Expansion Method, its general properties and some applications

Workshop on Quantum Gravity in the Southern Cone VI
Maresias, São Paulo, Brasil - September 2013

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*arXiv:1308.4832: c*olaboration with L. Andrianopoli,
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1) A Lie group G, its asociated algebra $\mathcal{G}$.

2) A G **invariant polynomial** of order n is a multilineal map

$$\langle ... \rangle : \mathcal{G}^n \to \mathbb{C}$$

which satisfy the **invariant condition**

$$\forall \mathbf{T}_A \in \mathcal{G}, g \in G, \langle \mathbf{T}_1, ..., \mathbf{T}_n \rangle = \left\langle g\mathbf{T}_1 g^{-1}, ..., g\mathbf{T}_n g^{-1} \right\rangle$$

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3) A gauge connection one form $\mathbf{A} = A^A \mathbf{T}_A$ and its asociated strength field $\mathbf{F} = d\mathbf{A} + \mathbf{A} \wedge \mathbf{A}$

Examples:

Yang-Mills Theories:

$$\mathcal{L}_{\mathrm{YM}}^{(d)}(\boldsymbol{A}) = \langle \boldsymbol{F} \wedge * \boldsymbol{F} \rangle ,$$
$$= F^{A} \wedge * F^{B} \langle \boldsymbol{T}_{A}, \boldsymbol{T}_{B} \rangle$$

or a Chern-Simons Theories in $d = 3$:

$$\mathcal{L}_{\mathrm{CS}}^{(3)}(\boldsymbol{A}) = \left\langle \boldsymbol{A} \wedge \left(\mathrm{d}\boldsymbol{A} + \frac{2}{3} \boldsymbol{A}^{\wedge 2} \right) \right\rangle$$
$$= A^{A} \wedge \left(\mathrm{d}A^{B} + \frac{1}{3} C_{MN}{}^{B} A^{M} \wedge A^{N} \right) \langle \boldsymbol{T}_{A}, \boldsymbol{T}_{B} \rangle$$

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Non trivial relations between **groups** and **Lie algebras**

↔

Non trivial relations between **physical theories** wich are invariants under that symmetries

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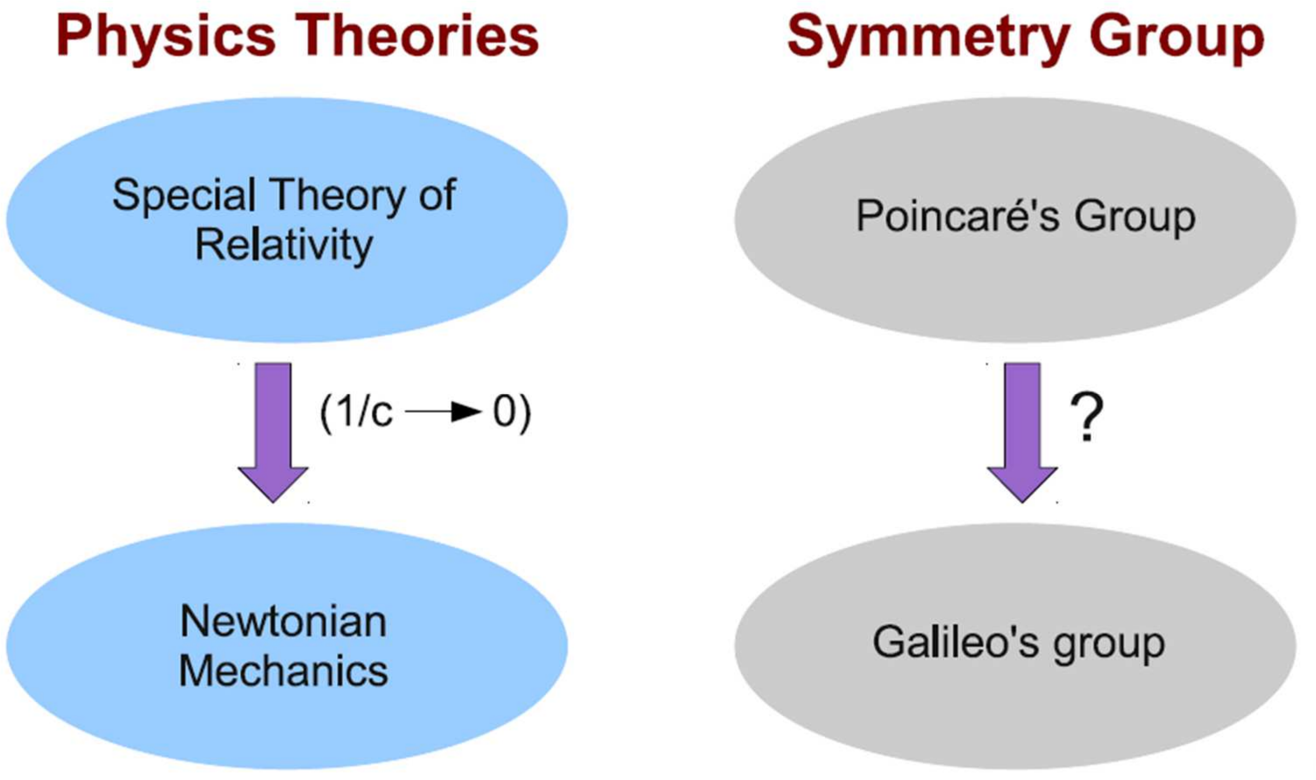
Non trivial relations between **physical theories** wich are invariants under that symmetries

First example:
I.E. Segal
(1951)



During the second half of the 20th century appeared in the literature:

Mechanism	**Example**		
1. Contraction	(Anti)de-Sitter	$\longrightarrow$	Poincaré
2. Deformation	Galileo	$\longrightarrow$	Poincaré
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We focus on its generalizations (particularly one called **S-expansion**)

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Ex: *ISO(3,1)* from *SO(3,2)*

$$[M_{ab}, M_{cd}] = g_{ac}M_{bd} - g_{bc}M_{ad} - g_{ad}M_{bc} + g_{bd}M_{ac}$$
$$a, c = 1, \ldots 5$$

Defining $M_{5\mu} = RP_\mu$ with $\mu, \nu = 1, \ldots, 4$ it leads to

$$[P_\mu, P_\nu] = \frac{1}{R^2}[M_{5\mu}, M_{5\nu}] = \frac{1}{R^2} g_{55} M_{\mu\nu}$$
$$[M_{\mu\nu}, P_\rho] = \frac{1}{R}[M_{\mu\nu}, M_{5\rho}] = \frac{1}{R}(-g_{\mu\rho}M_{\nu 5} + g_{\nu\rho}M_{\mu 5})$$
$$= \frac{1}{R}(-g_{\mu\rho}RP_\nu + g_{\nu\rho}RP_\mu) = -g_{\mu\rho}P_\nu + g_{\nu\rho}P_\mu$$

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$$[P_\mu, P_\nu] = \frac{1}{R^2}[M_{5\mu}, M_{5\nu}] = \frac{1}{R^2}g_{55}M_{\mu\nu} \rightarrow 0$$

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Expanded algebras with higher dimension

