sum_{i,j}\ln(a_i - a_j)^2 \quad \to \quad \sum_{j\neq i}\frac{1}{a_i - a_j} = \frac{8\pi^2 N}{\lambda} a_i$$

Introducing the eigenvalue density

$$\rho(x) = \frac{1}{N}\sum_{i=1}^{N}\delta(x - a_i)$$



The saddle-point equation becomes

$$\int_{-\mu}^{\mu} dy\, \rho(y)\, \frac{1}{x-y} = \frac{8\pi^2}{\lambda} x \quad \Rightarrow \quad \rho(x) = \frac{8\pi}{\lambda}\sqrt{\frac{\lambda}{4\pi^2} - x^2}$$

Eigenvalues are distributed in a semicircle (Wigner's law)

$\mathcal{N}$= 4 SYM : Wilson loop

$$W(C) = \int_{-\mu}^{\mu} dx\, \rho(x)\, \mathrm{e}^{2\pi x} = \frac{8\pi}{\lambda} \int_{-\mu}^{\mu} dx \sqrt{\frac{\lambda}{4\pi^2} - x^2}\, \mathrm{e}^{2\pi x} = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$$

$$\xrightarrow[\lambda\to\infty]{} \sqrt{\frac{2}{\pi}}\, \lambda^{-3/4}\, \mathrm{e}^{\sqrt{\lambda}}$$

String fluctuations

Minimal area in AdS_5

Expanding at small λ one gets a perturbative series

$$W(C) = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \xrightarrow[\lambda<<1]{} 1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \ldots$$

with **infinite** radius of convergence.

But the ½ susy W is very special because of a huge cancellation of diagrams.

In general, the planar expansion in $\mathcal{N}$= 4 theory is expected to have a **finite** radius of convergence.
Indeed, perturbation series for anomalous dimensions diverges at

$$|\lambda| > \pi^2$$

Can this radius of convergence be seen from the exact formulas computed by localization?

$\mathcal{N}$ = 2* SU(N) Super Yang-Mills theory

Deform the $\mathcal{N}$= 4 theory by a mass term preserving ½ of supersymmetries.

The theory has a vector multiplet and a massive hypermultiplet.
The partition function computed by localization is therefore given by

$$Z = \int d^{N-1}a \quad \prod_{i<j} \frac{(a_i - a_j)^2 H^2(a_i - a_j)}{H(a_i - a_j - M)H(a_i - a_j + M)} \, \mathrm{e}^{-\frac{8\pi^2}{g^2}\sum_i a_i^2} \quad \left| z_{inst}(a; g^2) \right|^2$$

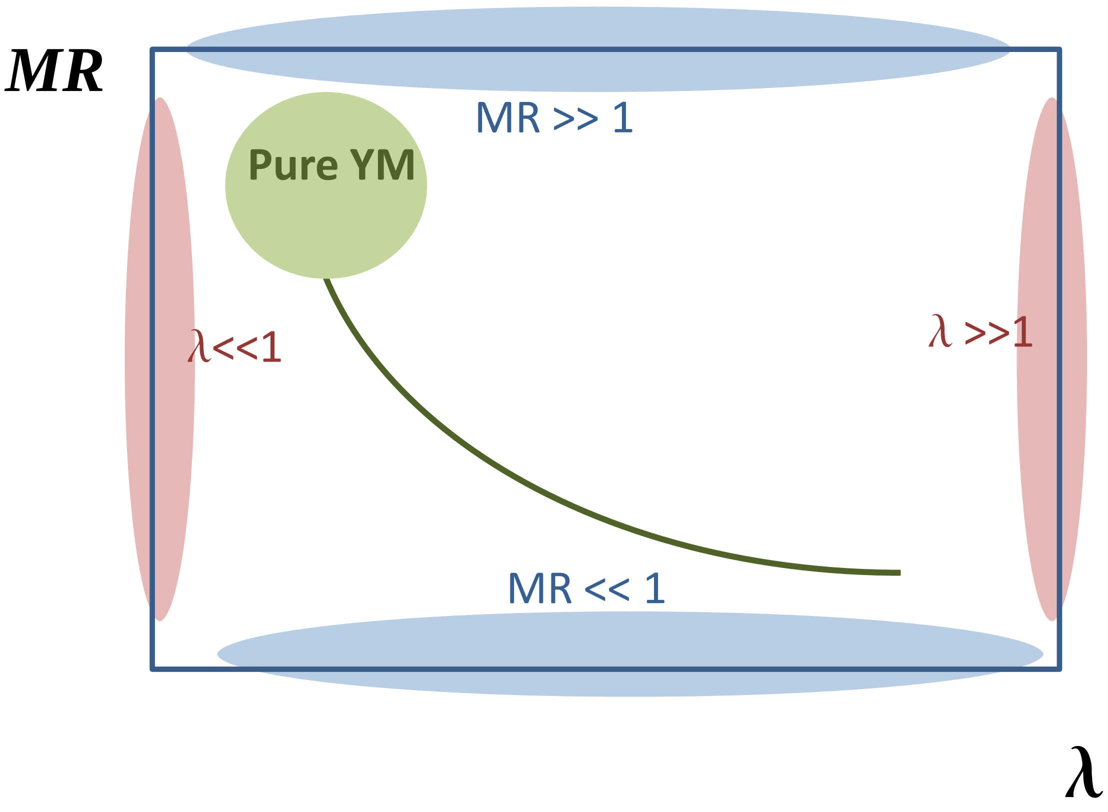
with

$$H(x) \equiv \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right)^n \mathrm{e}^{-\frac{x^2}{n}}$$

The instanton factor is more complicated, but we do not need it as in the large N limit turns out to be exponentially suppressed. (We proved it for the one-instanton contribution).

In the above formulas we have set $R = 1$, as all quantities depend on the combination MR.

