Supergravity Infinity Cancellations and Ultraviolet Puzzles

K.S. Stelle
Imperial College London

Quantum Gravity in the Southern Cone
Maresias, Brazil
Wednesday, September 11, 2013

G. Bossard, P.S. Howe & K.S.S. 0901.4661, 0908.3883, 1009.0743
G. Bossard, P.S. Howe, K.S.S. & P. Vanhove 1105.6087
G. Bossard, P.S. Howe & K.S.S., 1212.0841
G. Bossard, P.S. Howe & K.S.S., 1304.7753

Gell-Mann's Totalitarian Principle:

"Everything not forbidden is compulsory"

So why are expected UV divergences not occurring on schedule in maximal supergravity?

Are miracles happening?

(Quotation actually stolen from T.H. White, *The Sword in the Stone*)

Saint Augustine:

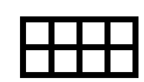
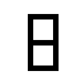
"Miracles are not contrary to nature, but only contrary to what we know about nature"

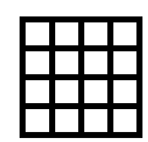
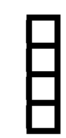
- A very useful approach to the analysis of UV divergence problems in supersymmetric theories has been the combination of <u>superspace techniques</u> with the <u>background field method</u> for calculating Feynman diagrams.
    - When one has an "off-shell" formalism for some degree of extended supersymmetry, the Feynman rules can be organised in such a way that background fields on *external* lines appear only through certain "geometrical" entities, e.g. superspace vielbeins and gauge connections.
    - The introduction of prepotentials for the quantum fields which correspond to the *internal* lines of superspace Feynman diagrams allows all terms used in the calculation of 1PI diagrams at loop orders L>1 to be written as full-superspace integrals.
    - Counterterms for L>1 must be writeable as full-superspace integrals of expressions involving the background fields.

- The degree of off-shell supersymmetry is the maximal supersymmetry for which the algebra can close without use of the equations of motion.
- Knowing the extent of this off-shell supersymmetry is tricky, and may involve formulations (*e.g.* harmonic superspace) with infinite numbers of auxiliary fields. Galperin, Ivanov, Kalitsin, Ogievetsky & Sokatchev
- For maximal N=4 Super Yang-Mills and maximal N=8 supergravity, the fraction of off-shell realizable supersymmetry is known to be at least *half* the full supersymmetry of the theory, but the maximum realizable fraction in harmonic superspace is not currently known. Assuming that the maximal fraction is 1/2 led originally to the expectation that the first allowable counterterms would have "1/2 BPS" structure.

- The 3-loop R^4 maximal supergravity candidate counterterm has a structure under linearized supersymmetry that is very similar to that of an F^4 N=4 super Yang-Mills invariant. Both of these are 1/2 BPS invariants, involving integration over just half the corresponding full superspaces:

Howe, K.S.S. & Townsend 1981
Kallosh 1981

$$\Delta I_{SYM} = \int (d^4\theta d^4\bar{\theta})_{105}\,\mathrm{tr}(\phi^4)_{105}$$  ***105*** ϕ_{ij}  ***6*** of $SU(4)$

$$\Delta I_{SG} = \int (d^8\theta d^8\bar{\theta})_{232848}(W^4)_{232848}$$  ***232848*** W_{ijkl}  ***70*** of $SU(8)$

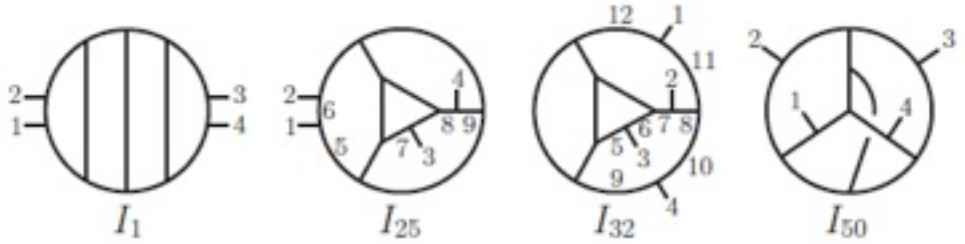
- Versions of these supergravity and SYM counterterms indeed do occur at one loop in D=8. This implies that, at least in that spacetime dimension, the full nonlinear structure of such counterterms exists and is consistent with all symmetries.

