(Canonical) Effective Equations for QFT

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arxiv: 1208.1242
arxiv: 13xx.xxxx

September 12, 2013

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Why use Effective Equations?

- Correlation functions are calculated with an absolutely generalized initial state, as required for cosmology.
- Can answer some questions even while avoiding several technical difficulties like the exact structure of inner products on the Hilbert space, or the non-unique nature of self-adjoint extensions.
- Systematic way to realize higher derivative corrections in the equations of motion for a canonically quantized system.
- New perspective on known features of QFT, like renormalization, which may prove to be useful while quantizing with a dynamical background.
- Being canonical, applicable to certain models of LQG and LQC.

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$(0+1)$-dimensional Field Theory - Quantum Mechanics

The New Variables

[M. Bojowald and A. Skirzewski, 2006]

- Define expectation values, with respect to some state, as:

$$\tilde{G}^{a,n} := \langle (\hat{p} - \langle \hat{p} \rangle)^a (\hat{q} - \langle \hat{q} \rangle)^{n-a} \rangle_{\text{Weyl}} \tag{2.1}$$

- Begin with a Hamiltonian operator: $\widehat{H} = \widehat{H}(\hat{q}, \hat{p})$
  Take its expectation value with respect to the same state to define an 'effective' Quantum Hamiltonian

$$H_Q := \langle \widehat{H} \rangle = \left\langle \widehat{H}\left(\langle \hat{q} \rangle + (\hat{q} - \langle \hat{q} \rangle), \langle \hat{p} \rangle + (\hat{p} - \langle \hat{q} \rangle)\right) \right\rangle$$
$$= \sum_{n=0}^{\infty} \sum_{a=0}^{n} \frac{1}{n!} \binom{n}{a} \frac{\partial^n H(q,p)}{\partial p^a \partial q^{n-a}} \tilde{G}^{a,n} \tag{2.2}$$

- A point in this infinite dimensional space is completely specified by $(\langle \hat{q} \rangle, \langle \hat{p} \rangle, \tilde{G}^{a,n})$

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The Poisson Bracket

- Define Poission Bracket as:

$$\left\{\langle\hat{F}\rangle,\langle\hat{K}\rangle\right\}=\frac{1}{i\hbar}\left\langle[\hat{F},\hat{K}]\right\rangle \tag{2.3}$$

- Using (2.3), we have:

$$\left\{\langle\hat{q}\rangle,\langle\hat{p}\rangle\right\}=1$$
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$$\left\{\tilde{G}^{a,n},\tilde{G}^{b,m}\right\}=\sum_{r=0}^{\infty}\left[\left(\frac{\hbar}{2}\right)^{2r}K[a,b,m,n,r]\tilde{G}^{a+b-2r-1,m+n-4r-2}\right]$$
$$-b(n-a)\tilde{G}^{a,n-1}\tilde{G}^{b-1,m-1}+a(m-b)\tilde{G}^{b,m-1}\tilde{G}^{a-1,n-1}$$

where

$$K[a,b,m,n,r]=\sum_{0\leq f\leq 2r+1}(-)^{r+f}(f!(2r+1-f)!)^{-1}\binom{a}{f}\binom{n-a}{2r+1-f}\binom{b}{f}\binom{m-b}{2r+1-f}. \tag{2.4}$$

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$$K[a,b,m,n,r] = \sum_{0\le f\le 2r+1}(-)^{r+f}(f!(2r+1-f)!)^{-1}\tbinom{a}{f}\tbinom{n-a}{2r+1-f}\binom{b}{f}\binom{m-b}{2r+1-f}. \tag{2.4}$$

The Equations of Motion

Let $q := \langle \hat{q} \rangle$ and $p := \langle \hat{p} \rangle$.
The Hamilton's equations of motion gives us

$$\dot{q} = \{q, H_Q\} \tag{2.5}$$
$$\dot{p} = \{p, H_Q\} \tag{2.6}$$
$$\dot{\tilde{G}}^{a,n} = \{\tilde{G}^{a,n}, H_Q\} \tag{2.7}$$

Instead of solving the Schrödinger's partial differential equation, we have to solve this infinite set of coupled ordinary differential equations.

