Chern-Simons Gravity induces Conformal Gravity QGSC VI

Danilo Diaz and me

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Outline

Conformal Gravity

Four dimensional Conformal Gravity

$$\int \left(W^{\mu\nu\alpha\beta} W_{\mu\nu\alpha\beta} \right) \sqrt{g} dx^4$$

Conformal Gravity is interesting

It has been mentioned

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- It was also useful for constructing supergravity theories.
- It has recently emerged from the twistor string theory.
- It can have a rôle in AdS/CFT

Chern Simons Gravity

$2n+1$-dimensional transgression form

$$I_{2n+1} = (n+1)\int_{\mathcal{M}}\int_0^1 dt \left\langle (A_1 - A_0)\wedge \underbrace{F_t \wedge \ldots \wedge F_t}_{n} \right\rangle , \qquad (1)$$

where A_1 and A_0 are two (1-form) connections in the same fiber. $F_t = dA_t + A_t \wedge A_t$ with $A_t = tA_1 + (1-t)A_0$. $\langle\rangle$ stands for the trace in the group.

Chern Simons Gravity

The (Euler) Chern Simons density

Provided $A_0 = 0$ one gets Chern Simons action for A_1, or viceversa, in $d = 2n + 1$.

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The Chern Simons equation of motion

$$\langle F^n \delta A \rangle = 0$$

where $F = dA + A \wedge A$.

Chern Simons gauge theories are interesting

They are gauge theories

- different from YM

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- different from YM
- in a sense purely topological
- connected with gravitational theories in a non trivial or standard way
- full of surprises.

Rewritten as 1.5 formalism

In 3 dimensions conformal gravity

$$I_{CG} = \int_M w_i \wedge dw^i + \frac{2}{3} \varepsilon^{ijk} w_i \wedge w_j \wedge w_k \tag{2}$$

where $w_i = \varepsilon_{ijk}\, \omega^{kl}{}_{\mu} dx^{\mu}$ is the Levi Civita (spin) connection associated a given dreibein $e^i{}_{\mu} dx^{\mu}$.

The equations of motion

$$C_{\mu\nu\lambda} = \nabla_\mu \rho_{\nu\lambda} - \nabla_\lambda \rho_{\nu\mu} = 0, \tag{3}$$

equivalent to the vanishing of the Cotton-York tensor. Here

$$\rho_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu}, \tag{4}$$

with $\rho_{\mu\nu}$ is sometimes called the Schouten tensor or plainly *rho*-tensor.

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This means

the solution must a conformally flat space.

Rewritten as a gauge theory

This action principle $d = 3$

The previous action can be written in terms of a connection for conformal group in 3 dimensions ($CFT_3 \approx SO(3,2)$)

$$A_\mu = e^i{}_\mu P_i + w^i{}_\mu J_i + \lambda^i{}_\mu K_i + \phi_\mu D \ . \tag{5}$$

Written as a gauge theory

Provided

$$
\begin{aligned}
T^i &= de^i + \omega^i{}_j e^j = 0 \\
\lambda^i{}_\mu dx^\mu &= -\frac{1}{2} R^i{}_\mu dx^\mu = \rho^i \\
D\rho^i &= 0 \\
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The previous equations of motion can be rewritten as $F = dA + A \wedge A = 0$.

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Chern Simons on-shell

No surprise

The conformal gravity action can be written as 3d Chern Simons action

$$I_{CS} = \frac{k}{8\pi}\int_M \left\langle A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\rangle \quad (6)$$

for the conformal group.

The Tractor Connection arises

Conformal connection

In mathematical lore the connection for SO(3,2)≈ CFT_3

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Weyl Transformations

Weyl as Conformal

A Weyl transformation, $g_{ij} \to e^{2\xi} g_{ij}$, of A is

$$A \to e^{\xi(x)D} A e^{-\xi(x)D} + e^{\xi(x)D} d(e^{-\xi(x)D})$$

where $\xi(x)$ is an arbitrary function of the coordinates of the base space $\{x^{\mu}\}$.