The large N theory depends on two parameters (λ, MR)



Saddle-point equation for $\mathcal{N}$ = 2*

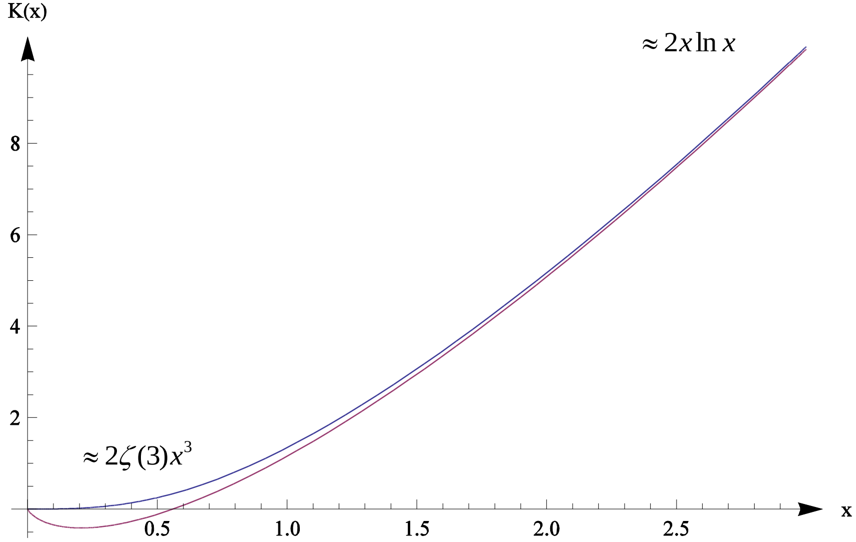
The planar limit can be exactly calculated by solving an integral equation

$$\frac{8\pi^2}{\lambda}x = \int_{-\mu}^{\mu} dy\, \rho(y)\left(\frac{1}{x-y} - K(x-y) + \tfrac{1}{2}K(x-y+M) + \tfrac{1}{2}K(x-y+M)\right)$$

attraction　　repulsion　　attraction　　repulsion

$$K(x) \equiv -\frac{H'(x)}{H(x)} \xrightarrow[x\to\infty]{} 2x\ln|x|$$

μ: width of eigenvalue distribution, $-\mu < x < \mu$



Strong coupling limit of $\mathcal{N}$ = 2* and AdS/CFT

λ >>1:

Gravity dual: Pilch-Warner solution, obtained as a deformation of AdS_5 x S^5.

As λ is increased, the attactive linear force become weaker and the eigenvalue distribution expands.
Since μ >> M, we can approximate

$$-K(x-y)+\tfrac{1}{2}K(x-y+M)+\tfrac{1}{2}K(x-y+M)\approx\tfrac{1}{2}K''(x)M^2\approx\frac{M^2}{x}$$

Hence

$$\frac{8\pi^2}{\lambda}x=\int_{-\mu}^{\mu}dy\,\rho(y)\,\frac{(1+M^2)}{x-y}\qquad\Rightarrow\qquad\rho(x)=\frac{2}{\pi\mu^2}\sqrt{\mu^2-x^2}\,,$$

$$\mu=\frac{\sqrt{\lambda}}{2\pi}\sqrt{\frac{1}{R^2}+M^2}\;\xrightarrow[R\to\infty]{}\;\frac{\sqrt{\lambda}\,M}{2\pi}$$

Having ρ(x), we can compute the Wilson loop:

$$W(C)=\int_{-\mu}^{\mu}dx\,\rho(x)\,\mathrm{e}^{2\pi x}=\frac{2}{\sqrt{\lambda(1+M^2R^2)}}I_1\left(\sqrt{\lambda(1+M^2R^2)}\right)$$

$$\xrightarrow[R\to\infty]{}\quad W(C)=\mathrm{e}^{\sqrt{\lambda}MR}$$

These results exactly match the supergravity prediction using Pilch-Warner solution [Buchel, J.R. and Zarembo, 2013]

Conformal perturbation theory in $\mathcal{N} = 2^*$

Consider $M \ll 1$, λ arbitrary.
The saddle-point integral equation can be solved in perturbation theory in M^2

For the free energy we find

$$F = -\frac{1}{2}\ln\lambda - M^2 f(\lambda)$$

$$f(\lambda) = \frac{4\pi^2}{\lambda}\int_0^\infty dw \, \frac{\frac{\lambda w^2}{4\pi^2} - J_1^2\left(\frac{\sqrt{\lambda}w}{\pi}\right)}{w\sinh^2 w}$$

We have the exact λ dependence.

• Expanding at weak coupling one finds a perturbation series that diverges at $\lambda > \pi^2$
Remarkably, this is the same radius of convergence found from computation of anomalous dimensions!!

Near $\lambda = -\pi^2$ the free energy has a logarithmic branch cut singularity

$$F = analytic - \frac{8M^2}{\pi^3}(\pi^2+\lambda)\ln(\pi^2+\lambda) \quad , \quad \lambda \approx -\pi^2$$

• At strong coupling

$$F = -\frac{1+M^2}{2}\ln\lambda - M^2\frac{2\pi^2}{3\lambda} + const.$$

$\mathcal{N}$= 2* in the decompactification limit at arbitrary λ

$\mathcal{N}$= 4 SYM: there is a smooth dependence with λ all the way from 0 to infinity.
No phase transition between the perturbative $\lambda << 1$ regime and the strong coupling $\lambda >> 1$ regime (described by AdS/CFT duality).

Now deform the theory by adding a mass term.
Is the interpolation between weak and strong coupling still smooth?
Could there be a phase transition in between?

In the limit *MR* >>1 at fixed λ, one can replace *K*(x) by 2x ln|x|.
As a result, the saddle-point equation simplifies:

$$\int_{-\mu}^{\mu} dy\, \rho(y) \ln \frac{M^2-(x-y)^2}{(x-y)^2} = \frac{8\pi^2}{\lambda}$$

Differentiating once, we obtain

$$\boxed{0 = \int_{-\mu}^{\mu} dy\, \rho(y) \left(\frac{2}{x-y} - \frac{1}{x-y+M} - \frac{1}{x-y-M} \right)}$$

This matrix model can be solved by following a method developed by [Kazakov, Kostov and Nekrasov, 9810035]

Note that the second and third terms do not have poles if *M* > |x-y|. This requires $\mu < M/2$
What is the physical origin of this bound?

μ increases with increasing λ
Can μ get to $M/2$ at finite coupling λ ? What happens above this coupling?

