Holography in Poincaré gauge theories

Olivera Mišković (PUCV, Valparaíso, Chile)

In collaboration with

Milutin Blagojević, Branislav Cvetković (IF, Belgrade, Serbia)

Rodrigo Olea (UNAB, Santiago, Chile)

Based on JHEP 1305 (2013) 103

Quantum Gravity in the Southern Cone VI

Maresias, Brazil, September 11-14, 2013

Plan of the talk

- Motivation
- General settings: Holography in Poincaré Gauge Theories
- Application: Quadratic parity-preserving 3d gravity
- Conclusions and final remarks

Motivation

- **3d gravity** is the simplest model for studying gravitational dynamics
  - Rich in both **bulk & boundary dynamics**
  - It has interesting both **classical & quantum theories**
  - It possesses exact solutions, e.g. **black holes** (BTZ), their thermodynamics can be studied
- **3d gravities in the context of Riemannian geometry**
  - **Einstein-Hilbert Gravity** is a topological theory
    – not a realistic feature of the gravitational dynamics
  - **Topologically Massive Gravity** (TMG)
    – with gravitational Chern-Simons term $\frac{1}{32\pi G\mu}\int d^3x\left(\Gamma\partial\Gamma+\frac{2}{3}\Gamma^3\right)$, massive graviton, violates the parity, 1 degree of freedom
  - **New Massive Gravity** (NMG), or *Bergshoeff-Hohm-Townsend* gravity
    – preserves parity, 2 degrees of freedom, $\frac{1}{m^2}\int d^3x\left(R_{\mu\nu}R^{\mu\nu}-\frac{3}{8}R^2\right)$
  - **General Massive Gravity** $=$ TMG $+$ NMG
    – the most general ghost-free masive gravity in 3D

Motivation

Field-theory approach to gravity as Poincarè gauge theory

- **Fundamental fields**

spin-connection	ω_μ^{AB}	gauge connection for the Lorentz group
vielbein	e_μ^A	compensating field for Poincaré traslations

- **Field-strenghts** are the curvature $R^{AB} = \left(d\omega + \omega^2\right)^{AB}$ and torsion $T^A = \nabla e^A$ of spacetime

- Characterized by a **Riemann-Cartan geometry** of spacetime: both the torsion and curvature carry the dynamics of gravity
- **Riemann space** is a *particular solution* $T^A = 0 \Rightarrow \tilde{\omega}^{AB} = \tilde{\omega}^{AB}(e)$

$\Longrightarrow$ In the framework of PGT, one is naturally motivated to study 3D gravitational models with **torsional degrees of freedom**

- *Example*: **Topological 3D Gravity (**i.e. **Mielke-Baekler (MB) model)**
  - Generalization of TMG that includes a torsional term $e^A \wedge T_A$
  - It can be formulated as a Chern–Simons gauge theory
  - The model posses **the black hole with torsion as a vacuum solution**
  - In the AdS sector, asymptotic symmetry is described by two independent Virasoro algebras with different central charges
  - Black hole entropy is found to depend on torsion

$\longrightarrow$ However, **the holographic FT of this theory is less interesting** (Riemannean CFT, Lorentz-violating, with truncated radial expansion)

Motivation

We study a class of gravities with propagating torsion, where Lagrangian contains **all invariants quadratic in the field strengths that preserve parity**.

- Generalization of NMG that includes torsional terms
- Possesses a **black hole with torsion as a vacuum solution**
- Torsional modes can propagate to the boundary

  $\Rightarrow$ we expect an **interesting holographic theory**
- There is a choice of couplings which defines a unitary massive gravity model in any dimension [Hernaski et al, 2009]
- Holography in RC spaces has been studied in:

  $D = 5$, CS gravity [Bañados, Miskovic, Theisen, 2006]
  $D = 4$, EH gravity [Petkou, 2010]
  $D = 3$, MB model [Klemm, Tagliabue, 2008]

General settings in the framework of Poincarè gauge theory

$\rightarrow$ all relations are *purely kinematical* (independent on the dynamics)

$\rightarrow$ *hatted quantities* are defined in the bulk

Fundamental fields

Vielbein $\hat{e}_\mu^A$ $\quad A = 0,1,2$ Lorentz indices

Spin-connection $\hat{\omega}_\mu^{AB} = -\varepsilon^{AB}{}_C\,\hat{\omega}_\mu^C$ $\quad x^\mu = (t,\rho,\varphi)$ local coordinates

Field-strenghts

Riemann tensor

$$\hat{R}^{AB} = (d\hat{\omega} + \hat{\omega}\wedge\hat{\omega})^{AB}$$

$$\hat{R}_{\mu\nu}^{AB} = -\varepsilon^{AB}{}_C\,\hat{R}_{\mu\nu}^C$$

Covariant derivative

$$\hat{\nabla}X^A = dX^A + [\hat{\omega}, X]^A$$

Torsion tensor

$$\hat{T}^A = \hat{\nabla}\hat{e}^A$$

(components $\hat{T}_{\mu\nu}^A = -\hat{T}_{\nu\mu}^A$)