Unitarity-based calculations

Bern, Carrasco, Dixon, Johansson, Roiban et al. 2007 … 2011

- The calculational front has now made substantial progress since the late 1990s.

- This has led to unanticipated and surprising cancellations at the 3- and 4-loop orders, yielding new lowest possible orders for super Yang-Mills and supergravity divergence onsets.



plus 46 more topologies

Max. SYM first divergences, current lowest possible orders (for integral spacetime dimensions).

Dimension D	10	8	7	6	5	4
Loop order L	1	1	2	3	6	∞
BPS degree	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
Gen. form	$\partial^2 F^4$	F^4	$\partial^2 F^4$	$\partial^2 F^4$	$\partial^2 F^4$	finite

Blue: known divergences

Max. supergravity first divergences, current lowest possible orders (for integral spacetime dimensions).

Dimension D	11	10	8	7	6	5	4
Loop order L	2	2	1	2	3	6?	5?
BPS degree	0	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	0	$\frac{1}{4}$
Gen. form	$\partial^{12} R^4$	$\partial^{10} R^4$	R^4	$\partial^4 R^4$	$\partial^6 R^4$	$\partial^{12} R^4$	$\partial^4 R^4$

Algebraic Renormalization and Ectoplasm

Dixon; Howe, Lindstrom & White; Piguet & Sorella; Hennaux; Stora;
Baulieu & Bossard; Voronov 1992; Gates, Grisaru, Knut-Whelau, & Siegel 1998
Berkovits and Howe 2008; Bossard, Howe & K.S.S. 0901.4661, 0908.3883, 1009.0743

- The construction of supersymmetric invariants is isomorphic to the construction of cohomologically nontrivial closed forms in superspace: $I = \int_{M_0} \sigma^* \mathcal{L}_D$ is invariant (where σ^* is a pull-back to a section of the projection map down to the purely bosonic "body" subspace M_0) if $\mathcal{L}_D$ is a closed form in superspace, and it is nonvanishing only if $\mathcal{L}_D$ is nontrivial.
- Using the BRST formalism, one can handle all gauge symmetries and space-time diffeomorphisms by the nilpotent BRST operator s. The invariance condition for $\mathcal{L}_D$ is $s\mathcal{L}_D + d_0 \mathcal{L}_{D-1} = 0$, where d_0 is the usual bosonic exterior derivative. Since $s^2 = 0$ and s anticommutes with d_0, one obtains from Poincaré's lemma $s\mathcal{L}_{D-1} + d_0 \mathcal{L}_{D-2} = 0$, etc.

- For example, the cocycle structure of the SYM Lagrangian turns out to match that of a full-superspace integral of a gauge-invariant integrand, showing that the latter are fully acceptable as counterterms.

- Examples of operators that are ruled out by the ectoplasm/ algebraic renormalization analysis include half-BPS counterterms such as the $\mathrm{tr}(F^4)$ or $(\mathrm{tr}(F^2))^2$ SYM counterterms. In D dimensions, the generator component of such a 1/2 BPS cocycle is an (0, D) form of dimension 8-D/2. Since the structure of this cocycle is different (i.e. it is *longer*) from than that of the SYM Lagrangian, the corresponding 1/2 BPS counterterm is *illegal*.

- Similar considerations allow one to analyse the R^4 counterterm in *N=8* supergravity, although the density character of supergravity invariants complicates analysis of their non-leading structure.