LIGHT STATES AT CRITICAL COUPLINGS

The mass spectrum in the scalar VEV background is given by

$$m_{ij}^{\mathrm{v}} = \left| a_i - a_j \right| \qquad , \qquad m_{ij}^{\mathrm{h}} = \left| a_i - a_j \pm M \right|$$

When μ > M/2, a new phenomenon takes place: there can be exceptionally light hypermultiplets for states with

$$a_i - a_j \approx \pm M$$

Extra light states whenever μ **= n M/2, n=2,3,4,…**

At which coupling these resonances occur?
For n >> 1 we can use the strong coupling expression for μ:

$$\mu = \frac{\sqrt{\lambda} M}{2\pi} \quad \Rightarrow \quad \lambda = \frac{4\pi^2 \mu^2}{M^2}$$

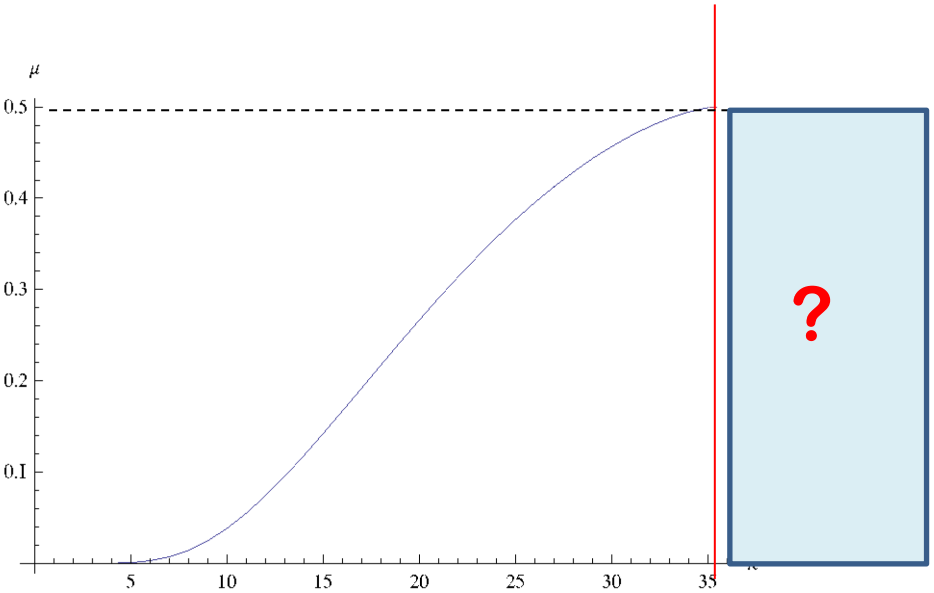
$$\mu_n = \frac{nM}{2} \quad \Rightarrow \quad \lambda_c^{(n)} \approx \pi^2 n^2 = 39.4\,,\quad 88.8,\quad 157.9,\quad 246.7, \ldots \quad , \quad n = 2, 3, 4, 5, \ldots$$

These resonance phenomena will produce with dramatic effects that we will observe shortly.

μ can be expressed in terms of elliptic functions.

It can be shown that μ is real provided:

$$\lambda < \lambda_c \quad , \qquad \lambda_c \approx 35.4$$



λ_c

The analytic solution terminates at λ_c . It cannot be extended for $\lambda > \lambda_c$

Near λ_c

$$\mu = \frac{M}{2}\left(1 - c\,(\lambda - \lambda_c)^{3/2} + \ldots\right)$$

At λ_c , $\mu = M/2$

Calculation of free energy

$$F=-\frac{1}{N^2}\ln Z \quad , \qquad F'(\lambda)=-\frac{1}{N^2}\frac{1}{Z}\frac{\partial Z}{\partial\lambda}=\frac{8\pi^2}{\lambda^2}\left\langle x^2\right\rangle$$

We find

$$F(\lambda)=2M^2\sum_{n=1}^{\infty}\ln(1-(-1)^n e^{-\frac{8\pi^2 n}{\lambda}})$$

$$=2M^2\sum_{k=1}^{\infty}\sigma_{-1}(k)\,(-1)^k\,e^{-\frac{8\pi^2 k}{\lambda}}$$

$$=M^2\left(2e^{-\frac{8\pi^2}{\lambda}}-3e^{-\frac{16\pi^2}{\lambda}}+\frac{8}{3}e^{-\frac{24\pi^2}{\lambda}}-\frac{7}{2}e^{-\frac{32\pi^2}{\lambda}}+\frac{12}{5}e^{-\frac{40\pi^2}{\lambda}}-4e^{-\frac{48\pi^2}{\lambda}}+\ldots\right)$$

This non-perturbative expansion represents the different terms in the OPE expansion
(they are not instantons!)

$$F=\sum_{n=1}^{\infty}C_n\left(\frac{\Lambda}{M}\right)^{2n}$$

Remarkably**, there is not any perturbative contribution**.
The OPE coefficients are rational numbers, the σ_{-1} (k) divisor function.

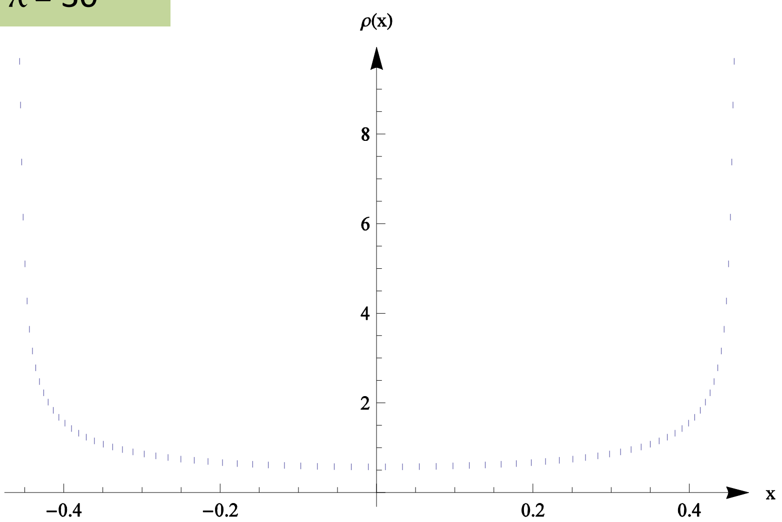
Wilson loop: $W=\exp(2\pi\mu)\,,\qquad \frac{\mu}{M}=2e^{-\frac{4\pi^2}{\lambda}}-4e^{-\frac{12\pi^2}{\lambda}}-16e^{-\frac{28\pi^2}{\lambda}}-58e^{-\frac{36\pi^2}{\lambda}}-324e^{-\frac{44\pi^2}{\lambda}}-1856\,e^{-\frac{52\pi^2}{\lambda}}+\ldots$