- The validity of the solutions to these equations of motion are subject to certain 'Uncertainty Relations', imposed on the moments.

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Example: A Quantum Anharmonic Oscillator

The Effective Quantum Hamiltonian

The Hamiltonian for an oscillator with a perturbation term is

$$\widehat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2 + \widehat{U}(\hat{q})$$

The corresponding 'effective' Quantum Hamiltonian is

$$H_Q = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2 + U(q) + \frac{\hbar\omega}{2}(G^{0,2} + G^{2,2}) + \sum_n \frac{1}{n!}(\hbar/m\omega)^{n/2} U^{(n)}(q) G^{0,n} \quad (2.8)$$

where $G^{a,n} = \hbar^{-n/2}(m\omega)^{n/2-a}\tilde{G}^{a,n}$ are now dimensionless quantities.

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The equations of motion generated by the effective Quantum Hamiltonian are:

$$\dot{q} = m^{-1}p$$

$$\dot{p} = -m\omega^2 q - U'(q) - \sum_n \frac{1}{n!}(m^{-1}\omega^{-1}\hbar)^{n/2} U^{(n+1)}(q) G^{0,n}$$

$$\begin{aligned}\dot{G}^{a,n} &= -a\omega G^{a-1,n} + (n-a)\omega G^{a+1,n} - \frac{aU''}{m\omega} G^{a-1,n} \\ &+ \frac{\sqrt{\hbar} aU'''(q)}{2(m\omega)^{\frac{3}{2}}} G^{a-1,n-1} G^{0,2} + \frac{\hbar aU''''(q)}{3!(m\omega)^2} G^{a-1,n-1} G^{0,3} \\ &- \frac{a}{2}\left(\frac{\sqrt{\hbar}U'''(q)}{(m\omega)^{\frac{3}{2}}} G^{a-1,n+1} + \frac{\hbar U''''(q)}{3(m\omega)^2} G^{a-1,n+2}\right) \\ &+ \frac{a(a-1)(a-2)}{3\cdot 2^3}\left(\frac{\sqrt{\hbar}U'''(q)}{(m\omega)^{\frac{3}{2}}} G^{a-3,n-3} + \frac{\hbar U''''(q)}{(m\omega)^2} G^{a-3,n-2}\right) + \cdots\end{aligned} \tag{2.9}$$

We need to make two approximations:

- Moments need to be solved perturbatively in $\left(\frac{\hbar}{L}\right)^{1/2}$. Here L is some angular momentum scale provided by the perturbing potential.
- Need to make an adiabatic approximation for the moments where we assume they are slowly varying with time but the evolution of q and p are free. Derivatives with respect to time in equations of motion are rescaled as $\frac{d}{dt} \to \lambda \frac{d}{dt}$. In the end, we shall set $\lambda = 1$

Thus, we can expand the moments as

$$G^{a,n} = \sum_e \sum_i G^{a,n}_{e,i} \left(\frac{\hbar}{L}\right)^{e/2} \lambda^i \tag{2.10}$$

At a given order in $\sqrt{\frac{\hbar}{L}}$, denoted by the index e, the adiabatic approximation gives

$$0 = \left\{ G^{a,n}_{e,0}, H_Q \right\} \tag{2.11}$$

to leading order, and

$$\dot{G}^{a,n}_{e,i} = \left\{ G^{a,n}_{e,i+1}, H_Q \right\} \tag{2.12}$$

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$O(\hbar^0, \lambda^0)$

The equation is:

$$0 = -a\omega G_{0,0}^{a-1,n} + (n-a)\omega G_{0,0}^{a+1,n} - \frac{U''(q)a}{m\omega} G_{0,0}^{a-1,n} \qquad (2.13)$$

subject to the constraint (coming from the first order equation) :

$$\frac{1}{\omega} \sum_{a \in \text{even}} \begin{pmatrix} n/2 \\ a/2 \end{pmatrix} \left(1 + \frac{U''(q)}{m\omega^2}\right)^{(n-a)/2} \dot{G}_0^{a,n} = 0 \qquad (2.14)$$

which gives the solution

$$G_{0,0}^{a,n} = \frac{(n-a)!a!}{2^n((n-a)/2)!(a/2)!} \left(1 + \frac{U''(q)}{m\omega^2}\right)^{(2a-n)/4} \qquad (2.15)$$

for even a and n, and $G_{0,0}^{a,n} = 0$ for odd a and/or n.

