RÉNYI ENTROPY ON ROUND SPHERES:
holographic and q-analog recipies

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Aim of the talk

- Re-derive few known results for entanglement and Rényi entropies for free conformal fields on round spheres
  (only in 1+1 overlaps with Ryu+Takayanagi's holographic recipe)

- Focus on universal (UV-cutoff independent, conformal invariant) log-term in even dimensions

$$S_{EE/RE} \sim \textit{coeff} \times \ln \epsilon$$

- Novelty: alternative holographic way to compute on the gravity side and use AdS/CFT-Conformal Geometry 'dictionary'

High hopes: holography in AdS (vulgarized)

1. anything *Conformal*$_d$ $\Rightarrow$ something in AdS_{d+1}
2. and if lucky, *AdS* side more tractable

Several celebrated examples

- Dirac'36, "Wave equations in conformal space"
  conformal group in Minkowski$_d$ $\Leftrightarrow$ isometries of $(A)dS_{d+1}$
- Fefferman-Graham'85, "Conformal invariants"
  conformal manifold $\mathcal{M}_d$ $\Leftrightarrow$ conformal infinity of Poincaré metric
- Maldacena'97, 16 years of AdS/CFT correspondence
  conformal field theory $\Leftrightarrow$ string/M-theory, (quantum) gravity

Outline

1. Entanglement entropy of free CFTs
2. Holographic formula and type-A trace anomaly
3. Rényi entropy (take I)
4. Rényi entropy (take II)
5. Summary and outlook

 entanglement entropy of free CFTs

Entanglement: measure of "quantum-ness"

- then: EPR paradox and Schrödinger's cat '35, Bell inequalities '64, ...
- now: quantum computation, quantum information processing, ...

Entanglement or geometric entropy (EE): von Neumann entropy of reduced density matrix

- $\rho_A = tr_B\{\rho_{A\oplus B}\}$
- $S_{EE} = -tr_A\{\rho_A \ln \rho_A\}$

EE and black hole entropy: Sorkin'84, Bombelli+Koul+Lee+Sorkin'86

- Massless scalar, $S_{EE} \sim \frac{\textit{Area of event horizon}}{\epsilon^{d-2}}$

 entanglement entropy of free CFTs

Numerical recipies: massless scalar, sphere in *flat space* (no bh!)

- Srednicki'93: $S_{EE} \sim \frac{Area}{\epsilon^2}$
- Lohmayer+Neuberger+Schwimmer+Theisen'10: $S_{EE} \sim \frac{Area}{\epsilon^2} + \frac{1}{90} \ln \epsilon$

Type-A trace anomaly coefficient:

$\langle T \rangle \sim a_d \cdot E_d$ det (conformal Laplacian)

- Casini+Huerta'10: modular hamiltonian and thermal entropy
- Dowker'10: heat kernel on the d-lune ($S^d / Cyclic_q$)
- Solodukhin'10: entropy of extreme black holes $\leftrightarrow$ brick wall

d	2	4	6	8	10
coeff.	$-\frac{1}{3}$	$\frac{1}{90}$	$-\frac{1}{756}$	$\frac{23}{113400}$	$-\frac{263}{7484400}$

Connection trace-anomaly with log-term, by now is well established (Solodukhin,Myers,...) but this offers no direct way to compute it.

holographic formula and type-A trace anomaly

Our favorite entry of AdS/CFT dictionary:

$$\frac{\det_-(\Delta_X + m^2)}{\det_+(\Delta_X + m^2)} = \det \langle O_\lambda O_\lambda \rangle_M$$

- bulk scalar: $\phi \leftrightsquigarrow O_\lambda$: operator at boundary $\quad m^2 = \lambda(\lambda - d)$
- Δ_X: Laplacian on X (ALAdS)
- M: compact conformal infinity
- $+$: computed with resolvent/Green function in Euclidean signature
- $-$: analytic continuation to the smaller root $\lambda \to d - \lambda$ (alternate b.c.)

many people:Gubser+Mitra,Gubser+Klebanov,Hartman+Rastelli,Dorn,Aros,DD
Patterson+Perry,Guillarmou

holographic formula and type-A trace anomaly

Heuristic derivation via AdS/CFT :

$$Z_{gravity} = Z_{CFT}$$

- Quantum (1-loop) contribution of a bulk scalar field to the partition function (Gubser+Mitra'02)
- RG flow triggered by a double-trace deformation (Gubser+Klebanov'02)
- A 'kinematical' understanding, based on Gelfand-Yaglom formula (Hartman+Rastelli'06)
- Full agreement: anomaly and renormalized determinants in dimensional regularization (Dorn+DD'07)

$$\frac{\det_-(\Delta_H + m^2)}{\det_+(\Delta_H + m^2)} = \det \langle O_\lambda O_\lambda \rangle_S$$

A plausibility argument

- det ... = exp tr ln ... : coincidence limit and volume integral

$$\frac{e^{\infty_{ir} \cdot \infty_{uv}^-}}{e^{\infty_{ir} \cdot \infty_{uv}^+}} = e^{\infty_{uv}}$$

- $+/-$ have same short-distance behavior

$$e^{\infty_{ir} \cdot (\infty_{uv}^- - \infty_{uv}^+)} = e^{\infty_{ir}} = e^{\infty_{uv}}$$

- IR/UV-connection (Susskind+Witten): ∞_{ir} and ∞_{uv} are comparable, same degree of divergence!