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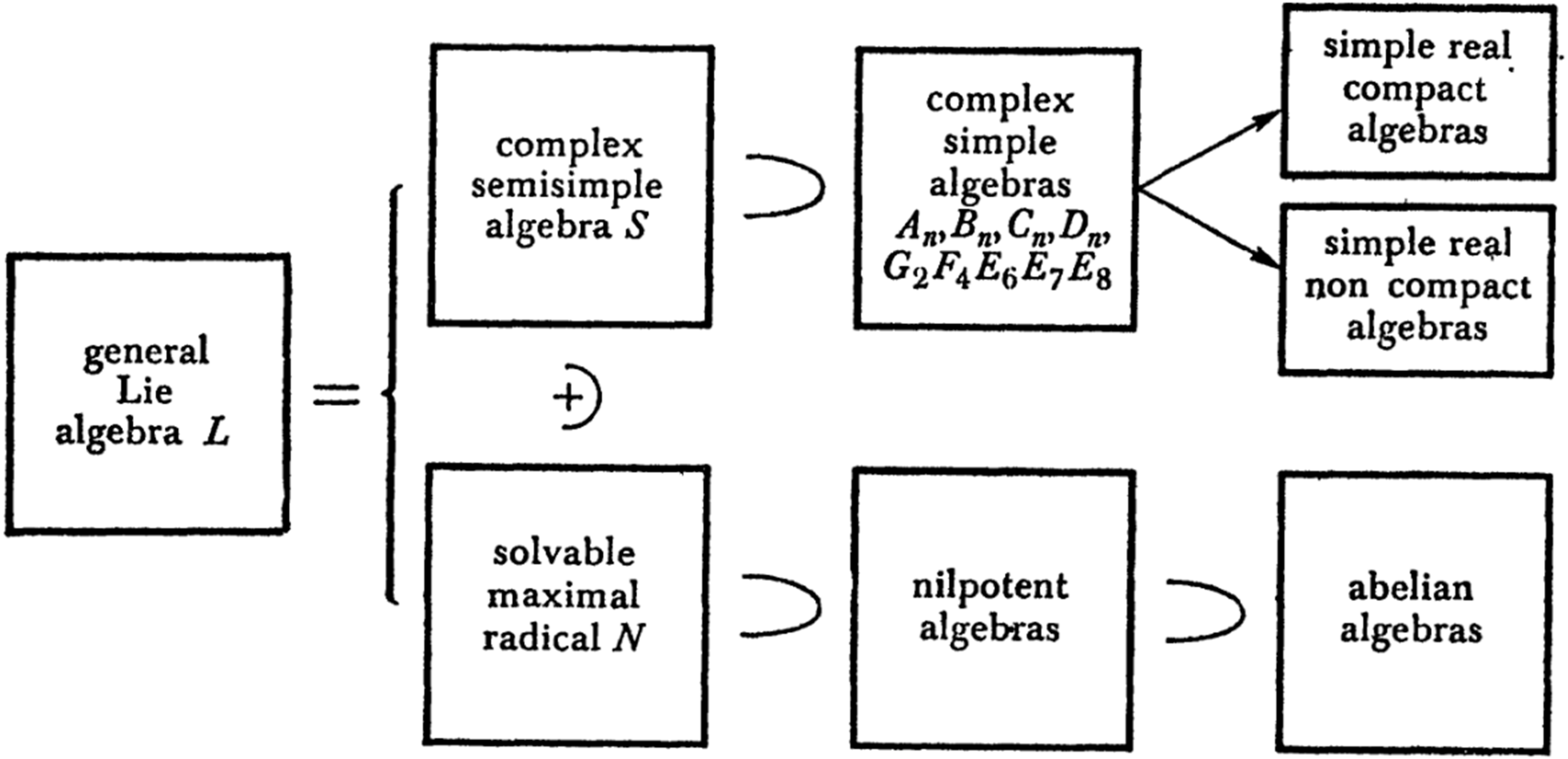
Expanded algebras with higher dimension

Classification of Lie algebras



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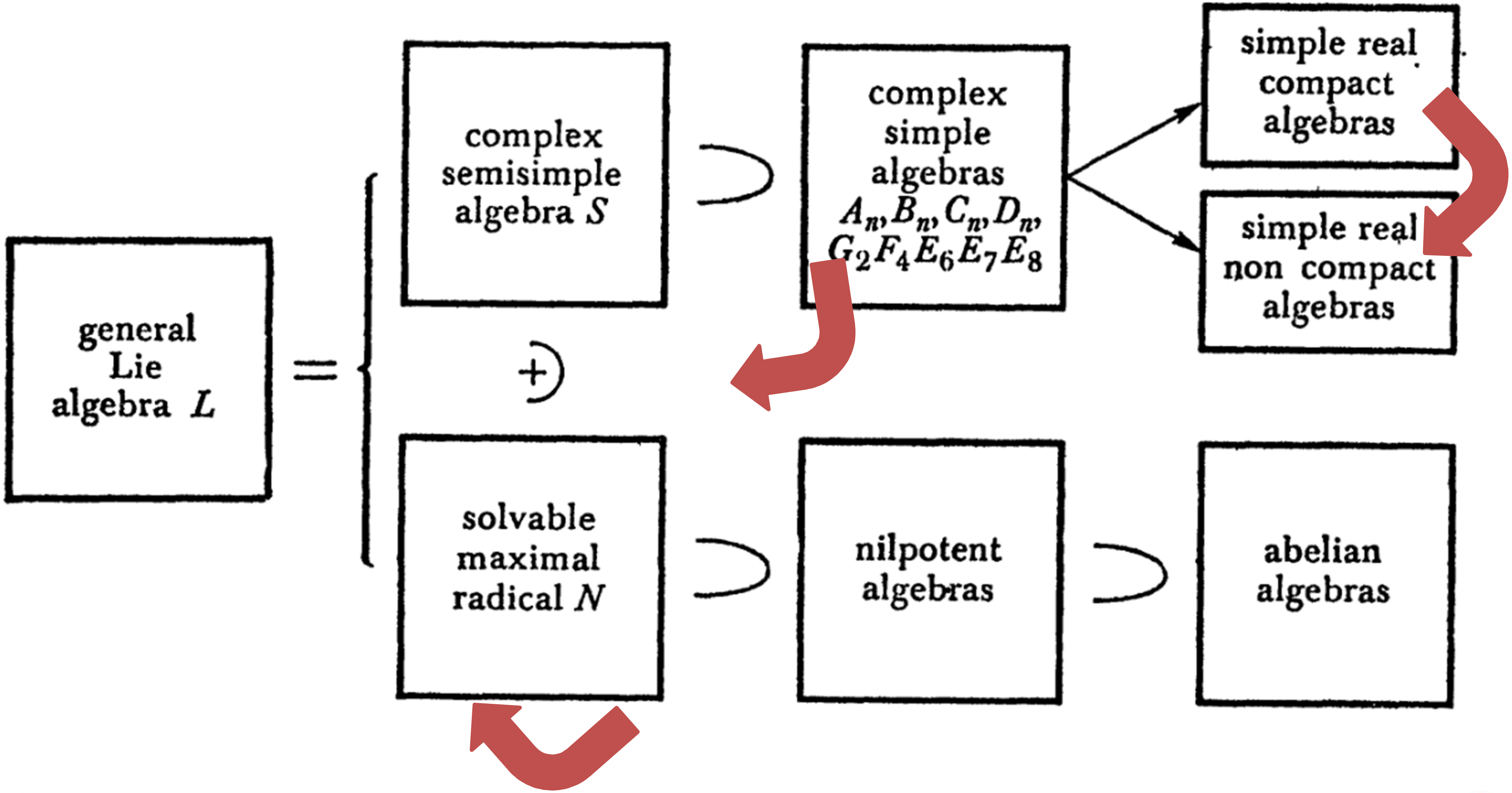

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→ This motivates the following question:

Given two algebras

A and B

Can these algebras be related by some contraction or expansion method?