Weyl Transformations

Weyl in components

The component of A transforms as

$$
\begin{aligned}
e^i &\rightarrow e^{\xi} e^i \\
\omega^{ij} &\rightarrow \omega^{ij} + \Upsilon^i e^j - \Upsilon^j e^i \\
\rho^i &\rightarrow e^{-\xi}(\rho^i + D\Upsilon^i + \Upsilon^i \Upsilon_\mu dx^\mu + e^i \Upsilon_\mu \Upsilon^\mu)
\end{aligned}
$$

with $\Upsilon_\mu = \partial_\mu \xi(x)$ and $\Upsilon^i = E^{i\mu}\Upsilon_\mu = E^{i\mu}\partial_\mu \xi(x)$

An AdS Gravity as Chern Simons

A theory of AdS gravity in $d = 3$

A Chern Simons theory for $\mathrm{AdS}_3 \approx \mathrm{SO}(2,2)$ written in terms of $A = \frac{1}{2}\omega^{AB} J_{AB}$ where J_{AB} (A,B=1...4) are the generator by

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1. splitting

$$A = \frac{1}{2}\hat{\omega}^{AB} J_{AB} = \frac{1}{2}\hat{\omega}^{ij} J_{ij} + \hat{q}^{i} J_{i\,4}, \quad (7)$$

where $i, j = 1, 2, 3$.

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where $i, j = 1, 2, 3$.
2. Next, identifying $\hat{q}^i$ and $\hat{\omega}^{ij}$ with $\hat{q}^i = l^{-1} e^i$, where e^i is a dreibein and $\hat{\omega}^{ij} = \omega^{ij}$ a Lorentz (spin) connection on the manifold to be considered.

Trace

How to take the trace

This is not a minor issue and most relevant results can be extract from the analysis of the different traces.

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This is not a minor issue and most relevant results can be extract from the analysis of the different traces. Nonetheless the trace can be defined as

$$\langle J_{A_1 A_2} J_{A_3 A_4} \rangle = \varepsilon_{A_1 \dots A_4} ,$$

which splits, throughout $A = (i, 4)$ with $i = 1 \dots 3$, as

$$\varepsilon_{A_1 \dots A_4} = \varepsilon_{i_1 i_2 i_3 4} = \varepsilon_{i_1 i_2 i_3} .$$

Rewritten action

Chern Simons action can be written as

$$I_{CS}^3 = l^{-1} \int \left(R^{ij} e^k + \frac{1}{3l^2} e^i e^j e^k \right) \varepsilon_{ijk} + BT$$

where $R^{ij} = d\omega^{ij} + \omega^i{}_k \omega^{kl}$ is called the curvature two form.

This is rather standard

This action is actually Einstein (Cartan) gravity in three dimensions.

Equation of Motion are

$F = 0$ means

$$
\begin{aligned}
F^{ij} &= R^{ij} + l^{-2} e^i \wedge e^j = 0 \\
F^{i4} &= T^i = de^i + \omega^i{}_k e^k = 0
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Solution

This allows only torsion free constant curvature manifolds, *i.e.*, AdS_3/Γ with $\Gamma \in AdS_3$.

We had one theory for two groups

Conformal

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Conformal is AdS somehow

It is quite appealing to try to connect both.

The previous can be generalized

A SO(2n,2) connection

Given $A = \frac{1}{2}\omega^{AB} J_{AB}$, where J_{AB} are the generator of $\mathrm{SO}(2n, 2)$

$$A = \frac{1}{2}\hat{\omega}^{AB} J_{AB} = \frac{1}{2}\hat{\omega}^{ab} J_{ab} + \hat{q}^{a} J_{a\,2n+2}, \tag{8}$$

where $a, b = 1 \ldots 2n + 1$.

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where $a, b = 1 \ldots 2n + 1$.

Traces

$$\langle J_{A_1 A_2} \ldots J_{A_{2n+1} A_{2n+2}} \rangle = \varepsilon_{A_1 \ldots A_{2n+2}} = \varepsilon_{a_1 \ldots a_{2n+1} 2n+2} \cdot,$$

This is the trace considered for the rest of this work.

Love-Chern-Simons

A more useful Chern Simons action

AdS-Chern-Simons gravity, module a boundary term, can be rewritten in the form of a Lovelock gravity as

$$\int \sum_{p=0}^{n} \frac{1}{2n-2p} \begin{pmatrix} n \\ p \end{pmatrix} \varepsilon_{a_1 \ldots a_{2n+1}} R^{a_1 a_2} \ldots R^{a_{2p-1} a_{2p}} q^{a_{2p+1}} \ldots q^{a_{2n+1}} \tag{9}$$

where $q^a = \omega^{a2n+1}$ and $R^{ab} = d\omega^{ab} + \omega^a{}_c \omega^{cb}$ with $a, b, c = 1, \ldots, 2n+1$.

