Outlook

Objective

Fermions

Scalar and Fermion Fields

Fermions, radiation and matter

Einstein-Cartan theory

Fermions as sources of accelerated regimes

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Objective

- Analysis of cosmological models with fermionic fields
- Description of accelerated regimes for
  (a) Early Universe
  (b) Old Universe

Classical Spinors

- Dirac spinor is a four-component object ψ that obeys Dirac's equation
- Under a local Lorentz transformation with parameters $\lambda_{ab}(x)$

$$\boxed{\psi \longrightarrow \widetilde{\psi} = \exp\left\{\frac{1}{2}\lambda_{ab}(x)\Sigma^{ab}\right\}\psi}$$

  $\Sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$ generators of the spinor representation of the Lorentz group, with $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ satisfying Clifford algebra.

- Canonical approach to QFT, spinors are operators $\hat{\psi}$, that satisfies DE

$$\boxed{\imath\gamma^\mu\partial_\mu\hat{\psi} - m\hat{\psi} = 0}$$

- Classical spinor: expectation value in an appropriate state $|s\rangle$

$$\boxed{\psi_{\rm cl} \equiv \langle s|\hat{\psi}|s\rangle \equiv \langle\hat{\psi}\rangle}$$

- Classical spinor satisfies Dirac's equation

$$\boxed{\imath\gamma^\mu\partial_\mu\psi_{\rm cl} - m\psi_{\rm cl} = 0}$$

  Expectation value of a spinor in a physical state is a complex number, not a Grassmannian number! (Armendáriz-Picón – Greene 2003)

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- Point transformation

$$\boxed{t \longrightarrow t' = t + \epsilon\beta(q(t),t), \qquad q(t) \longrightarrow q'(t') = q(t) + \epsilon\alpha(q(t),t)}$$

- Transf. of gen. coord. $Q = Q(q)$ induces a transf. of gen. vel. $\dot{Q} = \frac{\partial Q}{\partial q}\dot{q}$
- Point transf. $Q = Q(q,\epsilon)$ induces a transf. represented by vector field

$$\boxed{\mathbf{X} = \alpha(q)\frac{\partial}{\partial q} + \dot{\alpha}(q)\frac{\partial}{\partial \dot{q}}}$$

  in the tangent space and called infinitesimal generator of symmetry
- Lagrangian $\mathcal{L}(q,\dot{q})$ is invariant under the transformation **X** if

$$\boxed{L_{\mathbf{X}}\mathcal{L} = \mathbf{X}\mathcal{L} = \alpha(q)\frac{\partial \mathcal{L}}{\partial q} + \dot{\alpha}(q)\frac{\partial \mathcal{L}}{\partial \dot{q}} = 0}$$

- If $L_{\mathbf{X}}\mathcal{L} = 0$, **X** represents the symmetry for the dynamics described by $\mathcal{L}$ and $\alpha\frac{\partial \mathcal{L}}{\partial \dot{q}}$ is a constant of motion

Fermionic fields non-minimally coupled

- Action

$$\boxed{S=\int\sqrt{-g}d^4x\left\{F(\Psi)R+\frac{\imath}{2}\left[\overline{\psi}\Gamma^\mu D_\mu\psi-(\overline{D_\mu\psi})\Gamma^\mu\psi\right]-V(\Psi)\right\}}$$

Dirac spinor field ψ, adjoint $\overline{\psi}=\psi^\dagger\gamma^0$

Ricci scalar R

Pauli-Dirac matrices $\Gamma^\mu=e^\mu_a\gamma^a$

Tetrad (vierbein) e^μ_a

Coupling function $F(\Psi)$

Self-interaction potential of the fermionic field $V(\Psi)$

Functions analyzed:

1. Bilinear $\Psi=\overline{\psi}\psi$
2. Pseudo-scalar $\Psi=\overline{\psi}\gamma^5\psi$

Covariant derivatives $D_\mu\psi=\partial_\mu\psi-\Omega_\mu\psi,\ \ D_\mu\overline{\psi}=\partial_\mu\overline{\psi}+\overline{\psi}\Omega_\mu,$

Spin connection $\Omega_\mu=-\frac{1}{4}g_{\sigma\nu}[\,\Gamma^\nu_{\mu\lambda}-e^\nu_b(\partial_\mu e^b_\lambda)\,]\,\Gamma^\sigma\Gamma^\lambda$

- Spatially flat FLRW metric $ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2)$

$$\boxed{e_0^\mu = \delta_0^\mu, \qquad e_i^\mu = \frac{1}{a(t)}\delta_i^\mu, \qquad \Gamma^0 = \gamma^0,}$$

$$\boxed{\Gamma^i = \frac{1}{a(t)}\gamma^i, \qquad \Omega_0 = 0, \qquad \Omega_i = \frac{1}{2}\dot{a}(t)\gamma^i\gamma^0}$$