Trace of torsion

$$\hat{V}_\mu = \hat{T}^A{}_{A\mu} = \hat{T}^A{}_{\nu\mu}\,\hat{e}_A^\nu$$

Holographic ansatz

Poincare gauge transformations

$$\boxed{\begin{aligned} \delta \hat{e}^A_\mu &= -\varepsilon^{ABC}\hat{e}_{B\mu}\hat{\theta}_C - \mathcal{L}_{\hat{\xi}}\hat{e}^A_\mu \\ \delta \hat{\omega}^A_\mu &= -\hat{\nabla}_\mu \hat{\theta}^A - \mathcal{L}_{\hat{\xi}}\hat{\omega}^A_\mu \end{aligned}}$$

6 local parameters:

$\delta x^\mu = \hat{\xi}^\mu(x) =$ diffeomorphisms

$\hat{\theta}^A(x) =$ Lorentz rotations

Lie derivative:

$$\mathcal{L}_{\hat{\xi}} X_\mu \equiv \partial_\mu \hat{\xi} \cdot X + \hat{\xi} \cdot \partial X_\mu$$

Space is asymptotically AdS

- **Radial foliation**: $x^\mu = (\rho, x^i)$, $\quad A = (1, a)$

We set AdS radius: $\ell = 1$

- **AdS boundary**: in the asymptotic region, placed at the radius $\rho = 0$

Holographic ansatz

- We have 6 gauge parameters $\hat{\xi}^{\mu}, \hat{\theta}^{A}$, we can fix them partially.
- **6 gauge-fixing conditions**
$$\boxed{\begin{aligned} \hat{e}_{\rho}^{A} &= (\hat{e}_{\rho}^{1}, \hat{e}_{\rho}^{a}) = \left(\frac{1}{\rho}, 0\right) \\ \hat{\omega}_{\rho}^{A} &= (\hat{\omega}_{\rho}^{1}, \hat{\omega}_{\rho}^{a}) = \left(\frac{p}{2\rho}, 0\right) \end{aligned}}$$
- **2 extra conditions** $\boxed{\hat{e}_{i}^{1} = 0}$

- equivalent to "radial gauge" $\hat{e}_{A}^{\rho} = 0$, ensures that the radial direction coincides with the normal to ∂M

Holographic ansatz

In asymptotically AdS space

- Remaining components $\left\{\hat{e}_i^a, \hat{\omega}_i^a, \hat{\omega}_i^1\right\}$ of the fundamental fields can be written in the **Fefferman-Graham-like form** that factorizes the poles $\frac{1}{\rho}$ and leaves regular tensors near the boundary $\rho = 0$:

$$\begin{aligned}
\hat{e}_i^a &= \frac{1}{\rho}\, e_i^a(\rho, x) \\
\hat{\omega}_i^1 &= \omega_i(\rho, x) \\
\hat{\omega}_i^a &= \frac{1}{\rho}\, k^a{}_i(\rho, x)
\end{aligned}$$

In asymptotically AdS space

- Remaining components $\left\{\hat{e}_i^a, \hat{\omega}_i^a, \hat{\omega}_i^1\right\}$ of the fundamental fields can be written in the **Fefferman-Graham-like form** that factorizes the poles $\frac{1}{\rho}$ and leaves regular tensors near the boundary $\rho = 0$:

$$
\begin{aligned}
\hat{e}_i^a &= \frac{1}{\rho}\, e_i^a(\rho, x) \\
\hat{\omega}_i^1 &= \omega_i(\rho, x) \\
\hat{\omega}_i^a &= \frac{1}{\rho}\, k^a{}_i(\rho, x)
\end{aligned}
$$

- Regular functions $\left\{e_i^a, k^a{}_i, \omega_i\right\}$ are expanded in power series in ρ^2

$$
\begin{aligned}
e_i^a &= e^a_{(0)i}(x) + \rho^2\, e^a_{(2)i}(x) + \mathcal{O}_4 \\
\omega_i &= \omega_{(0)i}(x) + \rho^2 \omega_{(2)i}(x) + \mathcal{O}_4 \\
k^a{}_i &= k^a_{(0)i}(x) + \rho^2\, k^a_{(2)i}(x) + \mathcal{O}_4
\end{aligned}
$$

Holographic ansatz

- Bulk metric $\hat{g}_{\mu\nu} = \eta_{AB}\,\hat{e}^A_\mu \hat{e}^B_\nu$ in our ansatz corresponds to the Fefferman-Graham frame

$$
\begin{aligned}
ds^2 &= \hat{g}_{\mu\nu}\,dx^\mu dx^\nu = \frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2}\,g_{ij}(\rho, x)\,dx^i dx^j \\
g_{ij} &= g_{(0)ij} + \rho^2 \underbrace{\left(e_{(2)ij} + e_{(2)ji}\right)}_{g_{(2)ij}} + \mathcal{O}_4
\end{aligned}
$$

Holographic ansatz

Geometric quantities on the 2d boundary

Lorentz connection (Abelian): $\omega_i^{ab} = -\varepsilon^{ab}\omega_i$

Riemann curvature (Abelian): $R_{ij}^{ab} = -\varepsilon^{ab}F_{ij} = -\varepsilon^{ab}\left(\partial_i\omega_j - \partial_j\omega_i\right)$

