Quasilocal vs Holographic Stress Tensor in AdS Gravity

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Outline

1. Quasilocal (Brown-York) stress tensor

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2. Holographic Renormalization and Local Counterterms

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5. No quasilocal stress tensor: two examples

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5. No quasilocal stress tensor: two examples
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Variational problem in EH gravity

- **EH gravity action in $D = d+1$ dimensions**

$$I_{EH} = \frac{1}{16\pi G}\int\limits_{M} d^{d+1}x\sqrt{-\mathcal{G}}\,\left(R-2\Lambda\right)$$

- **Variation of the action**:

$$\delta I_{EH} = \frac{1}{16\pi G}\int\limits_{M} d^{d+1}x\sqrt{-\mathcal{G}}\,\mathcal{E}_{\mu}^{\nu}\left(\mathcal{G}^{-1}\delta\mathcal{G}\right)_{\nu}^{\mu} + \int\limits_{\partial M} d^{d}x\,\Theta$$

- **Surface term:**

$$\Theta = \tfrac{1}{16\pi G}\sqrt{-h}\,n_{\mu}\delta_{[\alpha\beta]}^{[\mu\nu]}\mathcal{G}^{\beta\varepsilon}\delta\Gamma_{\nu\varepsilon}^{\alpha}$$

Variational problem in EH gravity

- **In Gauss-normal coordinates**

$$ds^2 = N^2(\rho)d\rho^2 + h_{ij}(\rho,x)dx^i dx^j$$

Extrinsic curvature $K_{ij} = -\frac{1}{2N}\partial_\rho h_{ij}$

$$\Gamma^\rho_{ij} = \frac{1}{N}K_{ij}, \qquad \Gamma^i_{\rho j} = -NK^i_j$$

- **On-shell variation of the action ($n_\mu = (N,\vec{0})$)**:

$$\delta I_{EH} = \frac{1}{16\pi G}\int_{\partial M} d^dx\sqrt{-h}\left(2\delta K + K^i_j(h^{-1}\delta h)^j_i\right)$$

- **Gibbons-Hawking term:**

$$I_{Dirichlet} = I_{EH} - \frac{1}{8\pi G}\int_{\partial M} d^dx\sqrt{-h}K$$

- **Dirichlet problem for the metric:**

$$\delta I_{Dirichlet} = \frac{1}{16\pi G}\int_{\partial M} d^dx\sqrt{-h}\left(K^i_j - K\delta^i_j\right)(h^{-1}\delta h)^j_i$$

Quasilocal stress tensor

- **Variation of the Dirichlet action respect to** h_{ij}

$$\delta I_{Dir} = \int_{\partial M} d^d x \, \frac{1}{2} \sqrt{-h} \, T^{ij}[h] \, \delta h_{ij}$$

- **Brown-York tensor is conserved, i.e.,** $\nabla_i T^{ij} = 0$.

- **Conserved current:**

$$J^i = T^{ij} \xi_j \implies Q[\xi] = \int_{\Sigma} \sqrt{\sigma} \, u_i T^{ij} \xi_j$$

- **In AAdS spacetimes** $Q[\xi]$ **is divergent, even in** $D = 3$.

- **We can always add local counterterms such that** $T^{ij}[h]$ **is regular**

$$I_{ren} = I_{EH} - \frac{1}{8\pi G} \int_{\partial M} d^d x \sqrt{-h} K + \int_{\partial M} d^d x \, \mathcal{L}_{ct}(h, \mathcal{R}, \nabla \mathcal{R})$$

Counterterm method

- **Regularized AdS gravity action** (holographic renormalization)
  [Henningson, Skenderis JHEP 9807:023(1998)]

$$I_{ren} = \frac{1}{16\pi G}\int_M d^{d+1}x\sqrt{-\mathcal{G}}\,(R-2\Lambda) - \frac{1}{8\pi G}\int_{\partial M} d^d x\sqrt{-h}\,K + \\ + \int_{\partial M} d^d x\,\mathcal{L}_{ct}(h,\mathcal{R},\nabla\mathcal{R})$$

$$\Lambda = -\frac{d\,(d-1)}{2\ell^2}$$

- **Renormalized quasi-local stress tensor**: $T^{ij}_{ren}[h] = \frac{2}{\sqrt{-h}}\frac{\delta I_{ren}}{\delta h_{ij}}$.