Strong-coupling phase

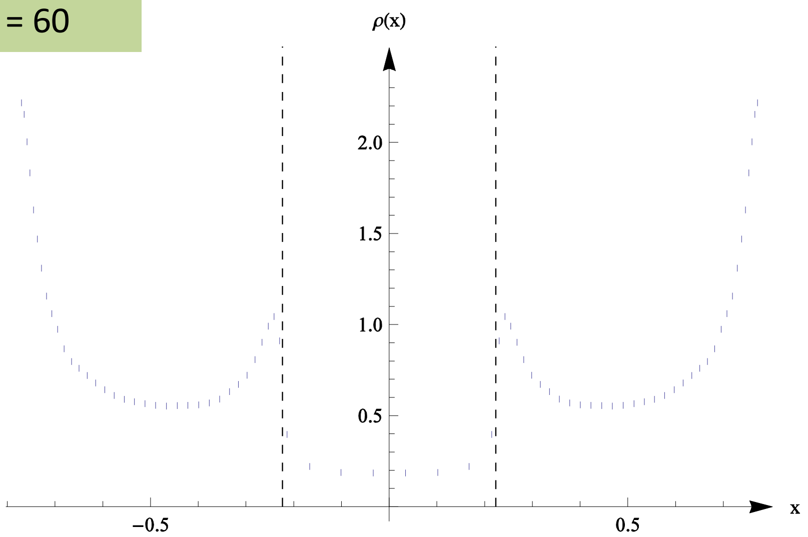
In the supercritical phase the eigenvalue density can be found by numeric methods.
Different phases emerge as λ is gradually increased.
They can be understood in terms of thresholds occurring at μ = nM/2, with n = 1,2,3,...

•For $\lambda < \lambda_c$, the eigenvalue density has a shape similar to 1/Sqrt[μ^2-x^2]

λ = 30



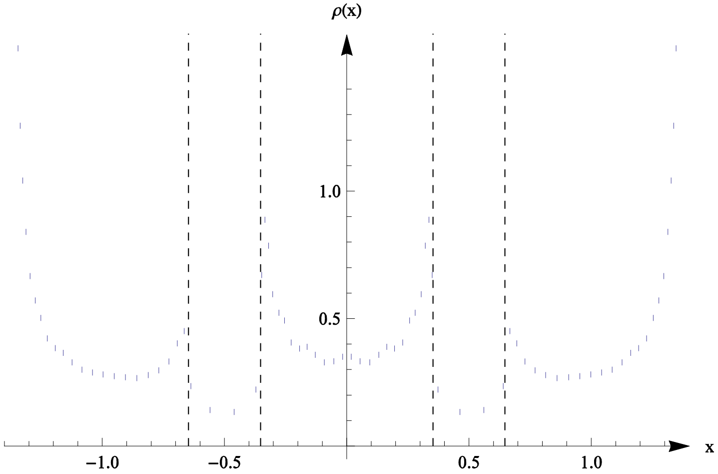
λ = 60



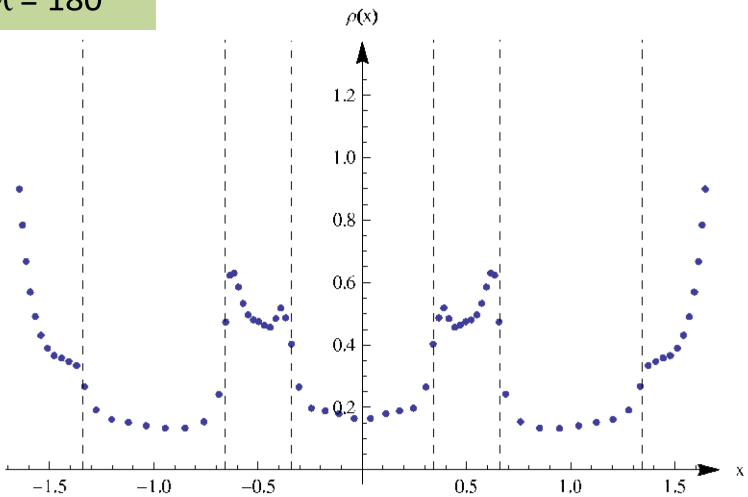
•When $\lambda > \lambda_c$, then μ overcomes the first threshold, M/2. The density develops two cusps at $-M + \mu$ and $M - \mu$, which travel towards $x = 0$ as is increased.

•The next phase transition happens when μ = M, at $\lambda \approx 83$.
At this point, the first two cusps collide at x = 0 and two new cusps at x = $-\mu$ + 2M and x = μ − 2M are formed near the boundary