Torsion tensor: $T_{ij}^a = \nabla_i e_j^a - \nabla_j e_i^a$

- completely determined by its trace $T_{ij}^a = e_i^a V_j - e_j^a V_i$

Extrinsic curvature:
(n_b = normal to ∂M) $K_{ab} = \hat{\nabla}_a n_b = \varepsilon_{cb}\, k^c{}_a$

- symmetric part of K $K_{(ij)} \propto \partial_\rho h_{ij}$

(derivative of the induced metric)

- antisymmetric part of K $K_{[ij]} \propto \varepsilon_{ij}\, A$

(axial torsion $A = \frac{1}{3!}\,\varepsilon^{ABC}\,\hat{T}_{ABC}$)

Residual gauge transformations

Residual symmetry that leaves the gauge conditions invariant

- **2D diffeomorphisms** (ξ^i): $\begin{cases} \delta e^a_{(0)i} & = -\mathcal{L}_\xi e^a_{(0)i} \\ \delta\omega_{(0)i} & = -\mathcal{L}_\xi \omega_{(0)i} \end{cases}$
- **2D Lorentz transformations** ($\varepsilon^{ab}\theta$): $\begin{cases} \delta e^a_{(0)i} & = -\theta\,\varepsilon^{ab} e_{(0)bi} \\ \delta\omega_{(0)i} & = -\partial_i\theta \end{cases}$
- **2D Weyl dilatations** (f): $\begin{cases} \delta e^a_{(0)i} & = \sigma\, e^a_{(0)i} \\ \delta\omega_{(0)\alpha} & = \varepsilon^{ab} e^j_{(0)a} e_{(0)bi}\,\partial_j\sigma \end{cases}$

Residual gauge transformations

Residual symmetry that leaves the gauge conditions invariant

- **2D diffeomorphisms** (ξ^i): $\begin{cases} \delta e^a_{(0)i} &= -\mathcal{L}_\xi e^a_{(0)i} \\ \delta\omega_{(0)i} &= -\mathcal{L}_\xi \omega_{(0)i} \end{cases}$
- **2D Lorentz transformations** ($\varepsilon^{ab}\theta$): $\begin{cases} \delta e^a_{(0)i} &= -\theta\,\varepsilon^{ab} e_{(0)bi} \\ \delta\omega_{(0)i} &= -\partial_i\theta \end{cases}$
- **2D Weyl dilatations** (f): $\begin{cases} \delta e^a_{(0)i} &= \sigma\, e^a_{(0)i} \\ \delta\omega_{(0)\alpha} &= \varepsilon^{ab} e^j_{(0)a} e_{(0)bi}\,\partial_j\sigma \end{cases}$
- Tensor $k^a{}_\alpha$ is not independent: $\boxed{k_{ab} = \frac{p}{2}\,\eta_{ab} - \varepsilon_{ab} + \rho\varepsilon_a{}^c e^i_c \partial_\rho e_{bi}}$

Residual gauge transformations

Residual symmetry that leaves the gauge conditions invariant

- **2D diffeomorphisms** (ξ^i): $\begin{cases} \delta e^a_{(0)i} &= -\mathcal{L}_\xi e^a_{(0)i} \\ \delta \omega_{(0)i} &= -\mathcal{L}_\xi \omega_{(0)i} \end{cases}$
- **2D Lorentz transformations** ($\varepsilon^{ab}\theta$): $\begin{cases} \delta e^a_{(0)i} &= -\theta\, \varepsilon^{ab} e_{(0)bi} \\ \delta \omega_{(0)i} &= -\partial_i \theta \end{cases}$
- **2D Weyl dilatations** (f): $\begin{cases} \delta e^a_{(0)i} &= \sigma\, e^a_{(0)i} \\ \delta \omega_{(0)\alpha} &= \varepsilon^{ab} e^j_{(0)a} e_{(0)bi}\, \partial_j \sigma \end{cases}$
- Tensor $k^a{}_\alpha$ is not independent: $\boxed{k_{ab} = \frac{p}{2}\eta_{ab} - \varepsilon_{ab} + \rho \varepsilon_a{}^c e^i_c \partial_\rho e_{bi}}$

$\Rightarrow$ **Residual symmetry transformations** $=$ Weyl group acting on ∂M

$\Rightarrow$ $e^a_{(0)i}$ and $\omega_{(0)i}$ are recognized as the vielbein and spin connection of the boundary RC geometry.

Residual gauge transformations

Comment

- Under local dilatations, the metric transforms as
$$\begin{aligned}\delta g_{(0)ij} &= 2\sigma\, g_{(0)ij}\\ \delta g_{(2)ij} &= 2\sigma\, g_{(2)ij} - e^{a}_{(0)(i}\nabla_{(0)j)}(e^{k}_{(0)a}\partial_k\sigma) + \nabla^{k}_{(0)}\sigma\, T_{\ (0)(ij)k}\end{aligned}$$
- When $T^{a}_{(0)ij} = 0$, this result matches the Penrose–Brown–Henneaux transformation, derived in Riemannian GR and used to study universal properties of trace anomalies

  [Imbimbo, Schwimmer, Theisen, Yankielwicz, 2000]