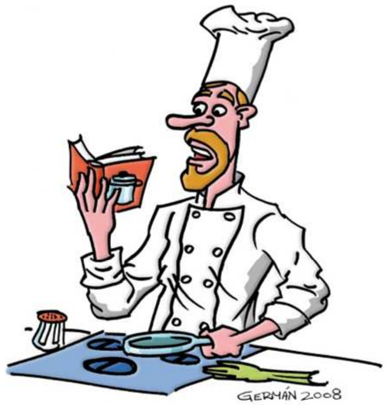
Contents

S-expansion method

Here we review the main characteristics of the method.
For further details, see the original article:

[1] F. Izaurieta, E. Rodríguez, P. Salgado, J.Math.Phys.47:123512,2006
arXiv: 0606215

S-expansion method



Ingredients:

1) A Lie algebra $\mathcal{G}$ with basis $\{\mathbf{T}_A\}_{A=1}^{\dim \mathcal{G}}$

2) A finite abelian semigroup $S = \{\lambda_\alpha\}_{\alpha=1}^{n}$
(there is a compostion law $\lambda_\alpha \lambda_\beta$ which is closed and asociative)

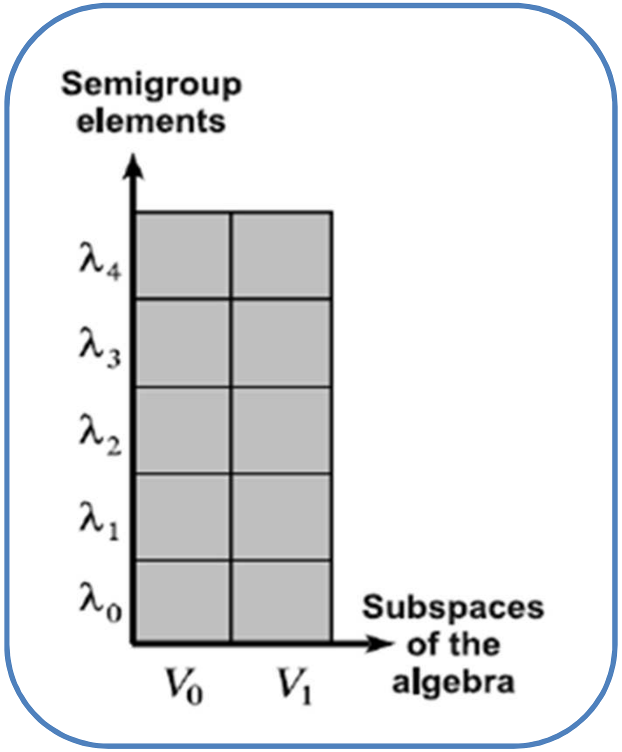
S-expansion method

Steps:

I) Construction of $\mathcal{G}_S = S \otimes \mathcal{G}$ with basis $\left\{\mathbf{T}_{(A,\alpha)} = \lambda_\alpha \otimes \mathbf{T}_A\right\}$ and define the induced Lie product:

$$\left[\mathbf{T}_{(A,\alpha)}, \mathbf{T}_{(B,\beta)}\right] = \lambda_\alpha \lambda_\beta \otimes \left[\mathbf{T}_A, \mathbf{T}_B\right]$$

First result: $\mathcal{G}_S = S \otimes \mathcal{G}$ is a Lie algebra, called the **expanded algebra**



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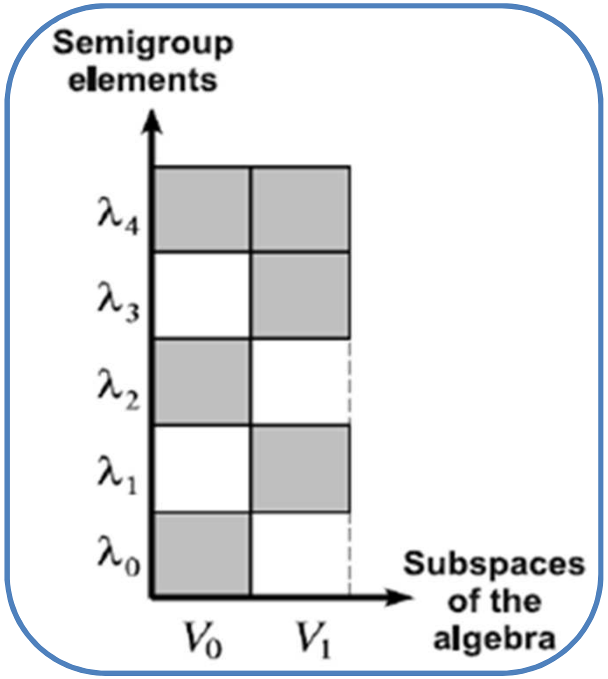


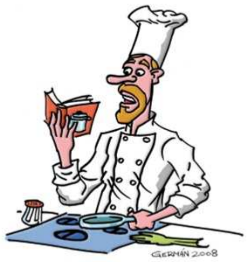
II) Supose that $\mathcal{G} = \bigoplus_{p\in I} V_p$, $S = \bigcup_{p\in I} S_p$
satisfying respectively:

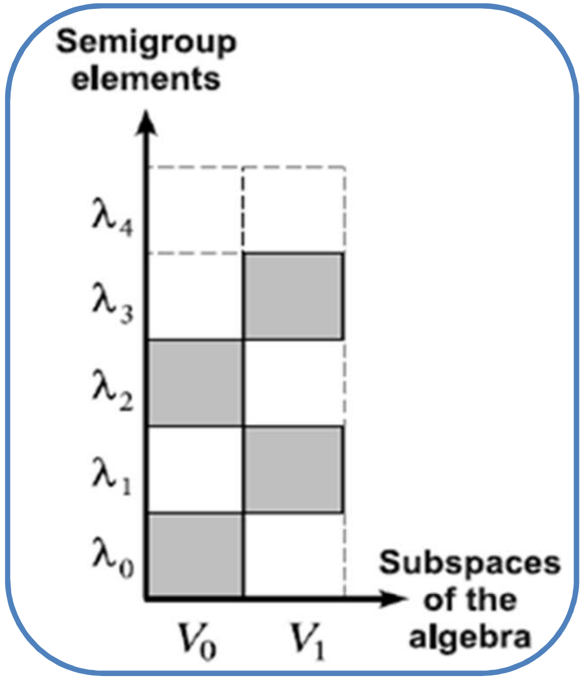
$$[V_p, V_q] \subset \bigoplus_{r\in i(p,q)} V_r \quad \text{and} \quad S_p \times S_q \subset \bigcup_{r\in i(p,q)} V_r$$

Perform the following construction $\mathcal{G}_{S,R} = \bigoplus_{p\in I} S_p \otimes V_p$

Second result: $\mathcal{G}_{S,R}$ is a subalgebra called the **resonant subalgebra**

S-expansion method





III) Supose that there is an element 0_S wich satisfies $0_S\lambda_\alpha = 0_S \ \forall\lambda_\alpha \in S$

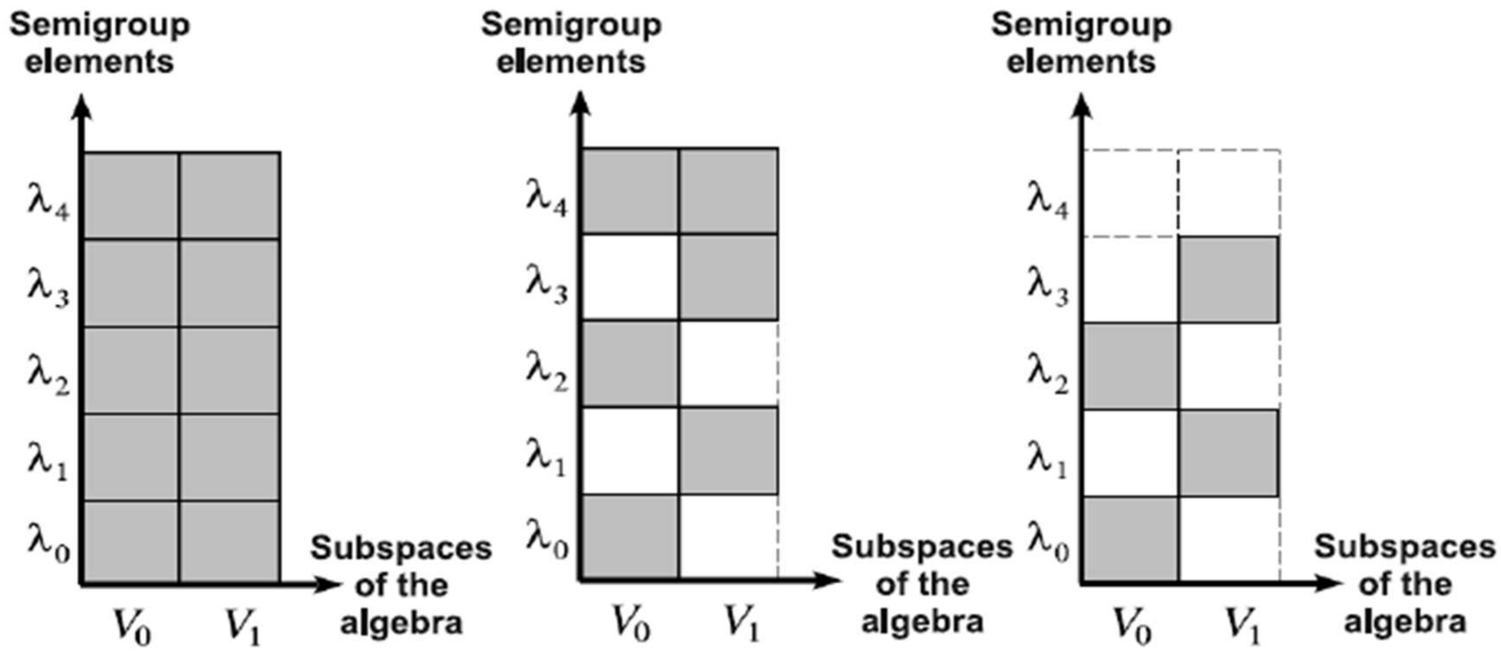
Third result: the sector $0_S \otimes \mathcal{G}$ can be removed from the expanded algebra to obtain the so called **reduced algebra.**