- **Background-independent charge definition**

Holographic Renormalization

- For Asymptotically AdS (AAdS) spacetimes, Fefferman-Graham (FG) form of the metric

$$ds^2 = \frac{\ell^2}{4\rho^2}\, d\rho^2 + \frac{1}{\rho}\, g_{ij}(x,\rho)\, dx^i dx^j \tag{1}$$

- the boundary of the spacetime is at $\rho = 0$
- $g_{ij}(x,\rho)$ accepts a regular expansion in powers of ρ

$$g_{ij}(x,\rho) = g_{(0)ij}(x) + \rho g_{(1)ij}(x) + \rho^2 g_{(2)ij}(x) + \cdots \,.$$

- $g_{(0)ij}$ is the boundary data for the *holographic reconstruction of the spacetime*, i.e., solving $g_{(k)}$ as a covariant funtional of $g_{(0)}$
- **Holographic stress tensor** $T^{ij}[g_{(0)}] = \lim_{\rho\to 0}\left(\frac{1}{\rho^{d/2-1}}\, T^{ij}[h]\right)$.

  Contains the holographic information of the theory (e.g., Weyl anomaly)

Counterterm method in AdS gravity

$$\mathcal{L}_{ct} = \frac{d-1}{\ell}\sqrt{-h} + \frac{\ell\sqrt{-h}}{2(d-2)}\mathcal{R} + \frac{\ell^3\sqrt{-h}}{2(d-2)^2(d-4)}\left(\mathcal{R}^{ij}\mathcal{R}_{ij} - \frac{d}{4(d-1)}\mathcal{R}^2\right)$$
$$+\frac{\ell^5\sqrt{-h}}{(d-2)^3(d-4)(d-6)}\left(\frac{3d-2}{4(d-1)}\mathcal{R}\mathcal{R}^{ij}\mathcal{R}_{ij} - \frac{d(d+2)}{16(d-1)^2}\mathcal{R}^3\right.$$
$$\left.-2\mathcal{R}^{ij}\mathcal{R}^{kl}\mathcal{R}_{ijkl} - \frac{d}{4(d-1)}\nabla_i\mathcal{R}\nabla^i\mathcal{R} + \nabla^k\mathcal{R}^{ij}\nabla_k\mathcal{R}_{ij}\right) + \ldots$$

- Full series for an arbitrary dimension is unknown.
- Terms do not seem to follow any pattern.
- Far more complicated for higher-curvature theories (e.g., Einstein-Gauss-Bonnet).

Kounterterms

- *Extrinsic* **regularization** $\tilde{I}_{ren} = \quad I_{EH} + c_d \int\limits_{\partial M} d^d x \, B_d(h, K, \mathcal{R})$

- **Inspired by a simple observation (EH+GB in $D = 4$)**

$$I = \frac{1}{16\pi G} \int\limits_M d^4x \sqrt{-\mathcal{G}} \left[(R - 2\Lambda) + \alpha (R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4R_{\mu\nu} R^{\mu\nu} + R^2) \right]$$

- **Euclidean action for Sch-AdS black hole:**

$$G = \beta^{-1} I^E = \frac{M}{2} \left(1 + \frac{4}{\ell^2} \alpha \right) - TS + \frac{\pi r^3}{4G\ell^2} \left(1 - \frac{4}{\ell^2} \alpha \right)$$

- **GB coupling has to be fixed as** $\alpha = \frac{\ell^2}{4}$

- **Finite Noether current also implies** $\alpha = \frac{\ell^2}{4}$
  [Aros, Contreras, Olea, Troncoso, Zanelli, PRL 84, 1647 (2000)]