Comment

- Under local dilatations, the metric transforms as
$$\begin{aligned}\delta g_{(0)ij} &= 2\sigma\, g_{(0)ij} \\ \delta g_{(2)ij} &= 2\sigma\, g_{(2)ij} - e^a_{(0)(i}\nabla_{(0)j)}(e^k_{(0)a}\partial_k\sigma) + \nabla^k_{(0)}\sigma T_{\ (0)(ij)k}\end{aligned}$$
- When $T^a_{(0)ij} = 0$, this result matches the Penrose–Brown–Henneaux transformation, derived in Riemannian GR and used to study universal properties of trace anomalies

  [Imbimbo, Schwimmer, Theisen, Yankielwicz, 2000]

- So far we have *kinematical* relations (free of any dynamical content); another kinematical relations are
$$\boxed{\hat{T}_{ijk} = p\varepsilon_{ijk} + \mathcal{O}_2\,, \qquad \hat{R}_{ijk} = q\varepsilon_{ijk} + \mathcal{O}_2} \qquad q = \frac{p^2}{4} - 1$$
- To lowest order in ρ, the parameters (p, q) define the torsion and the curvature of the **maximally symmetric spacetime** with constant $\hat{R}, \hat{T}$

Ward identities in dual CFT

Classical conservation laws: $\nabla_i J^i = 0$

Ward Identities: $\langle \nabla_i J^i \rangle_{\mathrm{CFT}} = 0$

Quantum anomaly: $\langle \nabla_i J^i \rangle_{\mathrm{CFT}} = \mathcal{A} \neq 0$

Application of AdS/CFT correspondence to calculate $\langle \nabla_i J^i \rangle_{\mathrm{CFT}}$

$J^i =$ energy-momentum tensor, spin current

Ward identities in dual CFT

$D = 3$, AdS_3 gravity in classical approximation

- Boundary conditions $\hat{e}_\mu^A(\rho, x)$, $\hat{\omega}_\mu^A(\rho, x) \to e_{(0)i}^a(x)$, $\omega_{(0)i}^a(x)$
- Action free of IR divergences

$$\boxed{\begin{array}{l} \textit{Gravitational partition function evaluated on-shell} \\ Z_{\mathrm{G}}[\hat{e}, \hat{\omega}]\big|_{\hat{e}, \hat{\omega} \to e_{(0)}, \omega_{(0)}} = Z_{\mathrm{G}}[e_{(0)}, \omega_{(0)}] \simeq e^{iI_{\mathrm{ren}}[e_{(0)}, \omega_{(0)}]} \end{array}}$$

$D = 2$, Quantum effective action of CFT_2 with field ψ

$$\boxed{\begin{array}{l} \textit{CFT partition function} \\ Z_{\mathrm{CFT}}[e_{(0)}, \omega_{(0)}] = \int D\psi \, e^{iI[\psi, e_{(0)}, \omega_{(0)}]} = e^{iW[e_{(0)}, \omega_{(0)}]} \end{array}}$$

AdS/CFT correspondence $\boxed{Z_{\mathrm{CFT}}[e_{(0)}, \omega_{(0)}] = Z_{\mathrm{G}}[e_{(0)}, \omega_{(0)}]}$

Ward identities in dual CFT

Gravitational constraints (bulk eqs. of motion without $\partial/\partial\rho$)
$=$ Ward identities for the 1-point functions derived from $W[e_{(0)},\omega_{(0)}]$

$$\delta I_{\mathrm{ren}} = -\int_{\partial M} d^2x \left(\tau^i{}_a\,\delta e^a_{(0)i} + \sigma^i\,\delta\omega_{(0)i}\right)$$

$e^a_{(0)i}$: source for the energy-momentum tensor (density)

$\omega_{(0)i}$: source for the spin current (density)

$$\left\langle \tau^i{}_a \right\rangle = -\frac{\delta I_{\mathrm{ren}}}{e^a_{(0)i}} \qquad \left\langle \sigma^i \right\rangle = -\frac{\delta I_{\mathrm{ren}}}{\omega_{(0)i}}$$

- We obtain the *quantum* results using just the *classical* calculation.
- **Ward identities:** *Conservation laws are obtained from the invariance condition* $\delta_{\mathrm{gauge}} I_{\mathrm{ren}} = 0$.

Traslations:	$\nabla_i \tau^i{}_a$	$= \tau^i{}_b T^b{}_{ai} + \sigma^j F_{aj} - \omega_a(\nabla\cdot\sigma + \varepsilon^{bc}\tau_{bc})$
Lorentz rotatons:	$\nabla\cdot\sigma$	$= -\varepsilon^{ab}\tau_{ab}$
Dilatations:	$\tau^a{}_a$	$= \nabla_i\left(\varepsilon_{ab}\sigma^a e^{bi}\right)$

Ward identities in dual CFT

Riemannian boundary: **Is the limit $T^a_{(0)ij} \to 0$ allowed?**

$$T^a_{(0)ij} = 0 \quad \Rightarrow \quad \tilde{\omega}_i = -\varepsilon_{ab}\varepsilon_{ij} e^{aj}_{(0)}\, \varepsilon^{km}\partial_k e^b_{(0)m}$$

The Levi-Civitá spin connection $\tilde{\omega} = \omega|_{T=0}$ is not a source!