λ = 130



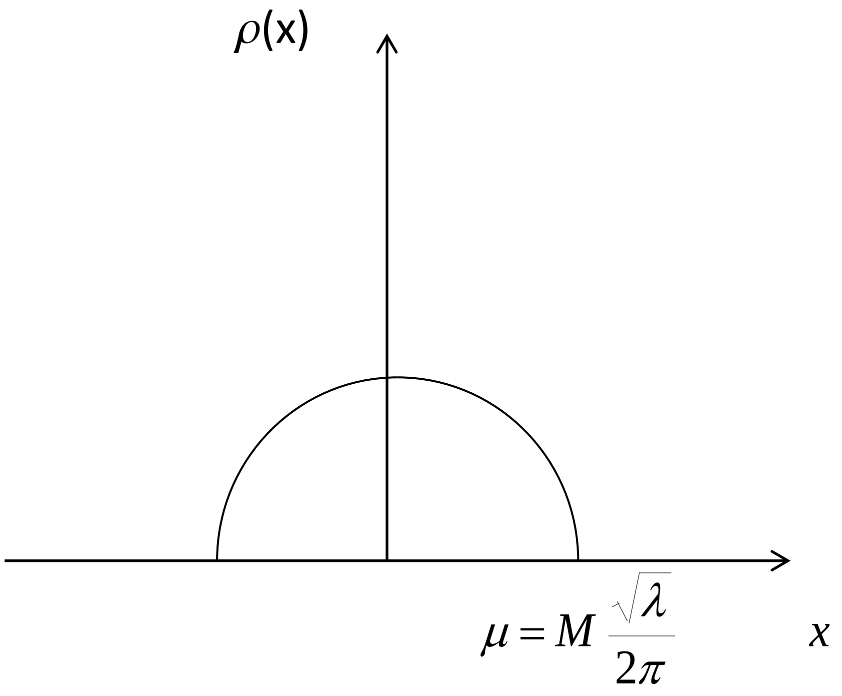
λ = 180



•As λ is further increased, the theory goes through new phases where more cusps are formed pairwise, whenever μ crosses nM/2, n = 1,2,3,…

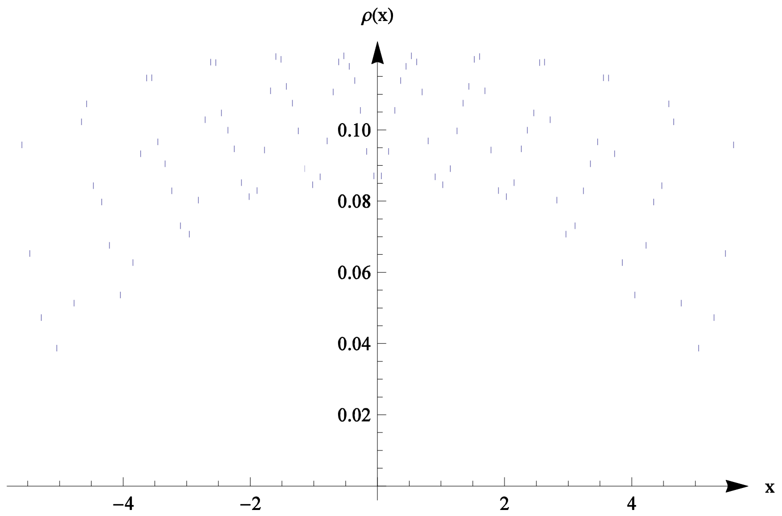
This looks different from what we hoped to find strong coupling:

$$\lambda >> 1, \quad MR >> 1$$

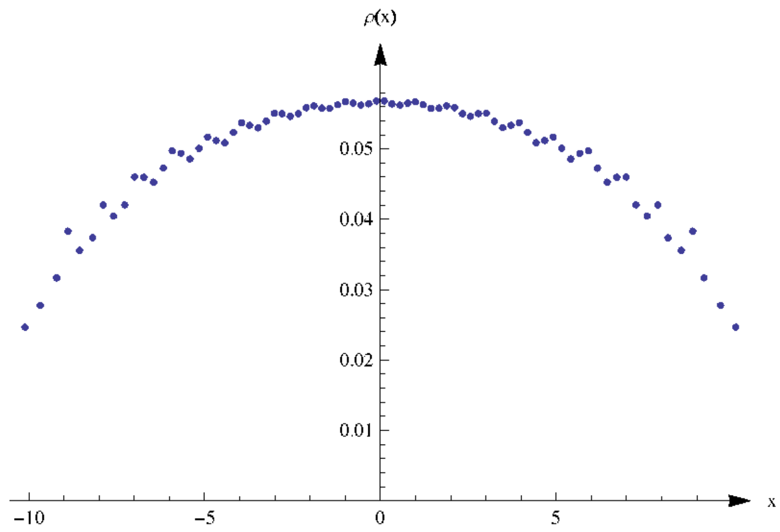


Getting the Wigner's semicircle distribution is necessary to match the AdS/CFT prediction.

λ = 1500



λ = 5000

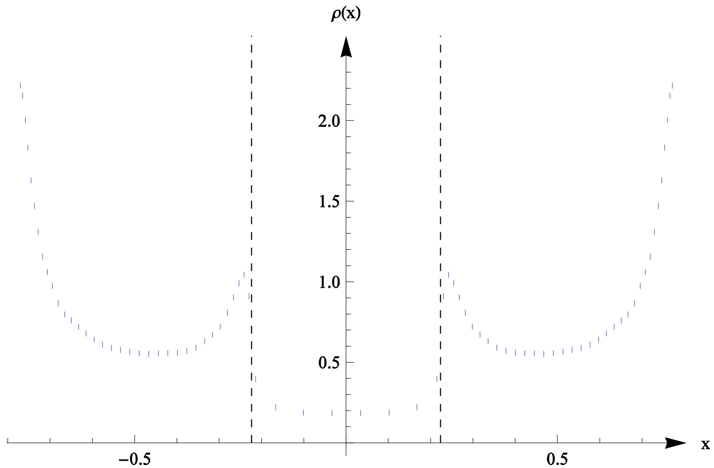


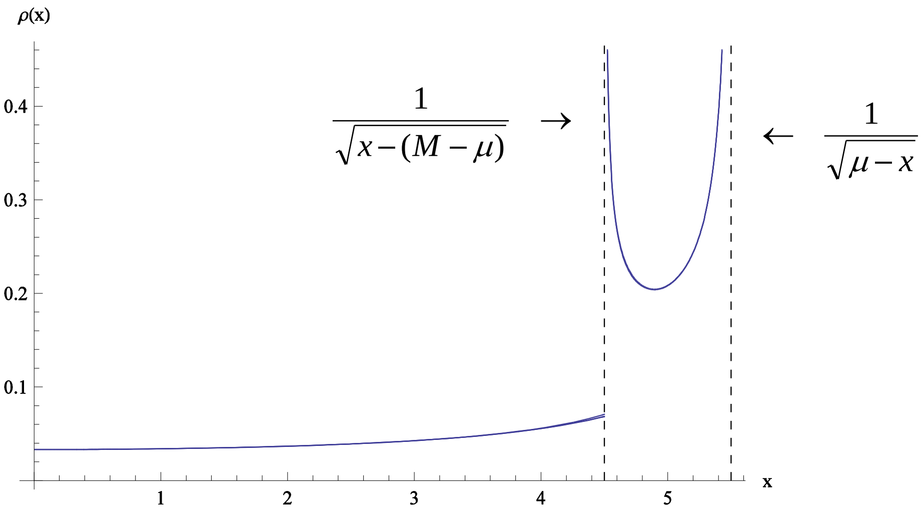
The Wigner distribution at strong coupling is the result of a coarse grain averaging over an infinite number of infinitely weak cusps.