Momentary lapse of reason: narrow bridge to conformal geometry

Holographic life of det (conformal Laplacian): conformal-tour from $S \times H^{d-1}$ to S^d

"Resonant masses" $\lambda \to d/2 + k$

$$\langle O_\lambda O_\lambda \rangle \Rightarrow \text{GJMS Laplacians:} P_{2k} = \Delta^k + lot$$

GJMS: conformal geometers (Graham,Jenne,Mason,Sparling'92)

$k = 1 \to$ conformal Laplacian (or Yamabe)
$k = 2 \to$ Paneitz operator Paneitz'83 but actually Fradkin+Tseytlin'82

Coming back to life: holographic formula and type-A trace anomaly

Now, turn to the bulk and compute for these resonant values:

- Polynomial in k times volume of $EAdS_{d+1} = H^{d+1}$
- In ϵ from volume anomaly (Henningson+Skenderis'98,Graham'99)
- heat kernel

$$anom.coeff.\,GJMS \sim \sum a_n \times \Gamma(\frac{d-2n-1}{2}) \times k^{2n+1}$$

- Green function leads to a neat formula, unexpected simple result (Dowker'10, DD'08)

$$\boxed{anom.coeff.\,GJMS \sim \int_0^k d\nu\ (\nu)_{d/2} \cdot (-\nu)_{d/2}}$$

(curiosity? for later use!)Casini+Huerta'10 by a formal recipe: in fact Cappelli+D'Appollonio'00 in disguise with Riemann zeta instead of Bernoulli numbers

$$\frac{d}{dz}\left(z\int_0^z d\nu\ (\nu)_{\frac{d-2}{2}} \cdot (-\nu)_{\frac{d-2}{2}}\right) \qquad \text{and} \quad z^l \rightarrow \zeta_R(-l)$$

A saucerful of secrets: q-entropies

Shannon $\rightarrow$ Rényi

$$S_q^{(R)} = \frac{1}{1-q} \ln tr\rho^q \qquad S_1^{(R)} = S_{EE}$$

Rényi entropy

- Casini+Huerta'10: replica trick (Riemann surface, 1/q sheets)
- Dowker'10: heat kernel on the d-lune ($S^d / Cyclic_q$)
- Solodukhin'10: conical defect $\theta \sim \theta + 2\pi q$
- also related works by Myers et al., Fursaev, Headrick, Takayanagi et al.

Mutatis mutandis (take I)

- boundary $M = S^1 \times H^d$ with different radii, bulk $X = H^{d+2}$ with conical singularity
- use Sommerfeld formula to work out the heat kernel
- holography fine! ... but yet tedious computation (unless you MAPLE it)

Mutatis mutandis (take II)

EE:

$$\frac{d}{dz}\left(z\int_0^z d\nu\;(\nu)_{d/2}\cdot(-\nu)_{d/2}\right) \qquad \text{and} \quad z^l \to \zeta_R(-l)$$

RE: q-derivative

$$\left[\frac{d}{dz}\right]_q \left(z\int_0^z d\nu\;(\nu)_{d/2}\cdot(-\nu)_{d/2}\right) \qquad \text{and} \quad z^l \to \zeta_R(-l)$$

Example, $d=2$:

$$S_q = \frac{1}{6}\cdot(1+q)\cdot\ln\frac{\ell}{\epsilon}$$

A 'gracious' result given that Rényi entropies (and Tsallis entropy!) are connected with entanglement entropy via a q-derivative.

Summary and outlook

Hope to shed some light on holographic approaches to EE and RE, but have to keep walking...
Several open issues:

- geometry of entangling surface and type-B trace anomaly (Myers et al.)
- finite contributions (Klebanov et al.) $\Leftrightarrow$ q-deformed Patterson-Selberg zeta (Floyd L. Williams)
- c-theorem, candidate for odd dimensions (Ryu+Takayanagi,Myers+Sinha,Casini+Huerta,...)
- weak form $a_{UV} > a_{IR}$ in terms of 'geometric' inequalities (Branson, Oersted, Beckner, Okikiolu,...)

keep walking ...

MANY THANKS!!!

Universal scaling at one-dimensional (d=1) conformal critical points

(Holzhey+Larsen+Wilczek'94: probably the most ubiquitous formula in last decade's literature)

$$S_{EE} = \frac{c}{3} \cdot \ln \frac{\ell}{\epsilon}$$

Holographic recipe at work (Ryu+Takayanagi'06):
length of the geodesic in AdS_3 "homologous" to a segment of length ℓ (entangling "surface") at the boundary and divide by four times Newton's constant G_N

- Regularized geodesic length: $L \ \ln \frac{\ell^2}{\epsilon^2}$
- Need entry of AdS/CFT dictionary : central charge of the algebra of asymptotic diffs (PBH) in AdS_3 (Brown+Henneaux'86)

$$c = \frac{3\,L}{2\,G_N}$$