- Renormalized action
$$\begin{aligned} \delta\tilde{I}_{\mathrm{ren}}[e_{(0)}] &= -\int_{\partial M} d^2x\, \tilde{\tau}^i{}_a\, \delta e^a_{(0)i} \\ \tilde{\tau}^i{}_a &= \tau^i{}_a - \tilde{\nabla}_j \left(e^{-1}\varepsilon^{ij}\sigma_a\right) \\ \tilde{\sigma}_a &= 0 \end{aligned}$$

- Conservation laws

Traslations:	$\left(e^{-1}\tilde{\tau}^j{}_i\right)_{;j} = 0$
Lorentz rotatons:	$\varepsilon^{ab}\tilde{\tau}_{ab} = 0$
Dilatations:	$\tilde{\tau}^a{}_a = 0$

$\Rightarrow$ **Riemannian Ward identities coincide with those obtained from the Riemann-Cartan ones in the limit** $T^a_{(0)ij} \to 0$

We can calculate the quantum Ward identities in First Order formalism, regardless the space is Riemannian or RC.

Dynamics – AdS Gravity Action

- **AdS_3 gravity with propagating torsion that preserves the parity**

- The Weyl tensor identically vanishes in 3D that reduces a number of independent curvature invariants
- Ricci tensor $\hat{R}_{\mu\nu}$ is not symmetric in RC space

$$
\begin{aligned}
I[\hat{e},\hat{\omega}] \;=\; & \int d^3x\,\hat{e}\,\Big(-a\,\hat{R} - 2\Lambda \\
& +\beta_1\,\hat{R}_{\mu\nu}\hat{R}^{\mu\nu} + \beta_2\,\hat{R}_{\mu\nu}\hat{R}^{\nu\mu} + \beta_3\,\hat{R}^2 \\
& +\alpha_1\,\hat{T}^a{}_{\mu\nu}\,\hat{T}_a{}^{\mu\nu} + \alpha_2\,\hat{T}^a{}_{\mu\nu}\,\hat{T}^{\nu\mu}{}_a + \alpha_3\,\hat{V}_\mu\hat{V}^\mu\Big)
\end{aligned}
$$

$a = 1/16\pi G$	gravitational constant
$a^{-1}\Lambda < 0$	cosmological constant
$\beta_1, \beta_2, \beta_3$	curvature coupling constants
$\alpha_1, \alpha_2, \alpha_3$	torsion coupling constants

Dynamics – AdS Gravity Action

- **AdS_3 gravity with propagating torsion that preserves the parity**
- The Weyl tensor identically vanishes in 3D that reduces a number of independent curvature invariants
- Ricci tensor $\hat{R}_{\mu\nu}$ is not symmetric in RC space

$$
\begin{aligned}
I[\hat{e},\hat{\omega}] \;=\; \int d^3x\, \hat{e}\, \Big(& -a\,\hat{R} - 2\Lambda \\
& +\beta_1\,\hat{R}_{\mu\nu}\hat{R}^{\mu\nu} + \beta_2\,\hat{R}_{\mu\nu}\hat{R}^{\nu\mu} + \beta_3\,\hat{R}^2 \\
& +\alpha_1\,\hat{T}^a{}_{\mu\nu}\,\hat{T}_a{}^{\mu\nu} + \alpha_2\,\hat{T}^a{}_{\mu\nu}\,\hat{T}^{\nu\mu}{}_a + \alpha_3\,\hat{V}_\mu\hat{V}^\mu\Big)
\end{aligned}
$$

- **Comment**: Sometimes it is more practical to use the coupling constants that correspond to the *irreducible components* of the curvature and torsion tensors

$$
\begin{aligned}
\alpha_1 &= \tfrac{1}{6}(2a_1 + a_3) & \alpha_2 &= \tfrac{1}{3}(a_1 - a_3) & \alpha_3 &= \tfrac{1}{2}(a_2 - a_1) \\
\beta_1 &= \tfrac{1}{2}(b_4 + b_5) & \beta_2 &= \tfrac{1}{2}(b_4 - b_5) & \beta_3 &= \tfrac{1}{12}(b_6 - 4b_4)
\end{aligned}
$$

Equations of motion

Covariant momenta:

$$\begin{aligned}\pi_{ab} &= \beta_1 \hat{R}_{ab} + \beta_2 \hat{R}_{ba} + \beta_3 \eta_{ab} \hat{R} \\ \pi_{abc} &= \alpha_1 \hat{T}_{abc} + \tfrac{\alpha_2}{2} \left(\hat{T}_{cba} - \hat{T}_{bca} \right) + \tfrac{\alpha_3}{2} \left(\eta_{ab} \hat{V}_c - \eta_{ac} \hat{V}_b \right)\end{aligned}$$

Lagrangian density: $L = -a\,\hat{R} + \pi_{abc}\hat{T}^{abc} + \hat{R}_{ab}\pi^{ab}$

Equations of motion: second order in $\hat{e}^a_\mu$ and $\hat{\omega}^a_\mu$

$$\delta\hat{\omega}^a_\mu : \quad \begin{aligned} 0 &= \hat{\nabla}_a \left(\pi_{bc} - \eta_{bc}\pi \right) - \hat{\nabla}_b \left(\pi_{ac} - \eta_{ac}\pi \right) \\ &\hat{T}_{ade} \left(a\,\delta^a_c + \pi^a{}_c - \delta^a_c \pi \right) + 2\varepsilon^{ab}{}_c \varepsilon_{pde} \pi_{ab}{}^p \end{aligned}$$

$$\delta\hat{e}^a_\mu : \quad \begin{aligned} 0 &= 4\hat{\nabla}^c \pi_{abc} + 2\pi_{ade}\hat{T}_b{}^{de} - 4\pi_{abc}\hat{V}^c + 2\pi^{de}\hat{R}_{bdea} - 2\hat{R}_{ca}\pi^c{}_b \\ &-4\hat{T}_{dea}\pi^{de}{}_b + 2a\hat{R}_{ba} + \eta_{ab}\left(L - 2\Lambda \right) \end{aligned}$$

(*Comment*: The spectrum of excitations around Minkowski M_3 consists of 6 independent torsional modes: two spin-$0^{\mp}$ states, two spin-1 states and two spin-2 states.)