- Point-like Lagrangian

$$\boxed{\mathcal{L} = 6a\dot{a}^2 F + 6a^2\dot{a}\dot{\Psi}\frac{dF}{d\Psi} + \frac{\imath}{2}a^3(\dot{\overline{\psi}}\gamma^0\psi - \overline{\psi}\gamma^0\dot{\psi}) + a^3 V}$$

- Euler-Lagrange equations for ψ and $\overline{\psi} \Longrightarrow$ Dirac equations

$$\boxed{\dot{\psi} + \frac{3}{2}H\psi + \imath\gamma^0\psi\frac{dV}{d\Psi} - \imath 6(\dot{H} + 2H^2)\gamma^0\psi\frac{dF}{d\Psi} = 0}$$

$$\boxed{\dot{\overline{\psi}} + \frac{3}{2}H\overline{\psi} - \imath\overline{\psi}\gamma^0\frac{dV}{d\Psi} + \imath 6(\dot{H} + 2H^2)\overline{\psi}\gamma^0\frac{dF}{d\Psi} = 0}$$

- Euler-Lagrange equations for $a \Longrightarrow$ acceleration equation

$$\boxed{\frac{\ddot{a}}{a} = -\frac{\rho_f + 3p_f}{12F}}$$

- Energy density and pressure of the fermionic field

$$\boxed{\rho_f = V - 6H\frac{dF}{d\Psi}\dot{\Psi}}$$

$$\boxed{p_f = \frac{dV}{d\Psi}\Psi - V - 6(\dot{H} + 2H^2)\frac{dF}{d\Psi}\Psi + 2\left(\frac{dF}{d\Psi}\ddot{\Psi} + 2H\frac{dF}{d\Psi}\dot{\Psi} + \frac{d^2F}{d\Psi^2}\dot{\Psi}^2\right)}$$

- Imposing that the energy function $E_{\mathcal{L}} = 0 \Longrightarrow$ Friedmann equation

$$\boxed{E_{\mathcal{L}} = \frac{\partial\mathcal{L}}{\partial\dot{a}}\dot{a} + \frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\dot{\varphi} - \mathcal{L} \equiv 0} \Longrightarrow \boxed{H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho_f}{6F}}$$

Fermionic fields non-minimally coupled:

- Infinitesimal generator of symmetry $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$

$$\boxed{\mathbf{X} = C_0 \frac{\partial}{\partial a} + \dot{C}_0 \frac{\partial}{\partial \dot{a}} + \sum_{l=1}^{4} \left(C_l \frac{\partial}{\partial \psi_l^\dagger} + D_l \frac{\partial}{\partial \psi_l} + \dot{C}_l \frac{\partial}{\partial \dot{\psi}_l^\dagger} + \dot{D}_l \frac{\partial}{\partial \dot{\psi}_l} \right)}$$

$$\boxed{(C_0, C_l, D_l) = \mathcal{F}(a, \psi_l^\dagger, \psi_l)}$$

- Noether symmetry is satisfied if $\mathbf{X}\,\mathcal{L} = 0 \implies$
- Coupled system of 55 differential equations $\implies$

$$C_0 F + 2a \frac{\partial C_0}{\partial a} F + a^2 F' \sum_{j=1}^{4} \left(\frac{\partial C_j}{\partial a} \epsilon_j \psi_j + \frac{\partial D_j}{\partial a} \epsilon_j \psi_j^\dagger \right)$$

$$+ aF' \sum_{j=1}^{4} (C_j \epsilon_j \psi_j + D_j \epsilon_j \psi_j^\dagger) = 0,$$

$$F' \epsilon_j \psi_j \left(2C_0 + a \frac{\partial C_0}{\partial a} \right) + aF'' \epsilon_j \psi_j \sum_{i=1}^{4} (C_i \psi_i + D_i \psi_i^\dagger) \epsilon_i + aF' D_j \epsilon_j$$

$$+ 2F \frac{\partial C_0}{\partial \psi_j^\dagger} + aF' \sum_{i=1}^{4} \left(\frac{\partial C_i}{\partial \psi_j^\dagger} \epsilon_i \psi_i + \frac{\partial D_i}{\partial \psi_j^\dagger} \epsilon_i \psi_i^\dagger \right) = 0,$$

Fermionic fields non-minimally coupled:

$$F'\epsilon_j\psi_j^\dagger\left(2C_0+a\frac{\partial C_0}{\partial a}\right)+aF''\epsilon_j\psi_j^\dagger\sum_{i=1}^{4}(C_i\psi_i+D_i\psi_i^\dagger)\epsilon_i+aF'C_j\epsilon_j$$

$$+2F\frac{\partial C_0}{\partial \psi_j}+aF'\sum_{i=1}^{4}\left(\frac{\partial C_i}{\partial \psi_j}\epsilon_i\psi_i+\frac{\partial D_i}{\partial \psi_j}\epsilon_i\psi_i^\dagger\right)=0,$$