- **MacDowell-Mansouri form of the action**

$$I_{ren} = \frac{\ell^2}{256\pi G} \int_M d^4x \sqrt{-\mathcal{G}}\, \delta^{[\sigma\lambda\mu\nu]}_{[\gamma\delta\alpha\beta]} \left(R^{\gamma\delta}_{\sigma\lambda} + \frac{1}{\ell^2} \delta^{[\gamma\delta]}_{[\sigma\lambda]} \right) \left(R^{\alpha\beta}_{\mu\nu} + \frac{1}{\ell^2} \delta^{[\alpha\beta]}_{[\mu\nu]} \right).$$

- **The action has a minimum for global AdS spacetime**

$$I_{ren} = 0$$

- **Weyl tensor (on-shell)**

$$W^{\alpha\beta}_{\mu\nu} = R^{\alpha\beta}_{\mu\nu} + \frac{1}{\ell^2} \delta^{[\alpha\beta]}_{[\mu\nu]}$$

- **(On-shell) regularized action is equal to Conformal Gravity action**

$$I_{ren} = \frac{\ell^2}{64\pi G} \int_M d^4x \sqrt{-\mathcal{G}}\, W_{\mu\nu\alpha\beta} W^{\mu\nu\alpha\beta}$$

O.Mišković and R.O., [arXiv:0902.2082];
J.Maldacena, [arXiv:1105.5632]

Kounterterms

- Euler Theorem in $D=4$ dimensions

$$\int_M d^4x\, GB = 32\pi^2\chi(M) + \int_{\partial M} d^3x\, B_3$$

- Kounterterms = given polynomial in the extrinsic and intrinsic curvatures (K_{ij} and $\mathcal{R}^{kl}_{ij}(h)$)

$$\begin{aligned}
B_3 &= 4\sqrt{-h}\,\delta^{[i_1 i_2 i_3]}_{[j_1 j_2 j_3]} K^{j_1}_{i_1}\left(\tfrac{1}{2}\,\mathcal{R}^{j_2 j_3}_{i_2 i_3}(h) - \tfrac{1}{3}\,K^{j_2}_{i_2}K^{j_3}_{i_3}\right)\\
&= 4\sqrt{-h}\left[-2(\mathcal{R}^i_j - \tfrac{1}{2}\delta^i_j\mathcal{R})K^j_i - \tfrac{2}{3}K^i_jK^j_kK^k_i + K(K^i_jK^j_i - \tfrac{1}{3}K^2)\right]\\
c_3 &= \ell^2/64\pi G
\end{aligned}$$

Kounterterms

- $D = 2n$ dimensions [R.O., JHEP 0506: 023 (2005)]

$$
\begin{aligned}
B_{2n-1} &= 2n\sqrt{-h}\int_0^1 dt\ \delta^{[i_1\cdots i_{2n-1}]}_{[j_1\cdots j_{2n-1}]} K^{j_1}_{i_1}\left(\frac{1}{2}\mathcal{R}^{j_2 j_3}_{i_2 i_3} - t^2 K^{j_2}_{i_2} K^{j_3}_{i_3}\right)\times\cdots \\
&\cdots\times\left(\frac{1}{2}\mathcal{R}^{j_{2n-2}j_{2n-1}}_{i_{2n-2}i_{2n-1}} - t^2 K^{j_{2n-2}}_{i_{2n-2}} K^{j_{2n-1}}_{i_{2n-1}}\right) \\
c_{2n-1} &= (-\ell^2)^{n-1}/(16\pi G n(2n-2)!)
\end{aligned}
$$

Kounterterms

- Kounterterms in $D = 2n+1$ [R.O., JHEP 0704: 073 (2007)]

$$
\begin{aligned}
B_{2n} &= 2n\int_0^1 dt\int_0^t ds\,\delta^{[j_1\cdots j_{2n}]}_{[i_1\cdots i_{2n}]}\,K^{i_1}_{j_1}\delta^{i_2}_{j_2}\left(\frac{1}{2}\mathcal{R}^{i_3 i_4}_{j_3 j_4}-t^2 K^{i_3}_{j_3}K^{i_4}_{j_4}+\frac{s^2}{\ell^2}\delta^{i_3}_{j_3}\delta^{i_4}_{j_4}\right)\times\cdots \\
&\quad \cdots\times\left(\frac{1}{2}\mathcal{R}^{i_{2n-1} i_{2n}}_{j_{2n-1} j_{2n}}-t^2 K^{i_{2n-1}}_{j_{2n-1}}K^{i_{2n}}_{j_{2n}}+\frac{s^2}{\ell^2}\delta^{i_{2n-1}}_{j_{2n-1}}\delta^{i_{2n}}_{j_{2n}}\right).
\end{aligned}
$$