Near-boundary analysis: We have to solve these equations order by order in ρ^2.

Equations of motion

- **Finite order of the equations of motion ($\Rightarrow$ the AdS vacuum)**

Relations involving the coupling constants:
(the constant b_6 couples $\hat{R}^2$)

$$\boxed{\begin{aligned} & p(a+qb_6+2a_3)=0 \\ & aq-\Lambda+\tfrac{1}{2}p^2a_3-\tfrac{1}{2}q^2b_6=0 \end{aligned}}$$

- Quadratic terms $\Rightarrow$ Two different effective AdS vacua, $\hat{R}^{\lambda\sigma}_{\mu\nu}=-\frac{1}{\ell_\pm^2}\,\delta^{\lambda\sigma}_{\mu\nu}$
- Unique AdS vacuum $\ell_+^2=\ell_-^2$: $p=0$, $a-qb_6=0$

- **Linear order of the equations of motion ($\Rightarrow$ independent $V_{(0)}$s)**

$$\begin{aligned} 0 &= (2\alpha_1+\alpha_2+\alpha_3+q\beta_1)\,V_{(0)a} \\ 0 &= \left[a+qb_6+\alpha_3-\beta_2\left(1+\frac{p^2}{4}\right)\right]\epsilon_{ab}V^b_{(0)}+p\beta_2V_{(0)a} \\ 0 &= 2\,(a+qb_6+\alpha_3+q\beta_2)\,V_{(0)a}+p\,(a+qb_6+\alpha_3-q\beta_2)\,\epsilon_{ab}V^b_{(0)} \end{aligned}$$

- Torsion can be zero ($V_{(0)}=0$), with one independent component ($V_{(0)a}=\pm\epsilon_{ab}V^b_{(0)}$) or two ones (arbitrary $V_{(0)a}$)

Equations of motion

- **Quadratic order of equations of motion ($\Rightarrow$ vevs)**
- Algebraic system in $X = \varepsilon^{ab} e_{(2)ab}$ and $Y = e^{a}_{(2)a} - \frac{1}{4} R_{(0)}$
- Has non-trivial solutions, with X, Y functions of $V^2_{(0)}, \nabla_{(0)a} V_{(0)b}$
- **Particular solution**: $e_{(2)ab}$ =symmetric, $e^{a}_{(2)a} = \frac{1}{4} R_{(0)}$

  or $X = 0, Y = 0$ $\quad \longrightarrow$ *the torsionless case belongs here*
- Symmetric traceless part of $e_{(2)ab}$ is, in general, **a nonlocal function,** whose determination requires a global solution; physical objects, such as conformal anomaly, are always local – this is a general feature of the boundary currents in an effective theory.

We don't have to solve the equations of motion explicitly in order to obtain the Ward identities.

Renormalization of the action

- $\delta I_{\text{on-shell}}$ **is a boundary term**:

$$\delta I_{\text{on-shell}} = -2\int d^2x\, \varepsilon^{ij}\hat{e}_i^c \left[\delta\hat{e}_j^a\, \varepsilon^{bd}{}_c \pi_{abd} + \delta\hat{\omega}_j^a\, \left(a\,\eta_{ac} + \pi_{ca} - \eta_{ca}\pi\right)\right]$$

  - We have to identify all divergent and finite parts in $\delta I_{\text{on-shell}}$ near $\rho = 0$.

Renormalization of the action

- $\delta I_{\text{on-shell}}$ **is a boundary term**:

$$\delta I_{\text{on-shell}} = -2\int d^2x\,\varepsilon^{ij}\,\hat{e}_i^c\left[\delta\hat{e}_j^a\,\varepsilon^{bd}{}_c\pi_{abd} + \delta\hat{\omega}_j^a\,\left(a\,\eta_{ac} + \pi_{ca} - \eta_{ca}\pi\right)\right]$$

  - We have to identify all divergent and finite parts in $\delta I_{\text{on-shell}}$ near $\rho = 0$.

- **Divergent terms can be written as a total variation**

  $\Rightarrow$ *the CFT is renormalizable and the action has the form*

$$\boxed{\delta I_{\text{on-shell}} = -\delta I_{\text{div}} + \text{finite terms} + \mathcal{O}_2}$$

Renormalization of the action

- $\delta I_{\text{on-shell}}$ **is a boundary term**:

$$\delta I_{\text{on-shell}} = -2\int d^2x\,\varepsilon^{ij}\,\hat{e}_i^c\left[\delta\hat{e}_j^a\,\varepsilon^{bd}{}_c\,\pi_{abd} + \delta\hat{\omega}_j^a\,\left(a\,\eta_{ac} + \pi_{ca} - \eta_{ca}\pi\right)\right]$$

  - We have to identify all divergent and finite parts in $\delta I_{\text{on-shell}}$ near $\rho = 0$.