- The numerical constant chosen here is such that our expectation values are about the ground state of the harmonic oscillator.

$O(\hbar^0, \lambda^1)$

The solutions are:

$$G^{a,n}_{0,1} = 0 \quad \text{for odd } n$$
$$G^{a,n}_{0,1} = 0 \quad \text{for even } a \text{ and } n \text{ (once again to match with the ground state)}$$
$$G^{a,n}_{0,1} = C_{a,n}\frac{U'''(q)\dot{q}}{m\omega^3}\left(1+\frac{U''(q)}{m\omega^2}\right)^{\frac{2a-n-6}{4}} \quad \text{for odd } a \text{ and even } n$$

where $C_{a,n}$ is a dimensionless prefactor given by:

$$C_{a-1,n} = -\frac{(n-a)!(a-1)!}{2^{n+2}(\frac{n-a}{2})!(\frac{a}{2})!}(2a-n) - 2^{-n-2}\sum_{b=0}^{\frac{n-a-2}{2}}\left[\prod_{c=0}^{b}\frac{n-(a+2c)}{a+2c}\right]\frac{(n-a')!(a'-1)!}{(\frac{n-a'}{2})!(\frac{a'}{2})!}(2a'-n)$$

for even a, where $a' = a + 2(b+1)$.

$O(\hbar^1, \lambda^0)$

The solutions are:

$$G^{a,n}_{1,0} = 0 \text{ for odd } a$$
$$G^{a,n}_{1,0} = 0 \text{ for even } a \text{ and } n \text{ (vacuum state considerations)}$$
$$G^{a,n}_{1,0} = \sqrt{L}\, D_{a,n} \frac{U'''(q)}{m^{3/2}\omega^{5/2}} \left(1 + \frac{U''(q)}{m\omega^2}\right)^{\frac{2a-n-5}{4}} \quad \text{for even } a \text{ and odd } n$$

where

$$D_{a,n} = \begin{cases} \frac{(-1)^b \Gamma\left(\frac{n}{2}\right)}{12\pi(1-\frac{n}{2})_b}\left((n-1)b!\sqrt{\pi} + (n-8b-1)\Gamma\left(b+\frac{1}{2}\right)\right. \\ \left. -\sum_{c=0}^{b-2}(-1)^c(n-8(b-c-1)-1)\Gamma\left(b-c-\frac{1}{2}\right)(-b)_{c+1}\right) & \text{if } n \geq 5,\ b \geq 2 \\ \frac{n-1}{12\pi}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right) & \text{if } n \geq 3,\ b = 0 \\ \frac{3n-11}{12\pi(n-2)}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right) & \text{if } n \geq 3,\ b = 1 \end{cases}$$

is a dimensionless prefactor that depends on a and n. In the above expression, $b = (n-a-1)/2$ and $(x)_n = x(x+1)...(x+n-1)$ is the Pochhammer symbol.

Equation of motion for q is thus:

$$\ddot{q} = -\omega^2 q - U'(q)/m - \frac{\hbar}{2m^2\omega} U'''(q) \left[\sum_{\lambda=0}^{4} G^{0,2}_{0,\lambda} + \left(\frac{\hbar}{L}\right)^{1/2} \sum_{\lambda=0}^{4} G^{0,2}_{1,\lambda} \right]$$

where the relevant moments are

$$G^{0,2}_{0,0} = \frac{1}{2}\left(1 + \frac{U''(q)}{m\omega^2}\right)^{-\frac{1}{2}}$$

$$G^{0,2}_{0,2} = \frac{U'''(q)\ddot{q} + U''''(q)\dot{q}^2}{16m\omega^4}\left(1 + \frac{U''(q)}{m\omega^2}\right)^{-\frac{5}{2}} - \frac{5(U'''(q)\dot{q})^2}{64m^2\omega^6}\left(1 + \frac{U''(q)}{m\omega^2}\right)^{-\frac{7}{2}}$$