More complex equations of motion

The new concept

En 2+1 dimensions $F = 0$ is a simple equation of motion, in higher odd dimensions this complicates. For instance in 5 the equation of motion is

$$F \wedge F = 0$$

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or for SO(4,2)

$$\begin{aligned} \mathcal{E}_f &= \varepsilon_{abcdf}\left(R^{ab} + q^a \wedge q^b\right) \wedge \left(R^{cd} + q^c \wedge q^d\right) = 0 \\ \mathcal{E}_{df} &= \varepsilon_{abcdf}\left(R^{ab} + q^a \wedge q^b\right) \wedge \left(dq^c + \omega^c{}_e \wedge q^e\right) = 0 \end{aligned}$$

A tractor like connection

AdS group in d dimensions

$$[J_{AB}, J_{CD}] = -\delta^{EF}_{AB}\delta^{GH}_{CD}\eta_{EG}J_{FH}, \tag{10}$$

with $A, B = 0 \dots d+1$.

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Conformal Group in $d-1$ dimensions

$$\begin{array}{ll} [M_{ij}, M_{kl}] = -\delta^{mn}_{ij}\delta^{op}_{kl}\eta_{mo}M_{np} & \\ [M_{ij}, P_k] = -(\eta_{ik}P_j - \eta_{jk}P_i) & [D, P_i] = P_i \\ [M_{ij}, K_k] = -(\eta_{ik}K_j - \eta_{jk}K_i) & [D, K_i] = -K_i \\ [P_i, K_j] = 2M_{ij} - 2\eta_{ij}D & [D, M_{ij}] = 0 \end{array} \tag{11}$$

with $i, j = 0 \dots d-1$.

The most provocative relation ever

AdS is Conformal provided

$$
\begin{array}{ll} J_{ij} = M_{ij} & J_{id-1} = \frac{1}{2}(P_i + K_i) \\ J_{d-1d} = D & J_{id} = \frac{1}{2}(P_i - K_i). \end{array} \tag{12}
$$

AdS tractor connection

Generalization

The d-dimensional tractor connection is

$$A = \frac{1}{2}\omega^{ij}J_{ij} + e^{i}P_{i} + \rho^{i}K_{i} \tag{13}$$

where ω^{ij} and e^{i} are a spin connection and a vielbein on the manifold considered.

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where ω^{ij} and e^i are a spin connection and a vielbein on the manifold considered. On the other hand,

$$\rho^i = e^i{}_\nu \rho^\nu{}_\mu dx^\mu$$

with $\rho^\mu{}_\nu$ is given by

$$\rho^\nu{}_\mu = \frac{1}{d-3}\left(R^\nu{}_\mu - \frac{1}{2(d-2)}\delta^\nu_\mu R\right) \tag{14}$$

AdS tractor connection

Some algebra

$$\begin{aligned} A &= \frac{1}{2}\omega^{ij} J_{ij} + \rho^i (J_{id-1} + J_{id}) + e^i (J_{id} - J_{id-1}) \\ &= \frac{1}{2}\omega^{ij} J_{ij} + (e^i - \rho^i) J_{id} + (e^i + \rho^i) J_{id-1} \end{aligned} \tag{15}$$

Conformal Gravity from Chern Simons

Add a dimension and wrap it

The idea is to show that a conformal theory of gravity can be written as a Chern Simons gauge theory with the help of an extension of tractor connection mentioned above.

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The idea is to show that a conformal theory of gravity can be written as a Chern Simons gauge theory with the help of an extension of tractor connection mentioned above.

This is not direct

A tractor connection for $SO(d-1,2)$ exist on a $d-1$ dimensions manifold while a $SO(d-1,2)$-CS density exist in $d=2n+1$ dimensions.

AdS tractor connection

Solution proposed

Dimensional reduction of a 2n+1-CS density on $\mathcal{M}' = \mathcal{M} \times S^1$ to produce an effective $2n$-dimensional theory.

AdS tractor connection

The generalization

On space $\mathcal{M}' = \mathcal{M} \times S^1$

$$A_{2n+1} = \frac{1}{2}\omega^{ij}(x^\mu)J_{ij} + e^i(x^\mu)P_i + \rho^i(x^\mu)K_i + \Phi(x^\mu)d\varphi D$$

where $i, j = 1, 2, \ldots 2n$ and a system of coordinates $X^M = (x^\mu, \varphi)$ has been considered on $\mathcal{M}'$ with φ parametrizing S^1.

AdS tractor connection

This is sound

The presence of $\Phi d\varphi$ along D does not changes the law of transformation under Weyl transformations. Furthermore $\Phi d\varphi$ transforms as

$$\Phi d\varphi \to \Phi d\varphi - d\xi.$$

This transformation has no effect on the CS action due to $d\xi$ has only projection on $\mathcal{M}$.