Observartion: Steps II and III are independent, but can also be applied simultaneously.

S-expansion

In general the S-expansion method consist in a serie of steps:

- Obtain the S-expanded algebra
- Find the (resonant)subalgebras
- Perform a reduction.



S-expansion method

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$$S_E^{(N)} = \{\lambda_0, \lambda_1, ..., \lambda_N, \lambda_{N+1}\} \text{ with}$$

$$\lambda_\alpha \lambda_\beta = \lambda_{\alpha+\beta}\,, \text{ if } \alpha+\beta < N+1$$
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$$S = S_0 \cup S_1 \text{ with } \begin{cases} S_0 = \left\{\lambda_{2m}, \text{ with } m = 0, \ldots, \left[\frac{N}{2}\right]\right\} \cup \{\lambda_{N+1}\} \\ S_1 = \left\{\lambda_{2m+1}, \text{ with } m = 0, \ldots, \left[\frac{N-1}{2}\right]\right\} \cup \{\lambda_{N+1}\} \\ S_0 \times S_0 \subset S_0 \\ S_0 \times S_1 \subset S_1 \\ S_1 \times S_1 \subset S_0 \end{cases}$$

S-expansion method

Example: Inönü-Wigner contraction as an $S_E^{(1)}$-expansion

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Example: Inönü-Wigner contraction as an $S_E^{(1)}$-expansion

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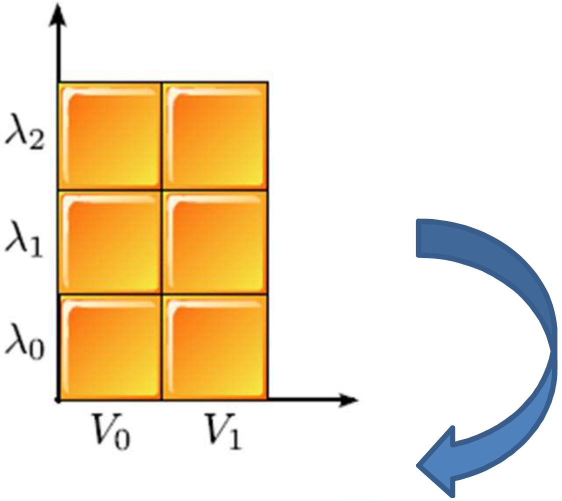
Consider the semigroup $S_E^{(1)}=\{\lambda_0,\lambda_1,\lambda_2\}$ with

$$\lambda_\alpha\lambda_\beta=\lambda_{\alpha+\beta}\,,\ \text{if}\ \alpha+\beta<2$$
$$\lambda_\alpha\lambda_\beta=\lambda_2,\quad \text{if}\ \alpha+\beta\geq 2$$

$$S_E^{(1)}=S_0\cup S_1\quad \left\{\begin{array}{l}S_0=\{\lambda_0,\lambda_2\}\\ S_1=\{\lambda_1,\lambda_2\}\\ S_0\times S_0\subset S_0\\ S_0\times S_1\subset S_1\\ S_1\times S_1\subset S_0\end{array}\right.$$

S-expansion method



$$\mathcal{G}^{S_E^{(1)}} = S_E^{(1)} \otimes \mathcal{G}$$

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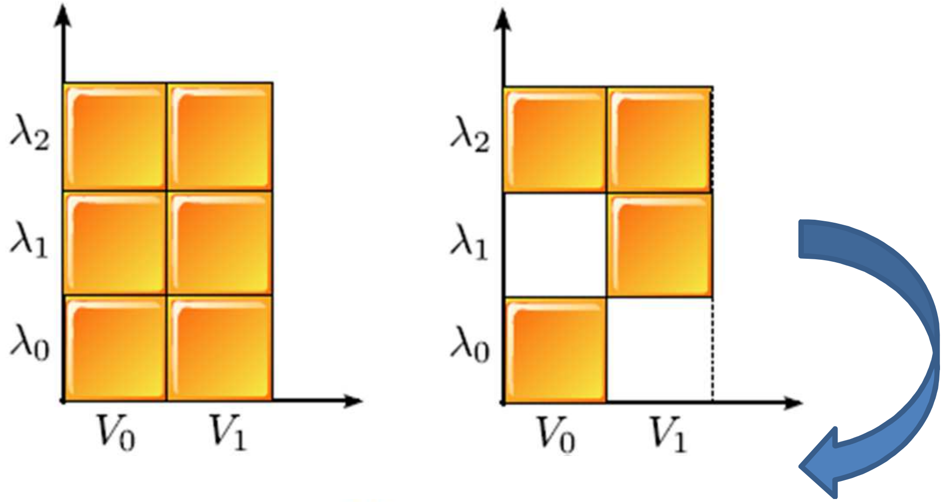

$$\mathcal{G}_R^{S_E^{(1)}} = (S_0 \otimes V_0) \oplus (S_1 \otimes V_1)$$

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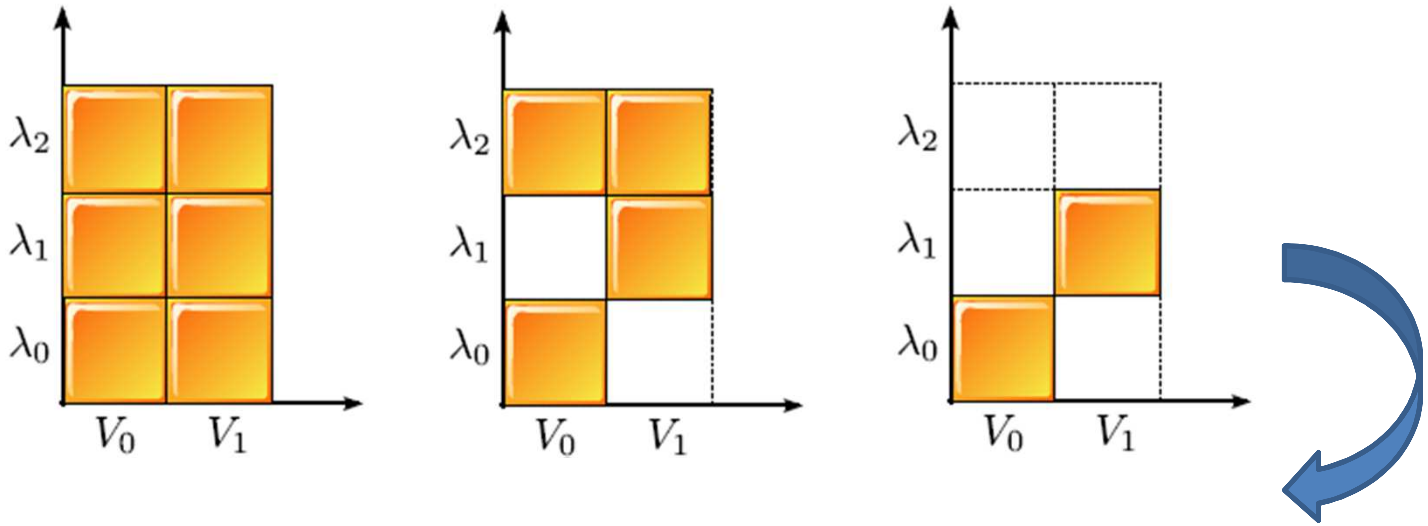

$$\mathcal{G}_{R,\mathrm{red}}^{S_E^{(1)}} = \left[(S_0 \otimes V_0) \oplus (S_1 \otimes V_1) \right]_{\mathrm{red}}$$