$$\begin{aligned}
G^{0,2}_{0,4} = &- \frac{U'''(q)\ddddot{q} + 4U''''(q)\dddot{q}\dot{q} + 3U''''(q)\ddot{q}^2 + 6U'''''(q)\dot{q}^2\ddot{q} + U''''''(q)\dot{q}^4}{64m\omega^6}\left(1 + \frac{U''(q)}{m\omega^2}\right)^{-\frac{7}{2}} \\
&+ \left[\frac{21(U''''(q)\dot{q}^2 + U'''(q)\ddot{q})^2}{256m^2\omega^8} + \frac{7(U'''(q)\dot{q})(U'''(q)\dddot{q} + 3U''''\ddot{q}\dot{q} + U'''''(q)\dot{q}^3)}{64m^2\omega^8}\right]\left(1 + \frac{U''(q)}{m\omega^2}\right)^{-\frac{9}{2}} \\
&- \frac{231(U'''(q)\ddot{q} + U''''(q)\dot{q}^2)(U'''\dot{q})^2}{512m^3\omega^{10}}\left(1 + \frac{U''(q)}{m\omega^2}\right)^{-\frac{11}{2}} \\
&+ \frac{1155(U'''(q)\dot{q})^4}{4096m^4\omega^{12}}\left(1 + \frac{U''(q)}{m\omega^2}\right)^{-\frac{13}{2}}
\end{aligned}$$

Equation of motion for q up to $\hbar^{3/2}$ and fourth adiabatic order

We may now rewrite the equation of motion as:

$$\begin{aligned}\ddot{q} &= -\omega^2 q - U'(q)/m \\ &\quad - \frac{\hbar}{2m^2\omega} U'''(q)\left[f(q,\dot{q}) + f_1(q,\dot{q})\ddot{q} + f_2(q)\ddot{q}^2 + f_3(q,\dot{q})\dddot{q} + f_4(q)\ddddot{q}\right] + \mathcal{O}(\hbar^2)\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}f(q,\dot{q}) &= \frac{1}{2}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-1/2} + \frac{U''''(q)\dot{q}^2}{16m\omega^4}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-5/2} - \frac{5(U'''(q))^2\dot{q}^2}{64m^2\omega^6}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-7/2} \\ &\quad - \frac{U''''''(q)\dot{q}^4}{64m\omega^6}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-7/2} + \frac{21(U''''(q))^2\dot{q}^4}{256m^2\omega^8}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-9/2} \\ &\quad + \frac{7U'''''(q)U'''(q)\dot{q}^4}{64m^2\omega^8}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-9/2} - \frac{231U''''(q)(U'''(q))^2\dot{q}^4}{512m^3\omega^{10}}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-11/2} \\ &\quad + \frac{1155(U'''(q))^4\dot{q}^4}{4096m^4\omega^{12}}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-13/2}\end{aligned} \tag{2.17}$$

$$
\begin{aligned}
f_1(q,\dot{q}) \quad &= \quad \frac{U'''(q)}{16m\omega^4}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-5/2} - \frac{3U'''''(q)\dot{q}^2}{32m\omega^6}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-7/2} \\
&+ \frac{63U''''(q)U'''(q)\dot{q}^2}{128m^2\omega^8}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-9/2} - \frac{231(U'''(q))^3\dot{q}^2}{512m^3\omega^{10}}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-11/2} \qquad (2.18)
\end{aligned}
$$

$$
\begin{aligned}
f_2(q) \quad &= \quad -\frac{3U''''(q)}{64m\omega^6}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-7/2} + \frac{21(U'''(q))^2}{256m^2\omega^8}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-9/2} \qquad (2.19) \\
f_3(q,\dot{q}) \quad &= \quad -\frac{U''''(q)\dot{q}}{16m\omega^6}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-7/2} + \frac{7(U'''(q))^2\dot{q}}{64m^2\omega^8}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-9/2} \qquad (2.20) \\
f_4(q) \quad &= \quad -\frac{U'''(q)}{64m\omega^6}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-7/2} \qquad (2.21)
\end{aligned}
$$

Coleman Weinberg Potential

CW Potential for a $(0+1)$-dimensional system

[S. Coleman and E. Weinberg, 1973]