Kounterterms

- Kounterterms in $D = 2n + 1$ [R.O., JHEP 0704: 073 (2007)]

$$B_{2n} = 2n \int_0^1 dt \int_0^t ds\, \delta^{[j_1 \cdots j_{2n}]}_{[i_1 \cdots i_{2n}]} K^{i_1}_{j_1} \delta^{i_2}_{j_2} \left(\frac{1}{2} \mathcal{R}^{i_3 i_4}_{j_3 j_4} - t^2 K^{i_3}_{j_3} K^{i_4}_{j_4} + \frac{s^2}{\ell^2} \delta^{i_3}_{j_3} \delta^{i_4}_{j_4} \right) \times \cdots$$

$$\cdots \times \left(\frac{1}{2} \mathcal{R}^{i_{2n-1} i_{2n}}_{j_{2n-1} j_{2n}} - t^2 K^{i_{2n-1}}_{j_{2n-1}} K^{i_{2n}}_{j_{2n}} + \frac{s^2}{\ell^2} \delta^{i_{2n-1}}_{j_{2n-1}} \delta^{i_{2n}}_{j_{2n}} \right) .$$

- Coupling constant

$$c_{2n} = \frac{1}{16\pi G} \frac{(-1)^n \ell^{2n-2}}{2^{2n-2} n (n-1)!^2} .$$

Quasilocal vs Holographic Stress Tensor

- **Kounterterms: No clear identification of the boundary quasilocal stress tensor**

$$\delta\tilde{I}_{ren}=\int\limits_{\partial M}d^{D-1}x\,\sqrt{-h}\left(\frac{1}{2}\,\tau_{i}^{j}\left(h^{-1}\delta h\right)_{j}^{i}+\Delta_{i}^{j}\,\delta K_{j}^{i}\right)$$

- **But what about holographic stress tensor?**

Holographic stress tensor not from a quasilocal one

- Example 1: AAdS sector in 3D Topologically Massive Gravity

$$I_{TMG} = I_{EH} + \frac{1}{32\pi G\mu}\int_M d^3x\left(\Gamma d\Gamma + \frac{2}{3}\Gamma^3\right)$$

- No Gibbons-Hawking term for $L_{CS}(\Gamma)$ $\Rightarrow$ No quasilocal stress tensor

$$\delta I_{TMG} = \int_{\partial M} d^2x\sqrt{-h}\left(\frac{1}{2}T^{ij}_{(EH)}\left(h^{-1}\delta h\right)^i_j + \frac{1}{16\pi G\mu}K_i{}^k\delta K_{kj}\,\epsilon^{ij}\right)$$

- However $K_{ij} = \frac{1}{\ell}\frac{g^{(0)}_{ij}}{\rho} + g^{(1)}_{ij} + \ldots$

- Holographic stress tensor

$$\hat{T}^{ij}[g_{(0)}] = T^{ij}_{(EH)}[g_{(0)}] - \frac{1}{8\pi G\mu}\left(g^{il}_{(1)}g^{(0)}_{lk}\,\epsilon^{kj} + i\leftrightarrow j\right)$$

Holographic stress tensor not from a quasilocal one

- Example 2: EH AdS+ GB term in 4D

$$\tilde{I}_{ren} = \frac{1}{16\pi G} \int\limits_{M} d^4x \sqrt{-\mathcal{G}} \left[(R - 2\Lambda) + \frac{\ell^2}{4} (R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2) \right]$$