Bossard, Howe & K.S.S. 0901.4661, 0908.3883

Duality invariance constraints

- Maximal supergravity has a series of duality symmetries which extend the automatic GL(11-D) symmetry obtained upon dimensional reduction down from D=11. The classic example is E_7 in the N=8, D=4 theory, with the 70=133-63 scalars taking their values in an $E_7/SU(8)$ coset target space.
- The N=8, D=4 theory can be formulated in a manifestly E_7 covariant (but non-manifestly Lorentz covariant) formalism. Bossard, Hillman & Nicolai 2010
Anomalies for SU(8), and hence E_7, cancel. Marcus 1985
- Combining the requirement of continuous duality invariance with the superspace cohomology requirements gives further powerful restrictions on counterterms.

Other approach to duality analysis from string amplitudes:
Broedel & Dixon 2010
Elvang & Kiermeier 2010;
Beisert, Elvang, Freedman, Kiermaier, Morales & Stieberger 2010

Supergravity Densities

- In a curved superspace, an invariant is constructed from the top (pure "body") component in a coordinate basis:

$$I = \frac{1}{D!}\int d^Dx\, \varepsilon^{m_D\ldots m_1}\; E_{m_D}{}^{A_D}\cdots E_{m_1}{}^{A_1}\; L_{A_1\ldots A_D}(x,\theta=0)$$

- Referring this to a preferred "flat" basis and identifying $E_M{}^A$ components with vielbeins and gravitinos, one has, *e.g.* in D=4

$$I = \frac{1}{24}\int \big(e^a{}_\wedge e^b{}_\wedge e^c{}_\wedge e^d\; L_{abcd} + 4e^a{}_\wedge e^b{}_\wedge e^c{}_\wedge \psi^{\underline{\alpha}} L_{abc\underline{\alpha}} + 6e^a{}_\wedge e^b{}_\wedge \psi^{\underline{\alpha}}{}_\wedge \psi^{\underline{\beta}}\; L_{ab\underline{\alpha}\,\underline{\beta}}$$

$$+4e^a{}_\wedge \psi^{\underline{\alpha}}{}_\wedge \psi^{\underline{\beta}}{}_\wedge \psi^{\underline{\gamma}} L_{a\underline{\alpha}\,\underline{\beta}\,\underline{\gamma}} + \psi^{\underline{\alpha}}{}_\wedge \psi^{\underline{\beta}}{}_\wedge \psi^{\underline{\gamma}}{}_\wedge \psi^{\underline{\delta}} L_{\underline{\alpha}\,\underline{\beta}\,\underline{\gamma}\,\underline{\delta}}\big)$$

  - Thus the "soul" components of the cocycle also contribute to the local supersymmetric covariantization.

- Since the <u>gravitinos do not transform</u> under the D=4 E_7 duality, the L_{ABCD} form components have to be *separately* duality invariant.

- At leading order, the $E_7/SU(8)$ coset generators of E_7 simply produce *constant shifts* in the 70 scalar fields. This leads to a much easier check of invariance than analyzing the full superspace cohomology problem.

Howe, K.S.S. & Townsend 1981

- Although the pure-body (4,0) component L_{abcd} of the R^4 counterterm has long been known to be shift-invariant at lowest order (since all 70 scalar fields are covered by derivatives), it is harder for the fermionic "soul" components to be so, since they are of lower dimension.

- Thus, one finds that the maxi-soul (0,4) $L_{\alpha\beta\gamma\delta}$ component is *not* invariant under constant shifts of the 70 scalars. Hence the D=4, N=8, 3-loop R^4 1/2 BPS counterterm is not E_7 duality invariant, so it is <u>ruled out</u> as an allowed counterterm.

G. Bossard, P.S. Howe & K.S.S. 1009.0743

L=7 and Vanishing Volume

G. Bossard, P.S. Howe, K.S.S. & P. Vanhove
1105.6087

- The above type of analysis knocks out all the candidates in D=4, N=8 supergravity through L=6 loops. This leaves 7 loops (Δ=16) as the first order where a fully acceptable candidate might occur, with the volume of superspace as a prime candidate: $\int d^4x d^{32}\theta E(x,\theta)$.
- Explicitly integrating out the volume into component fields using the superspace constraints implying the classical field equations would be an ugly task.
    - However, using an on-shell implementation of harmonic superspace together with a superspace implementation of the normal-coordinate expansion, one can in fact evaluate it, but one then finds that the volume *vanishes*:
    $$\int d^4x d^{32}\theta E(x,\theta) = 0 \quad \text{on-shell}$$