For a given Lagrangian $L(q, \dot{q}, t) = \frac{1}{2}m\dot{q}^2 - V(q)$, with a vev defined by $\langle 0| q |0\rangle := q_0$, the Effective Coleman-Weinberg potential is given by

$$V_{\text{eff}}(q) = V(q_0) + \frac{\hbar}{2\sqrt{m}} \int \frac{\mathrm{d}k}{2\pi} \log\left(\frac{k^2 + V''(q_0)}{k^2}\right) + O(\hbar^2) \qquad (2.22)$$

This integral is obviously convergent and it gives:

$$V_{\text{eff}}(q) = V(q_0) + \frac{\hbar}{2\sqrt{m}} \sqrt{V''(q_0)} + O(\hbar^2) \qquad (2.23)$$

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CW Potential for an anharmonic oscillator

Recall

$$H_Q = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2 + U(q) + \frac{\hbar\omega}{2}(G^{0,2} + G^{2,2}) + \sum_n \frac{1}{n!}(\hbar/m\omega)^{n/2} U^{(n)}(q) G^{0,n}$$

Plugging in the value for the moments in H_Q, the Coleman-Weinberg effective potential up to two loop order turns out to be

$$V_{\text{eff}}(q) = \frac{1}{2}m\omega^2 q_0^2 + U(q_0) + \frac{\hbar\omega}{2}\left(1 + \frac{U''(q_0)}{m\omega^2}\right)^{1/2} + \frac{\hbar^2}{8m^2\omega^2}\left(1 + \frac{U''(q_0)}{m\omega^2}\right)^{-1} \times \left[\frac{U''''(q_0)}{4} + \frac{[U'''(q_0)]^2}{9m\omega^2}\left(1 + \frac{U''(q_0)}{m\omega^2}\right)^{-1}\right] + O(\hbar^3) \quad (2.24)$$

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CW Potential using moment expansion

For a general Hamiltonian of the form $H(p,q,t) = \frac{p^2}{2m} + V(q)$, using a moment expansion we get the effective Coleman-Weinberg potential up to two loop order is:

$$V_{\text{eff}}(q) = V(q_0) + \frac{\hbar}{2}\sqrt{\frac{V''(q_0)}{m}} + \frac{\hbar^2}{8mV''(q_0)}\left[\frac{V''''(q_0)}{4} + \frac{[V'''(q_0)]^2}{9V''(q_0)}\right] + O(\hbar^3)$$

We reproduce the one-loop order correction and also have the two-loop order correction to the Coleman-Weinberg potential with a canonical kinetic term.

Effective Equations for $(3+1)$-dimensional field theory

The Setup

- Use the 'in-in' formalism to get equal-time correlation functions
- The 'phi-fourth' Hamiltonian
  $\widehat{H} = \int \mathrm{d}^3x \left[\frac{\hat{\pi}^2(x)}{2} + \frac{m^2}{2}\hat{\phi}^2(x) + \frac{1}{2}\left(\nabla\hat{\phi}(x)\right)^2 + \lambda\hat{\phi}^4(x)\right]$

Define

$$G^{a,b}(x_1,\ldots,x_a;y_1,\ldots,y_b,t) := \Big\langle \left(\hat{\pi}(x_1,t) - \langle\hat{\pi}(x_1,t)\rangle\right)\ldots\left(\hat{\pi}(x_a,t) - \langle\hat{\pi}(x_a,t)\rangle\right) \times \left(\hat{\phi}(y_1,t) - \langle\hat{\phi}(y_1,t)\rangle\right)\ldots\left(\hat{\phi}(y_b,t) - \langle\hat{\phi}(y_b,t)\rangle\right)\Big\rangle_{\text{Weyl}} \tag{3.1}$$

$$\nabla_{x_i}\nabla_{y_j}\left[G^{a,b}(x_1,\ldots,x_a;y_1,\ldots,y_b,t)\right] := \Big\langle \left(\hat{\pi}(x_1,t) - \langle\hat{\pi}(x_1,t)\rangle\right)\ldots\nabla_{x_i}\left(\hat{\pi}(x_i,t) - \langle\hat{\pi}(x_i,t)\rangle\right)\ldots \times\left(\hat{\phi}(y_1,t) - \langle\hat{\phi}(y_1,t)\rangle\right)\ldots\nabla_{y_j}\left(\hat{\phi}(y_j,t) - \langle\hat{\phi}(y_j,t)\rangle\right)\ldots\Big\rangle_{\text{Weyl}} \tag{3.2}$$