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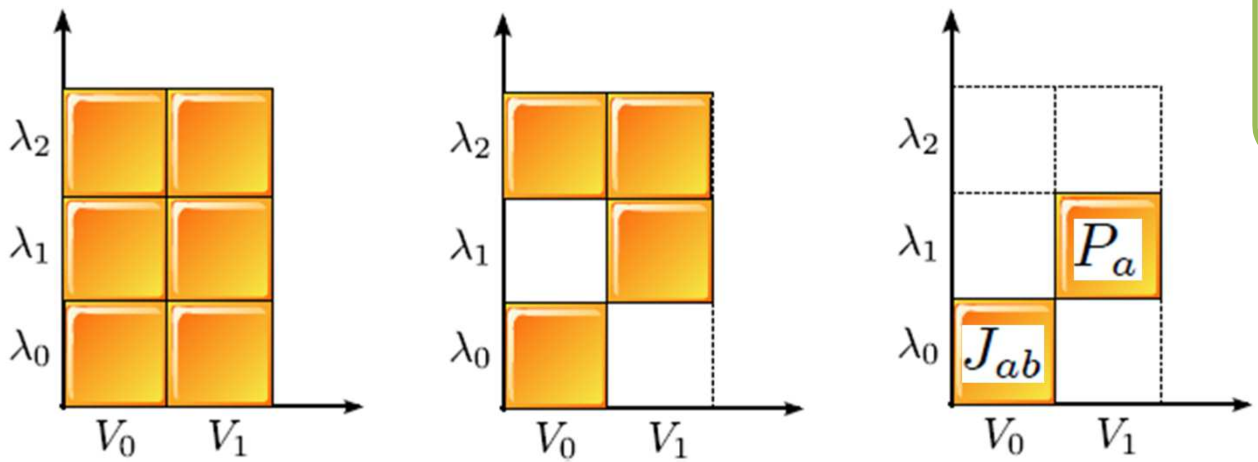

$$J_{ab} = \lambda_0 \otimes \bar{J}_{ab}$$
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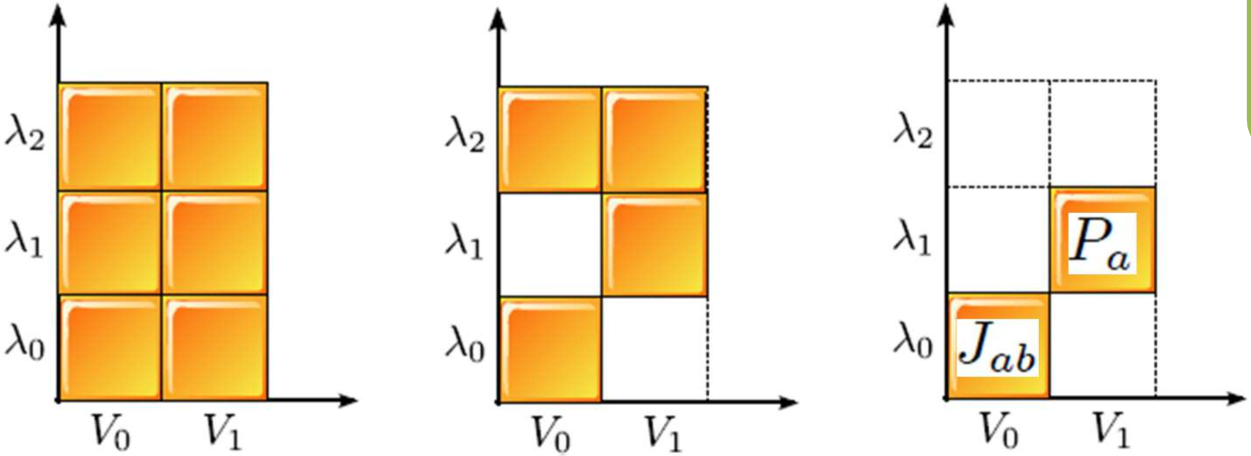

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$$\left[\mathbf{T}_{(A,\alpha)}, \mathbf{T}_{(B,\beta)} \right] = \lambda_\alpha \lambda_\beta \otimes [\mathbf{T}_A, \mathbf{T}_B]$$

$$\left.\begin{aligned} [J_{ab}, J_{cd}] &= \lambda_0\lambda_0 \otimes \left[\bar{J}_{ab}, \bar{J}_{cd}\right] \sim \lambda_0 \bar{J}_{ab} \sim J_{ab} \\ [J_{ab}, P_c] &= \lambda_0\lambda_1 \otimes \left[\bar{J}_{ab}, \bar{P}_c\right] \sim \lambda_1 \bar{P}_c \sim P_c \\ [P_a, P_b] &= \lambda_1\lambda_1 \otimes \left[\bar{P}_a, \bar{P}_b\right] \sim \lambda_2 \bar{J}_{ab} \sim 0 \end{aligned}\right\} \text{ IW contraction.}$$

S-expansion method

Example: The $\mathcal{B}$ algebra

Starting from $\mathcal{G}=SO(4,2)=V_0\oplus V_1$

where

$$V_0=\{\bar{J}_{ab}\}\;;\;V_1=\{\bar{P}_a\}\quad\text{and}\quad \begin{array}{l}[V_0,V_0]\sim V_0\\ [V_0,V_1]\sim V_1\\ [V_1,V_1]\sim V_0\end{array}$$

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Consider the semigroup $S_E^{(3)}=\{\lambda_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$ with

$$\lambda_\alpha\lambda_\beta=\lambda_{\alpha+\beta}\,,\ \text{if}\ \alpha+\beta<4$$
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Is easy to check that $S=S_0\cup S_1$ satisfy

$$\begin{array}{l}S_0=\{\lambda_0,\lambda_2,\lambda_4\}\\ S_1=\{\lambda_1,\lambda_3,\lambda_4\}\end{array}\quad\text{and}\quad\begin{array}{l}S_0\times S_0\subset S_0\\ S_0\times S_1\subset S_1\\ S_1\times S_1\subset S_0\end{array}$$