Normal coordinates for a 28+4 split

Kuzenko & Tartaglino-Mazzucchelli 2008
G. Bossard, P.S. Howe, K.S.S. & P. Vanhove 2011

- One can define normal coordinates
$$\zeta^{\hat{A}} = \{\zeta^\alpha = \delta^\alpha_\mu \theta^\mu_i u^i{}_1\,, \bar\zeta^{\dot\alpha} = \delta^{\dot\alpha}_{\dot\mu} u^8{}_i\, \bar\theta^{\dot\mu\, i}, z^r{}_1, z^8{}_r, z^8{}_1\}$$
associated to an involutive set of vector fields $\hat{E}_{\hat{A}}$ which allow for an on-shell harmonic superspace formalism based on the flag manifold $S(U(1)\times U(6)\times U(1))\backslash SU(8)$. Expanding the superspace Berezinian determinant in these, one finds the flow equation
$$\zeta^{\hat\alpha}\partial_{\hat\alpha}\ln E = -\frac{1}{3}B_{\alpha\dot\beta}\zeta^\alpha\bar\zeta^{\dot\beta} + \frac{1}{18}B_{\alpha\dot\beta}B_{\alpha\dot\alpha}\zeta^\alpha\zeta^\beta\bar\zeta^{\dot\alpha}\bar\zeta^{\dot\beta} \qquad B_{\alpha\dot\beta} = \bar\chi^{1ij}_{\dot\beta}\chi_{\alpha\, 8ij}$$
- Integrating this, one finds the expansion of the superspace determinant in the four fermionic coordinates $\zeta^{\hat\alpha} = (\zeta^\alpha, \zeta^{\dot\alpha})$:
$$E(\hat{x},\zeta,\bar\zeta) = \mathcal{E}(\hat{x})\left(1-\frac{1}{6}B_{\alpha\dot\beta}\zeta^\alpha\zeta^{\dot\beta}\right)$$
  - However, since this has only ζ^2 terms, integration over the four $\zeta^{\hat\alpha}$ *vanishes*, so $\int d^4x d^{32}\theta E(x,\theta) = 0\,.$

1/8 BPS E_7 invariant candidate notwithstanding

- Despite the vanishing of the full N=8 superspace volume, one can nonetheless use an on-shell harmonic superspace formalism to construct a different manifestly E_7 -invariant but 1/8 BPS candidate:

Bossard, Howe, K.S.S. & Vanhove 1105.6087

$$I^8 := \int d\mu_{(8,1,1)}\, B_{\alpha\dot\beta}\, B^{\alpha\dot\beta} \qquad\qquad B_{\alpha\dot\beta} = \bar\chi^{1ij}_{\dot\beta}\chi_{\alpha\, 8ij}$$

- At the leading 4-point level, this invariant, of generic $\partial^8 R^4$ structure, can be written as a full superspace integral with respect to the linearized N=8 supersymmetry. It cannot, however, be rewritten as a non-BPS full-superspace integral with a duality-invariant integrand at the nonlinear level.

- Non-BPS full-superspace and manifestly E_7-invariant candidates do exist in any case from 8 loops onwards.

Howe & Lindstrom 1981
Kallosh 1981

Current outlook for maximal supergravity

- So far, things seem under control for maximal supergravity from a purely field-theoretic analysis: what is prohibited does not occur, and what is not prohibited has occurred, as far as one can see.
- As far as one knows, the first acceptable D=4 counterterm for maximal supergravity occurs at L=7 loops ($\Delta = 16$); if not that, then they clearly exist at L=8 loops ($\Delta = 18$) and beyond.