It is not clear which implications this fractal structure has for holography.

Properties of the phase transition

1. Structure of the "cusps"





2. Order of phase transition

$$F'(\lambda) = -\frac{1}{Z}\frac{\partial Z}{\partial \lambda} = N^2 \frac{8\pi^2}{\lambda^2}\langle x^2 \rangle$$

$$\langle x^2 \rangle - \langle x^2 \rangle_{weak\,phase} \approx c\, M^2 (\lambda - \lambda_c)^3 \qquad (\lambda \rightarrow \lambda_c^+)$$

Therefore F'''' experiences a jump: the transition is fourth order.

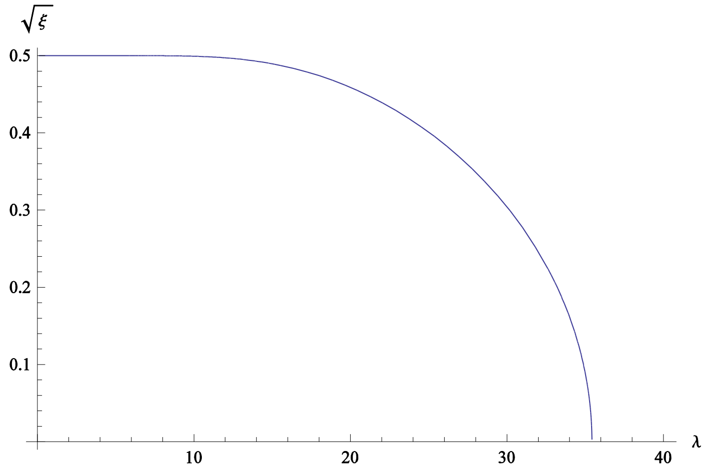
Is there anything special happening to the subcritical free energy at the critical point?

Defining $\beta = 1/\lambda$, at the critical point: F'''(β) = 0

3. Order parameter.

An order parameter that measures the scale of the mass gap of the light hypermultiplet excitations of masses $|x - y \pm M|$ is 1/L, with

$$L = \frac{1}{\sqrt{\xi}} = \left\langle \frac{1}{x^2 - \frac{M^2}{4}} \right\rangle^{1/2}$$



Instantons

Instanton contributions are usually believed to be negligible at large *N*, since

$$Z_k^{ins} \approx e^{-\frac{8\pi^2|k|}{g^2}} = e^{-\frac{8\pi^2 N|k|}{\lambda}} \xrightarrow[N\to\infty]{} 0$$

Proving this actually requires explicitly computing the volume of the instanton moduli space.
The one instanton contribution can be written as

$$Z_1^{ins} = \mathbf{e}^{-\frac{8\pi^2 N}{\lambda}} \quad \frac{2M^2}{M^2+1}\int dz \prod_{j=1}^{N} \frac{(z-a_j)^2 - M^2}{(z-a_j)^2+1}$$

$$Z_1^{ins} = \mathbf{e}^{-N S_{inst}}$$

$$\mathbf{S}_{inst} = \frac{8\pi^2}{\lambda} - \int_{-\mu}^{\mu} dx\, \rho(x) \ln \frac{(z-x)^2 - M^2}{(z-x)^2+1}$$

There are two competing saddle-points at z = 0 and z = infinity.

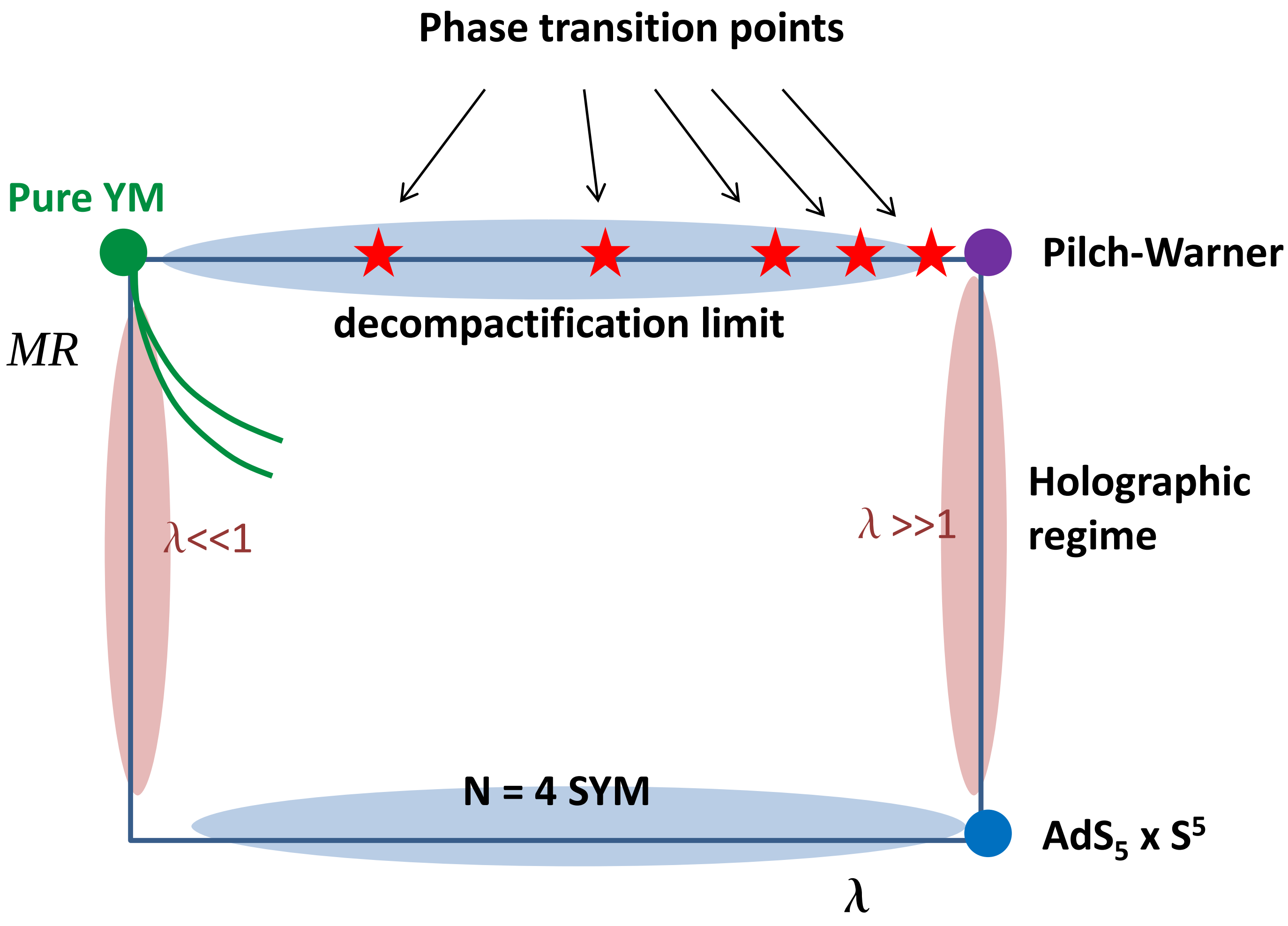
Instantons are suppressed provided $S_{inst} > 0$ in all the parameter space (λ, *MR*).