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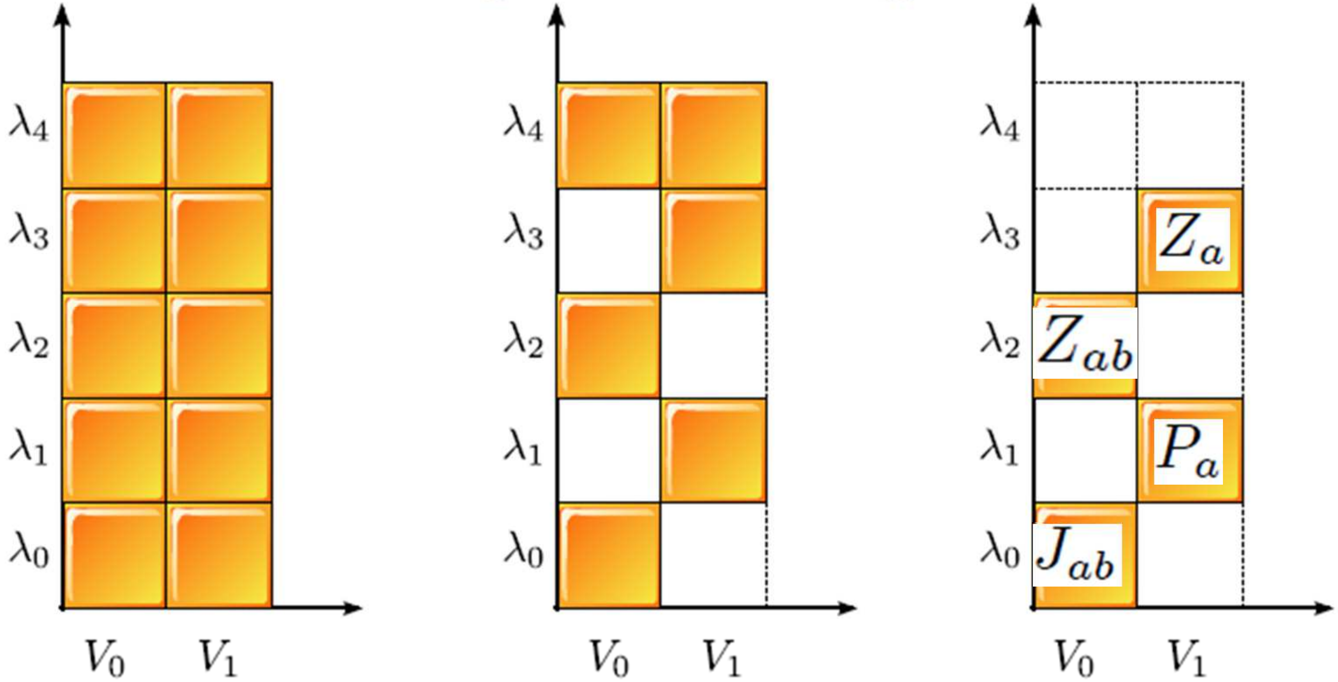

$$J_{ab} = \lambda_0 \otimes \bar{J}_{ab}$$
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$$Z_a = \lambda_3 \otimes \bar{P}_a$$

S-expansion method

Example: The $\mathcal{B}$ algebra

$$\mathcal{B} = \{J_{ab}, Z_{ab}, P_a, Z_a\}$$

$$\left[\mathbf{T}_{(A,\alpha)}, \mathbf{T}_{(B,\beta)}\right] = \lambda_\alpha \lambda_\beta \otimes \left[\mathbf{T}_A, \mathbf{T}_B\right]$$

$$
\begin{aligned}
&[P_a, P_b] = Z_{ab} \,, \qquad [J_{ab}, P_c] = P_a\,\eta_{bc} - P_b\,\eta_{ac} \,, \\
&[J_{ab}, J_{cd}] = -J_{ac}\,\eta_{bd} + J_{bc}\,\eta_{ad} - J_{bd}\,\eta_{ac} + J_{ad}\,\eta_{bc} \,, \\
&[Z_a, Z_b] = [Z_{ab}, Z_c] = [Z_{ab}, Z_{cd}] = [P_a, Z_b] = 0 \,, \\
&[Z_{ab}, P_c] = [J_{ab}, Z_c] = Z_a\,\eta_{bc} - Z_b\,\eta_{ac} \,, \\
&[J_{ab}, Z_{cd}] = -Z_{ac}\,\eta_{bd} + Z_{bc}\,\eta_{ad} - Z_{bd}\,\eta_{ac} + Z_{ad}\,\eta_{bc} \,.
\end{aligned}
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4) CS for AdS in D=3 is equivalent to GR
5) CS for Lorentz in D=3 defines *exotic gravity*.

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1) Exist in odd dimensions
2) Its fundamental field is a $\mathcal{g}$-valued gauge connection *A* with associate curvature *F*
3) Given an **invariant tensor** *g*, the dynamics of A is governed by the CS action whose Lagrangian is constructed from:

$$\langle F \wedge \cdots \wedge F \rangle_g = dL_{\mathrm{CS}}(A)$$

4) CS for AdS in D=3 is equivalent to GR
5) CS for Lorentz in D=3 defines *exotic gravity*.

For further details about CS therories see [2].

[2] J. Zanelli, *arXiv: 0502193*

Applications in Chern-Simons Theories

1) CS action for the M algebra:

In ref. [3] the M algebra is obtained from the Osp(32/1) algebra using the *dAIPV-expansion* method.

[3] J. A. de Azcárraga, J. M. Izquierdo, M. Picón, O. Varela, Nucl. Phys. B **662** (2003) 185 *arXiv: 0212347*
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→ A CS action for the M algebra is constructed.

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- on [*arXiv: 0606225*] it is shown that the supertrace is not a good choice

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- the *S*-expansion procedure provide an invariant tensor, different from the supertrace, which leads to a theory with a richer structure.
- the S-expansion method does not solve the problem of classifying all invariant tensors for non-semisimple algebras, it at least gives invariant tensors different from the supertrace that are useful for the construction of the mentioned gauge theories of gravity.

Applications in Chern-Simons Theories

2) Dual formulation:

A dual formulation of S-expansion method is given in [5].
permits to perform the expansion at the level of the Lagrangian.

[5] F. Izaurieta, A. Pérez, E. Rodríguez, P. Salgado, Jour.Math. Phys. **50** 7 073511 (2009) *arXiv: 0903.4712*

Applications in Chern-Simons Theories

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As an example, this permit to obtain 3D CS AdS gravity from *exotic gravity* (because SO(2,2) can be regarded as an expansion of SO(2,1))

[5] F. Izaurieta, A. Pérez, E. Rodríguez, P. Salgado, Jour.Math. Phys. **50** 7 073511 (2009) *arXiv: 0903.4712*

Applications in Chern-Simons Theories

3) Standard General Relativity from CS theories in d = 5

In ref. [6] it was observed that

$$L_{\mathrm{AdS}}^{(5)} = \kappa \varepsilon_{abcde} \left(\frac{1}{5\ell^5} e^a e^b e^c e^d e^e + \frac{2}{3\ell^3} R^{ab} e^c e^d e^e + \right.$$
$$\left. + \frac{1}{\ell} R^{ab} R^{cd} e^e \right) .$$

does not leave E-H term alone in any limit of the ℓ parameter!

[6] J. D. Edelstein, M. Hassaine, R. Troncoso, J. Zanelli, Phys. Lett. B **640** (2006) 278. *arXiv: 0605174*

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➡ It was proposed to construct a CS action for an expansion of AdS symmetry in D=5 and to study if there is a connection with GR.