- For any $D > 4$ (arbitrary GB coupling α)

$$\delta \mathrm{I}_{GB} = \frac{\alpha}{4\pi G} \int\limits_{\partial M} \mathrm{d}^{D-1}\mathrm{x} \sqrt{-h}\, \delta^{[jj_1 j_2]}_{[ii_1 i_2]} \left[\frac{1}{2} \left(h^{-1} \delta h \right)^i_k K^k_j + \delta K^i_j \right] \left(\frac{1}{2} \mathcal{R}^{i_1 i_2}_{j_1 j_2}(h) - K^{i_1}_{j_1} K^{i_2}_{j_2} \right)$$

- Gibbons-Hawking term for GB

$$\beta_{GB} = -\frac{\alpha}{4\pi G} \sqrt{-h} \delta^{[j_1 j_2 j_3]}_{[i_1 i_2 i_3]} K^{i_1}_{j_1} \left(\frac{1}{2} \mathcal{R}^{i_2 i_3}_{j_2 j_3} - \frac{1}{3} K^{i_2}_{j_2} K^{i_3}_{j_3} \right) ,$$

Holographic stress tensor not from a quasilocal one

- Dirichlet variation

$$\delta I_{GB}=\frac{\alpha}{8\pi G}\int\limits_{\partial M} d^{D-1}x\sqrt{-h}\,\delta^{[j\,j_1 j_2 j_3]}_{[i\,i_1 i_2 i_3]}(h^{-1}\delta h)^i_j\,K^{i_1}_{j_1}\left(\frac{1}{2}\mathcal{R}^{i_2 i_3}_{j_2 j_3}-\frac{1}{3}K^{i_2}_{j_2}K^{i_3}_{j_3}\right)$$

- No Gibbons-Hawking term for $4D$ GB $\Rightarrow$ No quasilocal stress tensor

$$\delta\tilde{I}_{ren}=\lim_{\rho\to 0}\int\limits_{\partial M} d^3x\,\sqrt{-h}\left(\frac{1}{2}\tau^j_i\,\left(h^{-1}\delta h\right)^i_j+\Delta^j_i\,\delta K^i_j\right)$$

where

$$\tau^j_i=\frac{1}{32\pi G}\delta^{[jkl]}_{[mnp]}K^m_i\left(R^{np}_{kl}+\frac{1}{\ell^2}\delta^{[np]}_{[kl]}\right),\quad \Delta^j_i=\frac{1}{32\pi G}\delta^{[jkl]}_{[inp]}\left(R^{np}_{kl}+\frac{1}{\ell^2}\delta^{[np]}_{[kl]}\right)$$

Holographic stress tensor not from a quasilocal one

- Performing FG expansion of the fields

$$\sqrt{-h}=\frac{\sqrt{-g}}{\rho^{3/2}}=\frac{\sqrt{-g_{(0)}}}{\rho^{3/2}}+\mathcal{O}(\rho^{-1/2})$$

$$\left(h^{-1}\delta h\right)_j^i=\left(g^{-1}\delta g\right)_j^i=\left(g_{(0)}^{-1}\delta g_{(0)}\right)_j^i+\mathcal{O}(\rho),\quad \delta K_j^i=\mathcal{O}(\rho),$$

$$\tau_i^j=\rho^{3/2}\,T_i^j[g_{(0)}]+\mathcal{O}(\rho^2)=\Delta_i^j+\mathcal{O}(\rho^2)$$

- Holographic stress tensor

$$\delta\tilde{I}_{ren}=\int\limits_{\partial M}d^3x\,\sqrt{-g_{(0)}}\frac{1}{2}\,T_i^j[g_{(0)}]\left(g_{(0)}^{-1}\delta g_{(0)}^{-1}\right)_j^i$$

Conclusions and prospects

- The result of $D = 4$ is always true in $D = 2n$
(requires the asymptotic behavior of the Weyl tensor)
- In $D = 2n + 1$ it is more subtle because counterterms ambiguity
(finite counterterms that do not modify the Weyl anomaly).
- Reading off the holographic (quasilocal) stress tensor from the variation of the action in other gravity theories (EGB, Lovelock, etc.) where Holographic Renormalization would drive you insane.