This is not a priori obvious, given that there is the negative term that might in principle be $> 8\pi^2/\lambda$.

Example: *M*>>1. Using the saddle-point equation, we can put S_{inst} in the form

$$\mathbf{M} >> \mathbf{1}, \textbf{fixed}\,\lambda: \quad \mathbf{S}_{inst} = \int_{-\mu}^{\mu} dx\, \rho(x) \ln(1+\frac{1}{x^2}) \quad > 0$$

PHASE DIAGRAM FOR $\mathcal{N} = 2^*$ THEORY ON S^4



$\mathcal{N} = 2$ SCFT*

A familiar superconformal theory is $\mathcal{N} = 2$ SU(N) SYM with 2N fundamental hypermultiplets.
What happens if we break superconformal invariance by introducing mass terms for the hypermultiplets?

Does the theory have a phase transition when the eigenvalues exceed the mass M?

The partition function computed by localization is given by

$$Z = \int d^{N-1}a \quad \frac{\prod_{i<j} (a_i - a_j)^2 H^2(a_i - a_j)}{\prod_i H(a_i + M)^N H(a_i - M)^N} \; \mathrm{e}^{-\frac{8\pi^2}{g^2}\sum_i a_i^2} \quad \left| z_{inst}(a; g^2) \right|^2$$

At large N, the integral is dominated by a saddle-point at the solution of the integral equation

$$\int_{-\mu}^{\mu} dy\, \rho(y) \left(\frac{1}{x-y} - K(x-y) \right) = \frac{8\pi^2}{\lambda} x - \tfrac{1}{2}\left(K(x+M) + K(x-M)\right)$$

$\mathcal{N}$ = 2 SCFT*: Large MR at fixed coupling λ

In the decompactification limit the model becomes exactly solvable

$$\int_{-\mu}^{\mu} dy\, \rho(y)\, \frac{2}{x-y} = \frac{1}{M+x} - \frac{1}{M-x}$$

$$\Rightarrow \quad \rho(x) = M\sqrt{M^2-\mu^2}\,\frac{1}{\pi\sqrt{\mu^2-x^2}}\,\frac{1}{M^2-x^2}$$

$$\mu = \frac{M}{\cosh(\frac{4\pi^2}{\lambda})}\,, \qquad 0<\mu<M$$

Thus μ < M for all λ.

There is no phase transition in this case: the massive theory exhibits a smooth interpolation between weak and strong coupling regime.

Having the eigenvalue density, we can easily compute the free energy. It is given by

$$F(\lambda) = -2N^2M^2 \quad \ln(1+e^{-\frac{8\pi^2}{\lambda}}) = 2N^2M^2\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\,e^{-\frac{8\pi^2 k}{\lambda}}$$

which again has the interpretation of OPE expansion in terms of the dynamically generated scale.

$\mathcal{N} = 2$ SQCD with $2N_f$ massive hypermultiplets

We assume $N_f < N$, in which case the theory is **asymptotically free**.

The partition function computed by localization is given by

$$Z = \int d^{N-1}a \quad \frac{\prod_{i<j} (a_i - a_j)^2 H^2(a_i - a_j)}{\prod_i H(a_i + M)^{N_f} H(a_i - M)^{N_f}} \; e^{-\frac{8\pi^2 N}{\lambda}\sum_i a_i^2} \quad \left| z_{inst}(a; g^2) \right|^2$$

Dynamically generated scale

$$\Lambda = \Lambda_0 \, e^{-\frac{4\pi^2}{\lambda(1-\zeta)}} \quad , \qquad \zeta \equiv \frac{N_f}{N}$$

At large N, the integral is dominated by a saddle-point at the solution of the integral equation

$$\int_{-\mu}^{\mu} dy\, \rho(y) \left(\frac{1}{x-y} - K(x-y) \right) = -4(1-\zeta)(\log \Lambda R)\, x - \zeta\, K(x+M) - \zeta\, K(x-M)$$

In the decompactification limit, the integral equation (differentiated twice) simplifies to

$$\int_{-\mu}^{\mu} dy \; \frac{\rho(y)}{x-y} = \frac{\zeta}{x+M} + \frac{\zeta}{x-M}$$

This is like the saddle-point equation for a one-matrix model with a logarithmic potential. It is a solvable model. It exhibits the same phenomenon as we encountered in N = 2*: poles at $x = \pm M$ which may or may not lie within the eigenvalue distribution.