- **Divergent terms can be written as a total variation**

  $\Rightarrow$ *the CFT is renormalizable and the action has the form*

$$\boxed{\delta I_{\text{on-shell}} = -\delta I_{\text{div}} + \text{finite terms} + \mathcal{O}_2}$$

Divergent terms	I_{div}	$= (a + qb_6)\int d^2x\,\sqrt{-h}\,K$
Gibbons-Hawking term	I_{GH}	$= 2\,(a + qb_6)\int d^2x\,\sqrt{-h}\,K$
Counterterms	I_{ct}	$= -2\,(a + qb_6)\int d^2x\,\sqrt{-h}$

  - Both the GH term and the Barasubramanian-Kraus counterterm have modified coefficients due to $qb_6 \neq 0$

Renormalization of the action

- $\delta I_{\text{on-shell}}$ **is a boundary term**:

$$\delta I_{\text{on-shell}} = -2\int d^2x\,\varepsilon^{ij}\hat{e}_i^c\left[\delta\hat{e}_j^a\,\varepsilon^{bd}{}_c\pi_{abd} + \delta\hat{\omega}_j^a\,\left(a\,\eta_{ac} + \pi_{ca} - \eta_{ca}\pi\right)\right]$$

  - We have to identify all divergent and finite parts in $\delta I_{\text{on-shell}}$ near $\rho = 0$.

- **Divergent terms can be written as a total variation**

  $\Rightarrow$ *the CFT is renormalizable and the action has the form*

$$\boxed{\delta I_{\text{on-shell}} = -\delta I_{\text{div}} + \text{finite terms} + \mathcal{O}_2}$$

Divergent terms	I_{div}	$= (a + qb_6)\int d^2x\,\sqrt{-h}\,K$
Gibbons-Hawking term	I_{GH}	$= 2\,(a + qb_6)\int d^2x\,\sqrt{-h}\,K$
Counterterms	I_{ct}	$= -2\,(a + qb_6)\int d^2x\,\sqrt{-h}$

  - Both the GH term and the Barasubramanian-Kraus counterterm have modified coefficients due to $qb_6 \neq 0$

  $\Rightarrow \qquad I_{\text{div}} = I_{\text{GH}} + I_{\text{ct}} + \text{Euler topological invariant}$

- Without torsion, the result is the same as for Einstein-Hilbert AdS gravity

Holographic currents

Renormalized CFT action $=$ finite terms in $\delta I_{\text{on-shell}}$

$$\begin{aligned}\delta I_{\text{ren}} &= \delta\left(I_{\text{on-shell}} + \delta I_{\text{div}}\right) \\ &= -\int d^2x \left(\tau^i{}_a\,\delta e^a_{(0)i} + \sigma^i\,\delta\omega_{(0)i}\right)\end{aligned}$$

$\langle\tau^i{}_a\rangle$,$\langle\sigma^i\rangle$ are the *finite* boundary energy-monentum and spin current
$=$ holographic currents whose sources are $e^a_{(0)i}$ and $\omega_{(0)i}$

Spin current: $\langle\sigma^i\rangle = 2\beta_2\,\varepsilon^{ij} e^a_{(0)j}\left(V_{(0)a} - \frac{p}{2}\,\varepsilon_{ab} V^b_{(0)}\right)$

Without torsion, we have $\langle\sigma^i\rangle = 0$.

Stress tensor:

$$\begin{aligned}\langle\tau^i{}_a\rangle &= -4\varepsilon^{ij}\left[\left(a+qb_6\right)\,\varepsilon_{ab} e_{(2)j}{}^b + 2\left(\alpha_1-\alpha_2\right)\,\varepsilon^{cd} e_{(2)cd}\, e_{(0)aj}\right] \\ &\quad -2\varepsilon^{ij} e^b_{(0)j}\left[\left(\tfrac{p}{2}\,\eta_{ab}+\varepsilon_{ab}\right)pb_6 - 2\left(\beta_1-\beta_2\right)\left(\eta_{ab}+\tfrac{p}{2}\,\varepsilon_{ab}\right)\right]\varepsilon^{cd} e_{(2)cd} \\ &\quad -\varepsilon^{ij} e^b_{(0)j}\left(4\beta_3+b_6\right)\left(\tfrac{p}{2}\,\eta_{ab}+\varepsilon_{ab}\right)\left(e^c_{(2)c} - \tfrac{1}{4}\,R_{(0)}\right)\end{aligned}$$

Holographic Ward identities

We have to check whether these currents satisfy the conservation laws.

- **Lorentz invariance**

  The spin current is conserved, as expected $\Rightarrow c_R = c_L$

- **Traslational invariance**

  After a tedious calculation, we obtain that the conservation law for $\tau^i{}_a$ is satisfied for a generic theory.

- **Conformal anomaly**

  Holografic energy-momentum tensor in odd dimensions usually exhibits the Weyl anomaly that arises due to breaking the radial diffeomorphisms by the boundary.