With $\langle\hat{\pi}(x)\rangle := \pi(x)$ and $\langle\hat{\phi}(x)\rangle := \phi(x)$,

$$H_Q = \frac{1}{2}\int \mathrm{d}^3x \Big[\pi^2(x) + G^{2,0}(x,x) + m^2\left(\phi^2(x) + G^{0,2}(x,x)\right) + \nabla_x^2\left(G^{0,2}(x,x)\right) + \left(\nabla\phi(x)\right)^2 + 2\lambda\{\phi^4(x) + 6\phi^2(x)G^{0,2}(x,x) + 4\phi(x)G^{0,3}(x,x,x) + G^{0,4}(x,x,x,x)\}\Big] \quad (3.3)$$

The (equal time) Poisson Algebra is defined as:

$$\{\phi(x),\pi(y)\} := \frac{1}{i\hbar}\left\langle \left[\hat{\phi}(x),\hat{\pi}(x)\right] \right\rangle = \delta^3(x-y) \quad (3.4)$$

The equations of motion are derived as:

$$\frac{d}{dt}[\mathcal{O}] := \{H_Q,\mathcal{O}\} \quad (3.5)$$

EOM $(\langle\widehat{\phi}(y)\rangle$ and $\langle\widehat{\pi}(y)\rangle)$

$$\begin{aligned}\dot{\phi}(y,t) &= -\pi(y,t)\\ \dot{\pi}(y,t) &= (m^2-\nabla_y^2)\phi(y,t)\\ &\quad +4\lambda\phi^3(y,t)+12\lambda\phi(y,t)G^{0,2}(y,y,t)\\ &\quad +4\lambda G^{0,3}(y,y,y,t) \qquad (3.6)\end{aligned}$$

EOM (Second Order Moments)

$$\begin{aligned}
\dot{G}^{0,2}(y,z,t) &= -[G^{1,1}(y,z,t) + G^{1,1}(z,y,t)] \\
\dot{G}^{1,1}(y,z,t) &= -G^{2,0}(y,z,t) + [m^2 - \nabla_y^2]G^{0,2}(y,z,t) + 4\lambda G^{0,4}(y,y,y,z,t) \\
\dot{G}^{2,0}(y,z,t) &= [m^2 - \nabla_z^2]G^{1,1}(y,z,t) + [m^2 - \nabla_y^2]G^{1,1}(z,y,t) \\
&\quad + 4\lambda[G^{1,3}(y,z,z,z,t) + G^{1,3}(z,y,y,y,t)] \qquad (3.7)
\end{aligned}$$

EOM (Higher Order Moments)

The general scheme for equations of higher order moments

$$\begin{aligned}
\dot{G}^{0,n}(y_1,\ldots,y_n,t) &\sim G^{1,n-1}(y_1,\ldots,y_n,t)\\
\dot{G}^{1,n-1}(y_1,\ldots,y_n,t) &\sim G^{2,n-2}(y_1,\ldots,y_n,t) + G^{0,n}(y_1,\ldots,y_n,t)\\
&\quad + \lambda G^{0,n+2}(y_1,\ldots,y_n,t)\\
&\vdots\\
\dot{G}^{n-1,1}(y_1,\ldots,y_n,t) &\sim G^{n-2,2}(y_1,\ldots,y_n,t) + G^{n,0}(y_1,\ldots,y_n,t)\\
&\quad + \lambda G^{n-2,4}(y_1,\ldots,y_n,t)\\
\dot{G}^{n,0}(y_1,\ldots,y_n,t) &\sim G^{n-1,1}(y_1,\ldots,y_n,t) + \lambda G^{n-1,3}(y_1,\ldots,y_n,t) \qquad (3.8)
\end{aligned}$$

Solving these equations

- Expand the moments in powers of the coupling constant, $G^{a,b} = \sum_e \bar{\lambda} G_e^{a,b}$
- Solve for the moments in lower orders in $\bar{\lambda}$, starting with the free field solutions.
- Plug the (solved) lower order $\bar{\lambda}$ moments, in the equations containing higher order in $\bar{\lambda}$.
- In this way, perturbatively solve for the moments, which shall give us the required correlation functions.