The model thus has two phases:

1. The weak-coupling phase with $\mu < M$, in which all hypermultiplets are heavy.
2. The strong-coupling phase at $\mu > M$, where light hypermultiplets appear in the spectrum.

1. Strong coupling phase $\mu > M$.

The poles sit within the eigenvalue distribution. The normalized eigenvalue density is then given by

$$\rho(x) = \frac{1-\xi}{\pi\sqrt{\mu^2 - x^2}} + \frac{\xi}{2}\delta(x+M) + \frac{\xi}{2}\delta(x-M)$$

$$\mu = 2\Lambda$$

The phase transition thus occurs at $M_c = 2\,\Lambda$

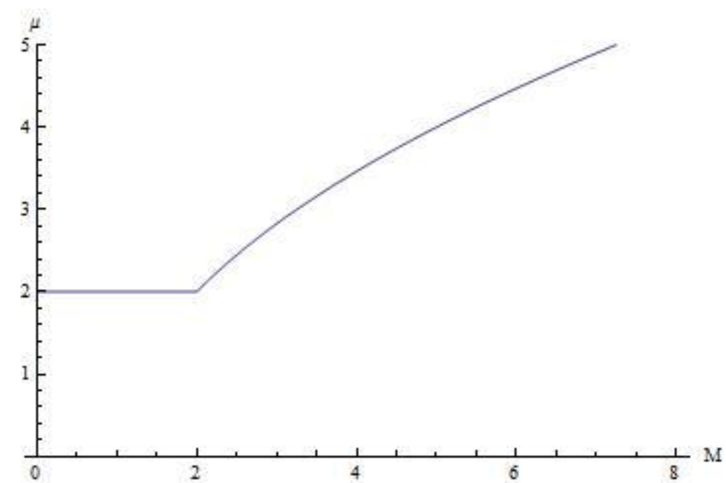
2. Weak coupling phase $\mu < M$.

The poles sit within the eigenvalue distribution. The normalized eigenvalue density is then given by

$$\rho(x) = \frac{1-\xi}{\pi\sqrt{\mu^2 - x^2}} + \xi\, M\sqrt{M^2 - \mu^2}\,\frac{1}{\pi\sqrt{\mu^2 - x^2}}\,\frac{1}{M^2 - x^2}$$

For concreteness, set, $N_f = N/2$, *i.e.* ξ=1/2, which is half the way between pure SYM ($N_f = 0$) and the SCFT* at $N_f = N$.

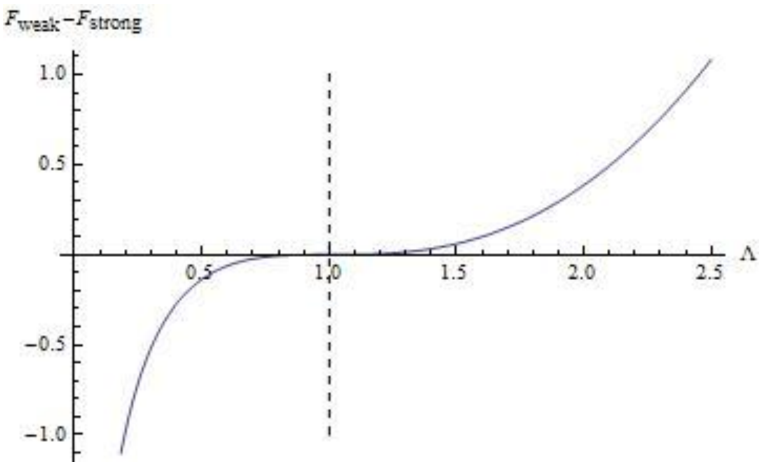
$$\mu = 2\sqrt{\Lambda(M - \Lambda)}$$



The Wilson loop W= exp(2 $\pi\,\mu$) is thus discontinuous

Free energy

$$\frac{\partial F}{\partial \log \Lambda} = -\langle x^2 \rangle$$



$$\langle x^2 \rangle = \begin{cases} \Lambda^2 + \dfrac{M^2}{2} , & M < 2\Lambda \\ \Lambda(2M - \Lambda) , & M > 2\Lambda \end{cases}$$

1. $\langle x^2 \rangle_{M=2\Lambda-\varepsilon} = \langle x^2 \rangle_{M=2\Lambda+\varepsilon} = 3\Lambda^2$

2. $\left.\dfrac{d\langle x^2 \rangle}{d\Lambda}\right|_{M=2\Lambda-\varepsilon} = \left.\dfrac{d\langle x^2 \rangle}{d\Lambda}\right|_{M=2\Lambda+\varepsilon} = 2\Lambda$

3. $\left.\dfrac{d^2\langle x^2 \rangle}{d\Lambda^2}\right|_{M=2\Lambda-\varepsilon} = 2 \quad \neq \quad \left.\dfrac{d^2\langle x^2 \rangle}{d\Lambda^2}\right|_{M=2\Lambda+\varepsilon} = -2$

Thus the transition is **third order**.

Conclusions

We have solved for the large-N master field of $\mathcal{N} = 2^*$ SYM, $\mathcal{N} = 2$ SCF* and $\mathcal{N} = 2$ SQCD using supersymmetric localization. The underlying physics of N = 2 gauge theories turns out to be very rich.

The main results are:

1. $\mathcal{N} = 2^*$ SYM and $\mathcal{N} = 2$ SQCD have quantum phase transitions at finite couplings.

2. $\mathcal{N} = 2^*$ theory exhibits an infinite number of phase transitions occurring as λ is increased and accumulating towards λ = infinity.

3. The strong coupling results agree with AdS/CFT. They are obtained by a coarse-grain average over a rather irregular fractal structure.

4. In the decompactification limit MR=infinity, the free energy and the expectation values of large Wilson loops have only non-perturbative terms in their weak-coupling expansion. These non-perturbative terms can be understood as an OPE expansion.

5. At small MR, $\mathcal{N} = 2^*$ theory has a perturbative series which diverges at $\lambda = -\pi^2$