[6] J. D. Edelstein, M. Hassaine, R. Troncoso, J. Zanelli, Phys. Lett. B **640** (2006) 278. *arXiv: 0605174*

Applications in Chern-Simons Theories

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AdS algebra → **Expanded AdS algebra**

using the *dAIPV-expansion*

Applications in Chern-Simons Theories

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Result:

- Equations of motion impose too restrictive conditions on the geometry and,

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Applications in Chern-Simons Theories

3) Standard General Relativity from CS theories in d = 5

Result:

- Equations of motion impose too restrictive conditions on the geometry and,

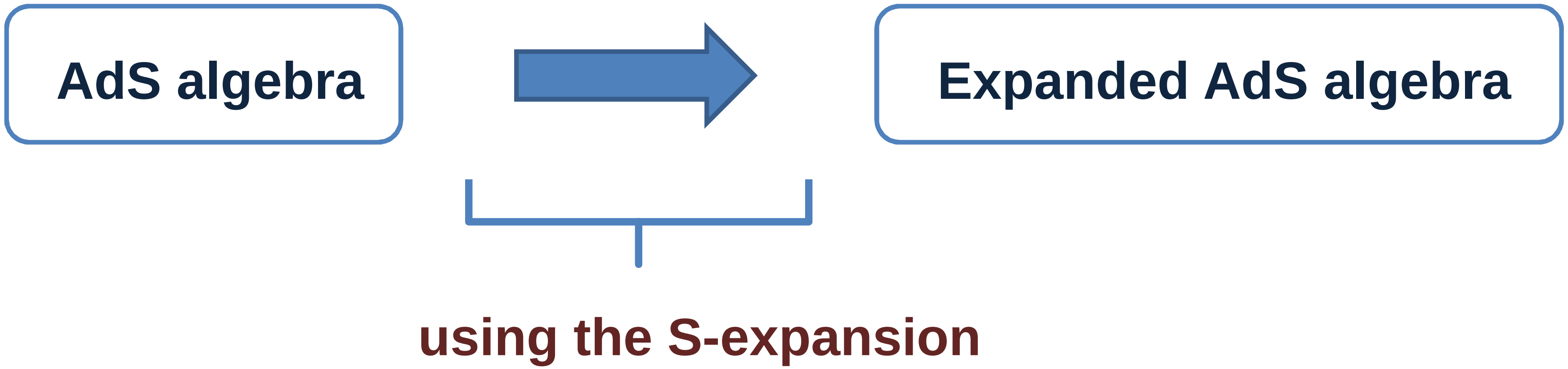
- there is no limit where those conditions desapear.

Therefore using the *dAIPV-expansion* method does not permit to connect a CS theory with 5D GR.

Applications in Chern-Simons Theories

3) Standard General Relativity from CS theories in d = 5

The same idea was followed in ref. [7], but using *S-expansion*



[7] F. Izaurieta, P. Minning, A. Pérez, E. Rodríguez and P. Salgado, Phys. Lett. B **678** (2009) 213 *arXiv: 0905.2187*

Applications in Chern-Simons Theories

3) Standard General Relativity from CS theories in d = 5

Result:

A CS action constructed in terms of the connection

$$\mathbf{A} \rightarrow \widetilde{\mathbf{A}} = e^a P_a + \frac{1}{2} \omega^{ab} J_{ab} + h^a Z_a + \frac{1}{2} \kappa^{ab} Z_{ab} .$$

valuated in the algebra $\mathcal{B}$ introduced before is given by

$$L_{\mathrm{CS}}^{(5)} = \alpha_1 \ell^2 \varepsilon_{abcde} R^{ab} R^{cd} e^e + \alpha_3 \varepsilon_{abcde} \left(\frac{2}{3} R^{ab} e^c e^d e^e + \right.$$
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This action leads to GR in the critic limit $\ell = 0$.

Applications in Chern-Simons Theories

4) Cosmological and Black hole solutions

In refs. [8,9] there were found a *black hole* and a *cosmological solution* for the Chern-Simons theory constructed in terms of the $\mathcal{B}$ algebra.

[8] C. A. C. Quinzacara and P. Salgado, Phys. Rev. D **85**, 124026 (2012)
[9] F. Gomez, P. Minning, and P. Salgado, Phys. Rev. D **84**, 063506 (2011)

Contents

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General properties of the expansion methods of Lie algebras

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Expansions with **other semigroups** can generate algebras that cannot be reached nor by any *contraction* neither by a *dAIPV expansion*.

General properties of the expansion methods of Lie algebras

An example is given in [10].

[10] R. Caroca, I. Kondrashuk, N. Merino and F. Nadal, J. Phys. A: Math. Theor. 46 (2013) 225201 (24pp), *arXiv: 1104.3541*

IOP Publishing — Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **46** (2013) 225201 (24pp) doi:10.1088/1751-8113/46/22/225201

Bianchi spaces and their three-dimensional isometries as *S*-expansions of two-dimensional isometries

Ricardo Caroca[1], Igor Kondrashuk[2], Nelson Merino[3] and Felip Nadal[4,5]

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[3] Instituto de Física, Pontificia Universidad Católica de Valparaíso, Av. Brasil 2950, Valparaíso, Chile
[4] Instituto de Física Corpuscular (IFIC), Edificio Institutos de Investigación., c/ Catedrático José Beltrán, 2. E-46980 Paterna, Spain
[5] Dipartimento di Scienza Applicata e Tecnologia (DISAT), Politecnico di Torino, Corso Duca degli Abruzzi, 24, I-10129 Torino, Italy

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Received 7 September 2012, in final form 12 April 2013
Published 16 May 2013
Online at stacks.iop.org/JPhysA/46/225201

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Principal Idea: can be related 2 and 3-dimensional isometries?

Considering the two set of algebras: $[X_1, X_2] = 0$ and $[X_1, X_2] = X_1$

Group	Algebra
→ type I	$[X_1, X_2] = [X_1, X_3] = [X_2, X_3] = 0$
→ type II	$[X_1, X_2] = [X_1, X_3] = 0, \quad [X_2, X_3] = X_1$
→ type III	$[X_1, X_2] = [X_2, X_3] = 0, \quad [X_1, X_3] = X_1$
type IV	$[X_1, X_2] = 0, \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = X_1 + X_2$
→ type V	$[X_1, X_2] = 0, \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = X_2$
type VI	$[X_1, X_2] = 0, \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = hX_2,$ where $h \neq 0, 1$
type VII$_1$	$[X_1, X_2] = 0, \quad [X_1, X_3] = X_2, \quad [X_2, X_3] = -X_1$
type VII$_2$	$[X_1, X_2] = 0, \quad [X_1, X_3] = X_2, \quad [X_2, X_3] = -X_1 + hX_2,$ where $h \neq 0$ $(0 < h < 2)$.
type VIII	$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3$
type IX	$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2$

General properties of the expansion methods of Lie algebras

Therefore, to answer the general question (given in *Introduction*):

> Given two algebras
>
> *A* and *B*
>
> Can these algebras be related by some contraction or expansion method?

we need to consider the **complete family of abelian semigrops**.

General properties of the expansion methods of Lie algebras

We need to consider the ***history about finite semigroup***:

order	$Q = \#$ semigroups	
1	1	
2	4	
3	18	
4	126	[Forsythe '54]
5	1,160	[Motzkin, Selfridge '55]
6	15,973	[Plemmons '66]
7	836,021	[Jurgensen, Wick '76]
8	1,843,120,128	[Satoh, Yama, Tokizawa '94]
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General properties of the expansion methods of Lie algebras

We have developed a procedure to answer that question in [11].

[11] L. Andrianopoli, N. Merino, F. Nadal, M. Trigiante, *General properties of the S-expansion method*, J. Phys. A: Math. Theor. 46 (2013) 365204 (33pp) *arXiv:1308.4832*

IOP PUBLISHING — JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

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General properties of the expansion methods of Lie algebras

L Andrianopoli[1,2], N Merino[3], F Nadal[1,4] and M Trigiante[1,2]

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Received 8 May 2013, in final form 22 July 2013
Published 21 August 2013
Online at stacks.iop.org/JPhysA/46/365204

General properties of the expansion methods of Lie algebras

1) First by establishing general properties of the expansion method (preservation of some properties of the Lie algebra)

Properties preserved by the action of the S-expansion process

Original $\mathcal{G}$	Expanded $\mathcal{G}_S$	Resonant $\mathcal{G}_{S,R}$	Reduced $\mathcal{G}_{S,R}^{\text{red}}$
Abelian	Abelian	Abelian	Abelian
Solvable	Solvable	Solvable	Solvable
Nilpotent	Nilpotent	Nilpotent	Nilpotent
Compact	Arbitrary	Arbitrary	Arbitrary
Semisimple $\mathcal{G} = S$	Arbitrary $\mathcal{G}_S = N_{\text{exp}} \uplus S_{\text{exp}}$	Arbitrary $\mathcal{G}_{S,R} = N_{\text{exp},R} \uplus S_{\text{exp},R}$	Arbitrary $\mathcal{G}_{S,R}^{\text{red}} = N_{\text{exp},R}^{\text{red}} \uplus S_{\text{exp},R}^{\text{red}}$
Arbitrary $\mathcal{G} = N \uplus S$	Arbitrary $\mathcal{G}_S = N_{\text{exp}} \uplus S_{\text{exp}}$	Arbitrary $\mathcal{G}_{S,R} = N_{\text{exp},R} \uplus S_{\text{exp},R}$	Arbitrary $\mathcal{G}_{S,R}^{\text{red}} = N_{\text{exp},R}^{\text{red}} \uplus S_{\text{exp},R}^{\text{red}}$