Renormalizability

When is such a moment expansion valid?

- The moments are expanded in the dimensionless perturbation parameter $\bar{\lambda} = \lambda/L$, where L is some parameter.
- For ϕ^4 theory, $L = 1$ and for ϕ^3 theory, $L \propto m/\hbar$.
- For any other ϕ^n theory, with $n > 4$, then $L \propto \hbar^{(n-4)}$, thus rendering the moment expansion ill-defined.

Cancellation of the tadpole term

For ϕ^4 theory,

$$\begin{aligned}\ddot{\phi}(y,t) &= -(m^2 - \nabla_y^2)\phi(y,t) \\ &\quad + 4\lambda\phi^3(y,t) + 12\lambda\phi(y,t)G^{0,2}(y,y,t) \\ &\quad + 4\lambda G^{0,3}(y,y,y,t) \end{aligned} \tag{3.9}$$

In this case, $\phi(y,t) = 0$ is easily a solution up to any order since all odd moments (including $G^{0,3}(y_1,y_2,y_3,t)$) are zero up to any order.
For ϕ^3 theory,

$$\begin{aligned}\ddot{\phi}(y,t) &= -(m^2 - \nabla_y^2)\phi(y,t) \\ &\quad + 3\lambda\phi^2(y,t) + 3\lambda G^{0,2}(y,y,t) \end{aligned} \tag{3.10}$$

In order for $\phi(y,t) = 0$ to be a solution of this equation, we require an additional term (proportional to ϕ) in the Hamiltonian (or equivalently, Lagrangian) which will cancel off the $G^{0,2}(y,y,t)$ up to whichever order we want.

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The Propagator

The free propagator is calculated from the set of equations:

$$\begin{aligned}
\dot{G}^{0,2}(y,z,t) &= -[G^{1,1}(y,z,t) + G^{1,1}(z,y,t)] \\
\dot{G}^{1,1}(y,z,t) &= -G^{2,0}(y,z,t) + [m^2 - \nabla_y^2]G^{0,2}(y,z,t) \\
\dot{G}^{2,0}(y,z,t) &= [m^2 - \nabla_z^2]G^{1,1}(y,z,t) + [m^2 - \nabla_y^2]G^{1,1}(z,y,t) \quad (3.11)
\end{aligned}$$

With the condition $\phi(y,t) = 0$, the propagator is $G^{0,2}(y,z,t)$. The most general solution turns out to be:

$$G^{0,2}(y,z,t) = \int \frac{\mathrm{d}^3 k_y}{(2\pi)^3} \int \frac{\mathrm{d}^3 k_z}{(2\pi)^3} \left[f(k_y,k_z) e^{i(\vec{k}_y.\vec{y}+\vec{k}_z.\vec{z}-\omega t)} + f^*(k_y,k_z) e^{-i(\vec{k}_y.\vec{y}+\vec{k}_z.\vec{z}-\omega t)} \right] \quad (3.12)$$

where $\omega = \omega_y - \omega_z$, with $\omega_i = \sqrt{m^2 + k_i^2}$.
The two arbitrary functions $f(k_y,k_z)$ & $f^*(k_y,k_z)$ are fixed by the initial value of the (second-order) moments, thus capturing the arbitrariness of the initial state.

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For a particular initial value of the moments, given by

$$G^{0,2}(y,z,0) = \frac{1}{2\pi^3}\int \frac{\mathrm{d}^3 k}{2\sqrt{k^2+m^2}} e^{i\vec{k}.(\vec{y}-\vec{z})} \quad \text{and}$$
$$\dot{G}^{0,2}(y,z,0) = 0 \qquad (3.13)$$

we reproduce the usual result from QFT, that is,

$$G^{0,2}(y,z,t) = \int \frac{\mathrm{d}^3 k}{2(2\pi^3)\sqrt{k^2+m^2}} e^{i\vec{k}.(\vec{y}-\vec{z})} \qquad (3.14)$$

The unique factorization of $\omega = \omega_y - \omega_z$ is why the two results (rightly) match up.

- The propagator has been calculated to agree up to one loop order with QFT.