- By definition, the "trace" anomaly is $\mathcal{A} = e\left[\tau^a{}_a - \nabla_i\left(\varepsilon^{ab} e^i_a \sigma_b\right)\right]$ (*we omit writing the index 'naught'*)

Holographic Ward identities

- Using the equations of motion, we can show that the **conformal symmetry is broken** and the quantum anomaly is
  $\boxed{-\mathcal{A} = c\,\mathcal{E}_2 + c_1 \mathcal{N}_2 + c_2 \mathcal{N}_2'}$

Holographic Ward identities

- Using the equations of motion, we can show that the **conformal symmetry is broken** and the quantum anomaly is

$$\boxed{-\mathcal{A} = c\,\mathcal{E}_2 + c_1\mathcal{N}_2 + c_2\mathcal{N}_2'}$$

$\mathcal{E}_2$	$= eR$	Euler topological invariant
$\mathcal{N}_2$	$= e(\nabla\cdot V - V^2)$	torsional topological invariant
$\mathcal{N}_2'$	$= e\,\varepsilon^{ab}\nabla_a V_b$	torsional topological invariant

Holographic Ward identities

- Using the equations of motion, we can show that the **conformal symmetry is broken** and the quantum anomaly is
  $\boxed{-\mathcal{A} = c\,\mathcal{E}_2 + c_1\mathcal{N}_2 + c_2\mathcal{N}_2'}$

  $$\begin{aligned} \mathcal{E}_2 &= eR && \text{Euler topological invariant} \\ \mathcal{N}_2 &= e(\nabla\cdot V - V^2) && \text{torsional topological invariant} \\ \mathcal{N}_2' &= e\,\varepsilon^{ab}\nabla_a V_b && \text{torsional topological invariant} \end{aligned}$$

- The second term is the 2D **Nieh-Yan-like invariant**, $\mathcal{N} = T^a * T_a - e^a \nabla * T_a$.
- In 4D, the Nieh-Yan torsional topological invariant, $\mathcal{N}_4 = T^a T_a - e^a \nabla T_a$, is related to the chiral anomaly.
- **Central charges in CFT** $$\begin{aligned} c &= a + b_6 q \\ c_1 &= 2\,(q\beta_2 + 2\beta_2 - \alpha_3) \\ c_2 &= -2p\beta_2 \end{aligned}$$

Final remarks

Additional comments

In theories with many parameters, e.g., $\{a, \Lambda, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\}$, there are critical values of the parameters where something happens.

In New Massive Gravity, there are two special points in the space of parameters:

(here $\beta_1 = \beta_2 = \frac{1}{16\pi G}\frac{1}{m^2}$ and $\beta_3 = -\frac{3}{8}\beta_1$)

- **Gravity with a unique AdS vacuum**: $m^2\ell^2 = +\frac{1}{2}$

- Relaxed boundary conditions allow hairy BTZ-like black holes.

- **Critical Gravity**= *gravity where all central charges vanish*, $\mathcal{A} = 0$

- Critical NMG: $m^2\ell^2 = +\frac{1}{2}$
- It contains **logarhitmic modes** in the metric, its dual CFT is so-called LCFT.

Final remarks

In AdS Gravity with propagating torsion: there are also critical points (here $b_6 = 4(\beta_1 + \beta_2 + 3\beta_3)$)

- **Gravity with a unique AdS vacuum** $p = 0$, $a - qb_6 = 0$
- **Critical Gravity**

  For example: $a + qb_6 = 0$, $\beta_2 = 0$, $\alpha_3 = 0$

Dual theories at critical points are still an open question

Final remarks

In AdS Gravity with propagating torsion: there are also critical points (here $b_6 = 4(\beta_1 + \beta_2 + 3\beta_3)$)

- **Gravity with a unique AdS vacuum** $p = 0$, $a - qb_6 = 0$
- **Critical Gravity**

  For example: $a + qb_6 = 0$, $\beta_2 = 0$, $\alpha_3 = 0$

Dual theories at critical points are still an open question

Other open questions:

- Any particular case of a CFT with non trivial torsion.
- Topological quantum numbers associated to new torsional invariants and particular solutions.
- Relaxed AAdS boundary conditions – Critical gravity, Unique AdS vacuum,...
- Non-AdS sectors of the theory, etc.

Conclusions

- We discussed **boundary conditions** and **coordinate frame** suitable for holographic description of Poincaré Gauge Theories with $\Lambda < 0$;
- We wrote general **Ward identities** for a dual CFT;
- We chose a particular 2+1 AdS gravity with (expected) interesting holographic properties and applied the above formalism;
- We found **the counterterms** for this theory and obtained the **renormalized action**.
- Then we calculated the holographic Ward identities and obtained **the Weyl anomaly** that contains torsional topological invariant.

Conclusions

- We discussed **boundary conditions** and **coordinate frame** suitable for holographic description of Poincaré Gauge Theories with $\Lambda < 0$;
- We wrote general **Ward identities** for a dual CFT;
- We chose a particular 2+1 AdS gravity with (expected) interesting holographic properties and applied the above formalism;
- We found **the counterterms** for this theory and obtained the **renormalized action**.
- Then we calculated the holographic Ward identities and obtained **the Weyl anomaly** that contains torsional topological invariant.

THANK YOU!