Howe & Lindstrom 1981
Kallosh 1981

- The current divergence expectations for maximal supergravity are consequently:

Dimension D	11	10	8	7	6	5	4
Loop order L	2	2	1	2	3	6	7
BPS degree	0	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	0	$\frac{1}{8}$
Gen. form	$\partial^{12}R^4$	$\partial^{10}R^4$	R^4	$\partial^4 R^4$	$\partial^6 R^4$	$\partial^{12}R^4$	$\partial^8 R^4$

Blue: known divergences

Green: anticipated divergences

The N=4 Supergravity L=3 surprise

- Not everything is perfect in the understanding of supergravity divergences, however. A surprize has occurred in an unexpected sector: D=4, N=4 supergravity at L=3. The expected 3-loop R^4 divergence (Δ=8) does not occur in that theory. Bern, Davies, Dennen & Huang 2012
  - Yet, the L=7 candidate counterterm of N=8 supergravity has a natural analogue here as a 1/4 BPS (4,1,1) G-analytic invariant: $I^4 = \int d\mu_{(4,1,1)} \, B_{\alpha\dot\beta} \, B^{\alpha\dot\beta} \quad B_{\alpha\dot\beta} = \chi^1_\alpha \bar\chi_{\dot\beta 4}$
  - Expanding the content of this N=4 invariant at linearized level, one finds a leading R^4 structure undressed by the $SL(2,\mathbb{R})/U(1)$ complex scalar field: it is perfectly duality invariant, just like the 1/8 BPS candidate 7-loop N=8 counterterm. Bossard, Howe, K.S.S. & Vanhove 1105.6087

- Some aspects of this N=4 case:

    Marcus 1985

    - There are anomalies at one loop in the U(1) R-symmetry. These eventually destroy the $SL(2,\mathbb{R})$ duality symmetry. Happily, these anomalies do not yet affect the structure of the 3-loop divergences, for which the requirement of duality invariance still holds. Bossard, Howe & K.S.S.1304.7753

    Tourkine & Vanhove 2012

    - Genus-1 and genus-2 asymmetric-orbifold string calculations likewise show that R^4 divergences do not appear in N=4 supergravity models coupled to $4\leq n_v\leq 22$ vector multiplets. Note that such matter-coupled models are already divergent at L=1, so there are subdivergence subtractions to worry about, but the absence of R^4 divergences at D=4, L=3 is nonetheless confirmed. Fischler 1979

Vanishing volumes and their consequences

- Another aspect of this story needs to be clarified. The vanishing of a superspace volume can open the door to another representation of candidate counterterms.
- Consider the cases where superspace volumes vanish on-shell:
  - The full superspace volumes of all D=4 pure supergravities vanish, for any extension N of supersymmetry.
  - In D=5, the volume of maximal (32 supercharge) supergravity does *not* vanish, but the volume of half-maximal (16 supercharge, i.e. N=2, D=5) supergravity *does*.

Half-maximal D=5, L=2

Bern, Davies, Dennen & Huang 2012

- Unitarity-based calculations in D=5 half-maximal supergravity show cancellation of R^4 divergences at the 2-loop level similar to those found in half-maximal D=4, L=3.

- This cancellation is equally surprising as in the N=4, D=4 case, because there is an available 1/4 BPS D=5 (4,1) G-analytic Sp(2)/(U(1)×Sp(1)) counterterm:

$$\int d\mu_{(4,1)} \Omega^{\alpha\beta}\Omega^{\gamma\delta}\left(\chi^1_\alpha \chi^1_\beta \chi^1_\gamma \chi^1_\delta\right)$$

  where $\Omega^{\alpha\beta}$ is the D=5 Lorentz Sp(1,1) symplectic matrix.

- Moreover, in D=5 there are no complications from anomalies to the "duality" shift symmetry for the single scalar ϕ of half-maximal D=5 supergravity, unlike in the D=4, N=4 case.

- The vanishing volume of half-maximal D=5 supergravity invites another way to write a candidate Δ=8 counterterm in D=5. One can write simply

$$I^{4\prime} = \int d^{16}\theta E \Phi$$

where Φ is the D=5 field-strength superfield containing the scalar ϕ as its lowest component field.

- Also, this candidate is clearly invariant under the rather minimalistic D=5 duality symmetry $\Phi \to \Phi$ + constant, since $\int d^{16}\theta E = 0$.