Important lessons and looking ahead

So, why Effective Equations?

- Using these canonical techniques for effective action, we recover the usual QFT results and also extend them, for instance, by including more general states.
- There is well defined systematic way to derive the higher derivative corrections while avoiding some technical difficulties.

Where are these useful?

- Currently being applied to certain models of isotropic, homogeneous cosmology and also to a de Sitter background.
- Current work is underway to include (perturbative) quantum corrections in the Scalar and Diffeomorphism constraints of spherical LQG, and see what effects they have on the hyperspace deformation algebra. In the high curvature regime, these might be of the same order as that of other non-perturbative corrections (like holonomy corrections), and hence they should be included for a full analysis.

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Hypersurface Deformation Algebra

Generically,

$$\{D[N^i], D[M^J]\} = D[\mathcal{L}_{M^j} N^i] \quad (5.1)$$
$$\{H[N], D[N^i]\} = -H[\mathcal{L}_{N^i} N] \quad (5.2)$$
$$\{H[N], H[M]\} = D[g^{ij}(N\partial_j M - M\partial_j N)] \quad (5.3)$$

For spherically symmetric LQG, the specific form of the constraints (after solving for the Gauss constant) is given by

$$H[N] = \frac{1}{2G}\int \mathrm{d}x\, N|E^x|^{-1/2}\left[K_\phi^2 E^\phi + 2K_\phi K_x E^x + (1-\Gamma_\phi^2)E^\phi + 2\Gamma'_\phi E^x\right]$$
$$D[N^x] = \frac{1}{2G}\int \mathrm{d}x\, N^x\left[2E^\phi K'_\phi - K_x E'^x\right]$$

where $\Gamma_\phi = -E'^x/(2E^\phi)$.

Hypersurface Deformation Algebra

Generically,

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- [M. Bojowald and J. Reyes, 2009; M. Bojowald and G. Pailey, 2012] Work has been done to see what kind of correction functions are required to include Holonomy-corrections and Inverse-Triad corrections in these constraints and how does that affect the Hypersurface Deformation Algebra.
- [M. Bojowald and S.B., *forthcoming*] Recent results show we do not need to modify the Hypersurface Deformation Algebra if we include only the higher derivative terms, given by these moment expansions (as is expected from previous results on higher curvature gravity).
- Aim is to unify non-perturbative corrections from LQG with these perturbative corrections and look at resulting effects on the Hypersurface Deformation Algebra, at least in the effective framework.

Isotropic and Homogeneous Cosmology

Starting with the Einstein-Hilbert action (with the FLRW metric), including a cosmological constant and matter, we can write the Friedmann Equation as (setting $\frac{8\pi G}{3} = 1$):

$$\frac{1}{4}\frac{p_a^2}{a^4} + \frac{k}{a^2} - \frac{\Lambda}{3} = \rho \tag{5.4}$$

where a is the scale factor, Λ is the cosmological constant, ρ is the energy density, $p_a = -\frac{2a\dot{a}}{N}$ is the momentum canonically conjugate to a and N is the usual lapse function.

For a closed universe $(k = 1)$ and the radiation-dominated era $\rho = \frac{p_t}{a^4}$, we have a Hamiltonian which generates evolution with respect to some time co-ordinate t, related to the proper time τ as $t = \int_0^\tau a(\tau')^{-1} d\tau'$, given by:

$$H = p_t = \frac{1}{4}p_a^2 + a^2 - \frac{\Lambda}{3}a^4 \tag{5.5}$$

So we have an Anharmonic Oscillator Hamiltonian with $m = 2$, $\omega = 1$ and $U(a) = -\frac{\Lambda}{3}a^4$.

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Results

- The quantum corrections to the scale factor prevents it from going back to zero where the classical solution did!!! Although the quantum corrections are small usually, they play a significant role when the classical solution goes to zero. This result indicates that the scale factor may be saved (or, at least, delayed) from going back to the singular point in the presence of quantum corrections.

- The acceleration from the classical solution is negative for the first half cycle (as expected during the radiation-dominated era). However the acceleration for the overall scale factor (including quantum corrections) turns positive at some points in this period. This also indicates that this positive acceleration, coming from the quantum corrections, may drive the scale factor away from zero!!

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