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This transformation has no effect on the CS action due to $d\xi$ has only projection on $\mathcal{M}$. This defines that Φ is actually a scalar field under Weyl transformations.

3d AdS 2d Conformal

A simple example is 3d to 2d

This leads to the identification

$$\begin{aligned}\hat{\omega}^{ij} &= \omega^{ij}\\ \hat{\omega}^{i3} &= \rho^i + e^i,\\ \hat{\omega}^{34} &= \Phi(x)d\varphi = q^3,\\ \hat{\omega}^{i4} &= e^i - \rho^i = q^i,\end{aligned}$$

This yields to the splitting of the three dimensional R^{ab} as

$$\begin{aligned}\hat{R}^{ij} &= R^{ij} - (\rho^i + e^i)(\rho^j + e^j) \text{ and}\\ \hat{R}^{i3} &= D(\rho^i - e^i),\end{aligned} \tag{16}$$

3d AdS 2d Conformal

A simple example en 3d to 2d

Finally the CS action given by

$$I_3 = \int_{\mathcal{M}'} \varepsilon_{abc} \left(\hat{R}^{ab} q^c + \frac{1}{3} q^a q^b q^c \right). \tag{17}$$

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becomes, upon integration along S^1,

$$I_3 = 2 \int_{\mathcal{M}} \Phi R \sqrt{g} d^2 x.$$

AdS tractor connection

Propeties $\rho_{\nu\beta}$

For $d > 3$ tensor $\rho_{\nu\beta}$ satisfies the relation

$$R_{\mu\nu\alpha\beta} = W_{\mu\nu\alpha\beta} + g_{\mu\alpha}\rho_{\nu\beta} - g_{\nu\alpha}\rho_{\mu\beta} - g_{\mu\alpha}\rho_{\nu\alpha} + g_{\nu\beta}\rho_{\mu\alpha}, \quad (18)$$

where $W_{\mu\nu\alpha\beta}$ is the Weyl tensor.

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where $W_{\mu\nu\alpha\beta}$ is the Weyl tensor.
This can be rewritten equivalently in differential forms formalism as

$$R^{ij} = \frac{1}{2}W^{ij}{}_{kl}e^k e^l - 2(e^i\rho^j - e^j\rho^i). \qquad (19)$$

2n+1 AdS 2n Conformal

Generically

With the identification

$$\begin{aligned} \hat{\omega}^{ij} &= \omega^{ij} \\ \hat{\omega}^{i\,2n+1} &= e^{i} + \rho^{i} \\ \hat{\omega}^{2n+1\,2n+2} &= \Phi(x) d\varphi = q^{2n+1} \\ \hat{\omega}^{i\,2n+2} &= e^{i} - \rho^{i} = q^{i}, \end{aligned} \tag{20}$$

with $i = 1, \ldots, 2n$.

2n+1 AdS 2n Conformal

Chern Simons

The CS action becomes

$$I_{CS}^{2n+1} = \int \varepsilon_{i_1 \ldots i_{2n}} \left(\left(R^{i_1 i_2} + 4\rho^{i_1} e^{i_2} \right) \ldots \left(R^{i_{2n-1} i_{2n}} + 4\rho^{i_{2n-1}} e^{i_{2n}} \right) \right) \Phi d\varphi$$

2n+1 AdS 2n Conformal

In terms of Weyl

The previous CS action, upon integration along S^1, becomes

$$I_{CS} = \int \Phi \delta^{j_1 \ldots j_{2n}}_{i_1 \ldots i_{2n}} \left(W^{i_1 i_2}_{j_1 j_2} \ldots W^{i_{2n-1} i_{2n}}_{j_{2n-1} j_{2n}} \right) |e| d^{2n} x. \tag{21}$$

2n+1 AdS 2n Conformal

To be noticed

- This is simpler than it seems as $W^{ij}{}_{ik} = 0$.
- This is very similar to the Euler density but where Riemann tensor has been replaced by Weyl tensor.

5 AdS and 4 Conformal

The usual conformal with a twist

$$I^4_{CS} = \int \Phi \left(W^{\mu\nu\alpha\beta} W_{\mu\nu\alpha\beta} \right) \sqrt{g} d^4 x$$

which is a generalization of the usual Weyl Gravity mentioned at the beginning.

Comments and Outlooks

Conclusion

Chern Simons theories can describe a simple generalization of Weyl Gravities.

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- These are mere the zero modes of the compactification. A lot to do.
- A higher spin version of this is calling on.