- Moreover, this candidate turns out to be just a rewriting of the above (4,1) G-analytic manifestly duality invariant 1/4 BPS candidate counterterm.

- In this sense, the D=5 Δ=8 (4,1) R^4 counterterm is of *marginal* F/D type.

G. Bossard, P.S. Howe & K.S.S., 1212.0841, 1304.7753

- The D=4 (4,1,1) G-analytic counterterm has the same marginal F/D character.

- The D=4, N=4 theory has as its lowest-dimension physical component a complex scalar field τ taking its values in the Kähler space $SL(2,\mathbb{R})/U(1)$. In terms of τ, the Kähler potential is

$$K[\tau] = -\ln(\mathrm{Im}[\tau])$$

and the N=4, Δ=8 (4,1,1) counterterm can equally well be written

$$\int d^{16}\theta E K[\tau]$$

- As in the D=5 case, although this full-superspace integral is duality invariant, its *integrand* is not duality invariant. The integrand varies as follows:

$$\delta\left(E\ln(\mathrm{Im}[\tau])\right) = 2hE + fE(\tau + \bar{\tau})$$

Superspace nonrenormalization theorems: refinement of the duality-invariance requirement

G. Bossard, P.S. Howe & K.S.S., 1212.0841, 1304.7753

- The marginal F/D structure of the Δ=8 counterm candidates in half-maximal D=4 and D=5 supergravities requires a more careful treatment of the Ward identies for duality.
- If one makes the *assumption* that there exist off-shell full 16-supercharge superfield formulations for the half-maximal theories, then one can derive a stronger invariance requirement: not only must the integrated counterterm be duality invariant, but also the counter-Lagrangian superfield *integrand* must itself be duality invariant.

Off-shell half-maximal supergravity

- From the point of view of field-theoretic nonrenormalization theorems, a key question is whether there exists an off-shell linearly realised formulation of half-maximal supergravity. If so, then the nonrenormalization theorem would require a full-superspace $\int d^{16}\theta$ integral with a duality-invariant integrand, thus ruling out the F/D marginal D=4 and D=5 R^4 counterterms.
- Unfortunately, the answer to this question is not currently known. But there is a closely related off-shell formulation for *linearized* D=10, N=1 supergravity, with a finite number of component fields:

Howe, Nicolai & Van Proeyen 1982

$$\mathcal{L}_{10} = \frac{1}{2} V_{abc} \Delta_{abc,def} V_{def} - V_{abc} \bar{D} \Gamma_{abc} D S \qquad \Delta : \quad D^{16}, \ \partial D^{14}, \ \text{etc.}$$

- Upon dimensional reduction to D=4, the N=1, D=10 theory yields D=4, N=4 supergravity plus 6 N=4 super-Maxwell multiplets. So one has something close to the required formalism, at least in the linearised theory.

The most recent developments

- The divergences in half-maximal supergravity in D=4 and D=5 have now been calculated up through 3 loops including also n_v "matter" vector multiplets. D=4 result: a mixture of $1/\varepsilon$ and $1/\varepsilon^2$ divergences. Some of these will be consequences of known divergences from matter inclusion at L=1, so the implications of this result are still under debate. More clear is the D=5 situation: in addition to the cancellations for $n_v = 0$ (pure half-maximal supergravity), there is just *one* other case with 2-loop cancellations: $n_v = 5$, giving precisely the dimensionally reduced content of off-shell D=10, N=1 supergravity.
- Out just today: pure half-maximal supergravity at L=4 loops has divergences $\sim \partial^2 R^4$. In D=4, there is a clearly expected duality invariant full superspace counterterm $\int d^{16}\theta E \chi^i \bar{\chi}_i \chi^j \bar{\chi}_j$. End of the miracles story? Maybe not: the one-loop U(1) anomaly can now be causing 4-loop divergences. So is this result purely caused by the anomaly?

Bern, Davies & Dennen, 1305.4876;
Bern, Davies, Dennen, Smirnov & Smirnov, out today