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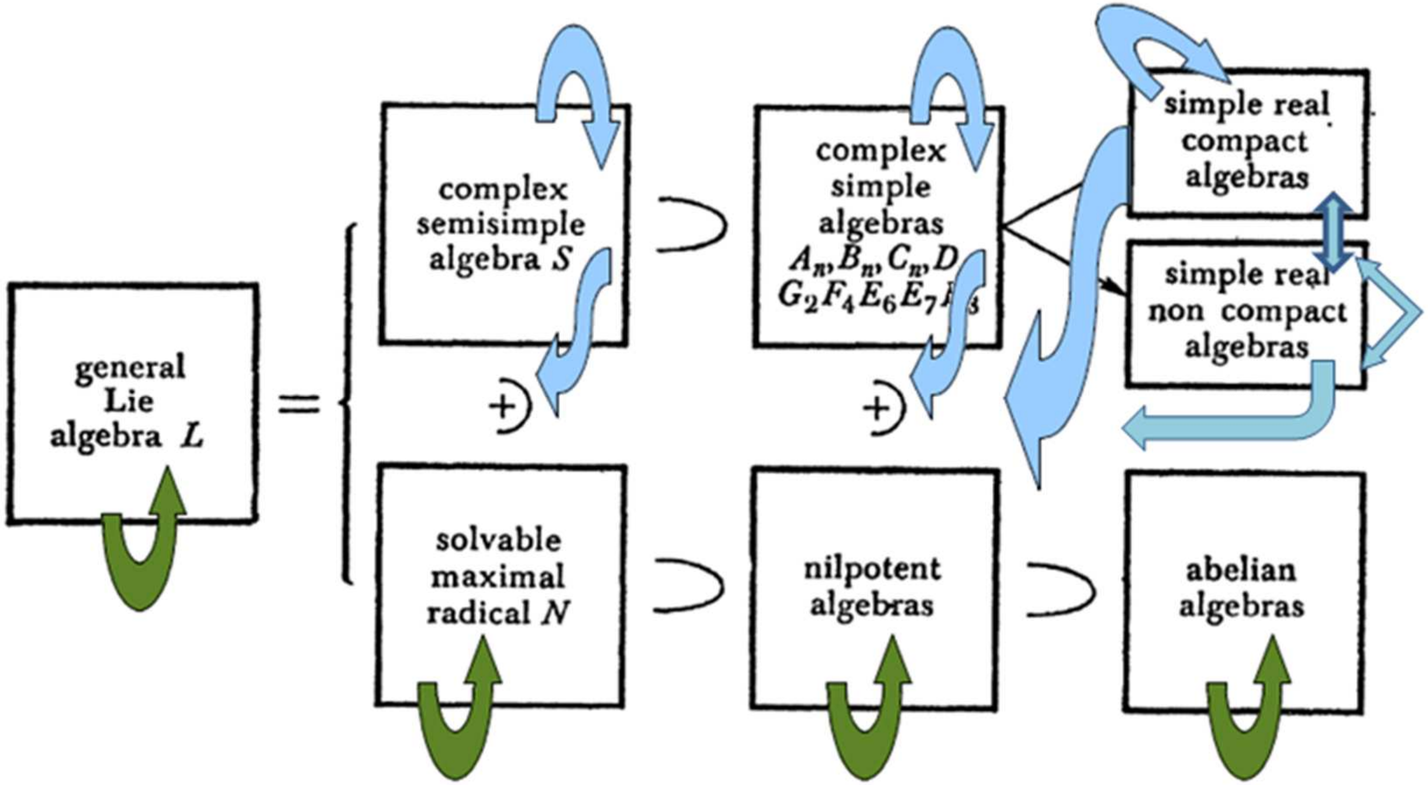

General properties of the expansion methods of Lie algebras

2) Then, by implementing computer programs to perform expansion with any semigroup (up to order 6).

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Those programs allow us:

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iv) To study further conditions for some specific problem

Final remarks

As mentioned in

Nesterenko M. 2012 S-expansions of three dimensional Lie algebras *arXiv:1212.1820*

it would be interesting to know whether *S*-expansions fit the classification of solvable Lie algebras of a fixed dimension by means of S-expansions of simple (semisimple) Lie algebras of the same dimension.

The theoretical results proposed in this work could be useful in solving that problem.

Final remarks

The S-expansion method have already been extended to the case of other mathematical structures (*higher order Lie algebras, Loop algebras*)

Caroca R, Merino N and Salgado P 2009 *S-Expansion of higher-order Lie algebras* J. Math. Phys. 50 013503

Caroca R, Merino N, Pérez A and Salgado P 2009 *Generating higher order Lie algebras by expanding Maurer-Cartan forms* J. Math. Phys. 50 123527

Caroca R, Merino N, Salgado P and Valdivia O 2011 *Generating infinite-dimensional algebras from loop algebras by expanding Maurer–Cartan forms* J. Math. Phys. 52 043519

Final remarks

We think that this mathematical tool could be useful other gauge theories such as *Yang-Mills theory, Wess-Zumino models, higher spin theories, gauge/gravity duality,* etc.

Thanks you for your attention

Implementing computer programs

History about finite semigroup programs:

HANDBOOK OF FINITE SEMIGROUP PROGRAMS

John A Hildebrant

TABLE OF CONTENTS

Implementing computer programs

History about finite semigroup programs:

For example, for n=2 the program com.f gives the following list:

$S_{(2)}^{1}$	λ_1	λ_2
λ_1	λ_1	λ_1
λ_2	λ_1	λ_1

,

$S_{(2)}^{2}$	λ_1	λ_2
λ_1	λ_1	λ_1
λ_2	λ_1	λ_2

,

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type VII_2	$[X_1, X_2] = 0, \quad [X_1, X_3] = X_2, \quad [X_2, X_3] = -X_1 + hX_2,$ where $h \neq 0$ $(0 < h < 2)$.
type VIII	$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3$
type IX	$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2$

Checking with computer programs

Isomorphisms and consistency of the procedure

Algebra	Semigroup used
Type I	many semigroups
Type II	$S^{10}_{(4)}$, $S^{12}_{(4)}$
Type III	$S^{13}_{(4)}$, $S^{28}_{(4)}$, $S^{42}_{(4)}$, $S^{43}_{(4)}$, $S^{44}_{(4)}$, $S^{45}_{(4)}$ and $S^{64}_{(4)}$
Type V	$S^{42}_{(4)}$

isomorphic to			isomorphism
S_{N1}	$\Longleftrightarrow$	$S^{44}_{(4)}$	$(\ \lambda_4\ \lambda_1\ \lambda_2\ \lambda_3\ )$
S_{N2}	$\Longleftrightarrow$	$S^{12}_{(4)}$	$(\ \lambda_4\ \lambda_3\ \lambda_2\ \lambda_1\ )$
S_{N3}	$\Longleftrightarrow$	$S^{42}_{(4)}$	$(\ \lambda_4\ \lambda_1\ \lambda_3\ \lambda_2\ )$
$S^{(2)}_{E}$	$\Longleftrightarrow$	$S^{43}_{(4)}$	$(\ \lambda_4\ \lambda_3\ \lambda_2\ \lambda_1\ )$
$S^{(3)}_{K}$	$\Longleftrightarrow$	$S^{45}_{(4)}$	$(\ \lambda_4\ \lambda_2\ \lambda_1\ \lambda_3\ )$