$$F'\left(\frac{\partial C_0}{\partial \psi_j^\dagger}\epsilon_i\psi_i+\frac{\partial C_0}{\partial \psi_i^\dagger}\epsilon_j\psi_j\right)=0,\qquad F'\left(\frac{\partial C_0}{\partial \psi_j}\epsilon_i\psi_i^\dagger+\frac{\partial C_0}{\partial \psi_i}\epsilon_j\psi_j^\dagger\right)=0,$$

$$F'\left(\frac{\partial C_0}{\partial \psi_j}\epsilon_i\psi_i+\frac{\partial C_0}{\partial \psi_i^\dagger}\epsilon_j\psi_j^\dagger\right)=0,\qquad \sum_{j=1}^{4}\left(\frac{\partial C_j}{\partial a}\psi_j-\frac{\partial D_j}{\partial a}\psi_j^\dagger\right)=0,$$

$$3C_0\psi_j+aD_j+a\sum_{i=1}^{4}\left(\frac{\partial C_i}{\partial \psi_j^\dagger}\psi_i-\frac{\partial D_i}{\partial \psi_j^\dagger}\psi_i^\dagger\right)=0,$$

$$3C_0\psi_j^\dagger+aC_j-a\sum_{i=1}^{4}\left(\frac{\partial C_i}{\partial \psi_j}\psi_i-\frac{\partial D_i}{\partial \psi_j}\psi_i^\dagger\right)=0.$$

Rest equality used for the determination of the potential density

$$3C_0V+aV'\sum_{j=1}^{4}\left(C_j\epsilon_j\psi_j+D_j\epsilon_j\psi_j^\dagger\right)=0.$$

Fermionic fields non-minimally coupled:

Result: $\boxed{C_0 = \frac{k}{a^{1/2}}}$ $\boxed{C_j = -\frac{3}{2}k\frac{\psi_j^\dagger}{a^{3/2}} + \beta\epsilon_j\psi_j^\dagger}$ $\boxed{D_j = -\frac{3}{2}k\frac{\psi_j}{a^{3/2}} - \beta\epsilon_j\psi_j}$

$$\boxed{V = \lambda\Psi} \Rightarrow \begin{cases} (a) F = \text{constant} \\ (b) F = \alpha\Psi \end{cases}$$

1. Bilinear $\Psi = \overline{\psi}\psi$
2. Pseudo-scalar $\Psi = \overline{\psi}\gamma^5\psi$

- From Dirac equations

$$\boxed{\dot{\Psi} + 3H\Psi = 0} \quad \Longrightarrow \quad \boxed{\Psi = \frac{\Psi_0}{a^3}}$$

- Case (a) $F =$ constant=1/2 $\Longrightarrow$
  fermionic field behaves as pressureless matter field

$$\boxed{a(t) = [K(t-t_0)]^{2/3}} \quad \boxed{\rho_f = \frac{m\Psi_0}{a^3}} \quad \boxed{p_f = 0}$$

- Case (b) $F = \alpha\Psi \Longrightarrow$
  fermionic field behaves as inflaton

$$\boxed{a(t) = \exp\left[\mathcal{K}(t-t_0)\right]} \quad \boxed{\rho_f = -\frac{m\Psi_0}{2a^3}} \quad \boxed{p_f = -\rho_f}$$

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- Action

$$\boxed{S=\int d^4x\sqrt{-g}\left\{\frac{R}{2}+\mathcal{L}_\psi+\mathcal{L}_\phi+\mathcal{L}_M\right\}}$$

Dirac Lagrangian

$$\boxed{\mathcal{L}_\psi=\frac{\imath}{2}\left[\overline{\psi}\Gamma^\mu D_\mu\psi-(\overline{D_\mu\psi})\Gamma^\mu\psi\right]-V(\Psi)}$$

Scalar field Lagrangian

$$\boxed{\mathcal{L}_\phi=\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-U(\phi)}$$

Matter field Lagrangian $\mathcal{L}_M$

- Point-like Lagrangian

$$\boxed{\mathcal{L}=3a\dot{a}^2-a^3\left(\frac{\dot{\phi}^2}{2}-U\right)+\frac{\imath}{2}a^3(\dot{\overline{\psi}}\gamma^0\psi-\overline{\psi}\gamma^0\dot{\psi})+a^3V+\rho_M^0}$$

- Dirac equations

$$\boxed{\dot{\psi}+\frac{3}{2}\left(\frac{\dot{a}}{a}\right)\psi+\imath\gamma^{0}\psi\frac{dV}{d\Psi}=0} \qquad \boxed{\dot{\overline{\psi}}+\frac{3}{2}\left(\frac{\dot{a}}{a}\right)\overline{\psi}-\imath\overline{\psi}\gamma^{0}\frac{dV}{d\Psi}=0}$$

- Klein-Gordon equation

$$\boxed{\ddot{\phi}+3\frac{\dot{a}}{a}\dot{\phi}+\frac{dU}{d\phi}=0}$$

- Acceleration and Friedmann equations

$$\boxed{\frac{\ddot{a}}{a}=-\frac{\rho_M+\rho_\psi+\rho_\phi+3(p_\psi+p_\phi)}{6}} \qquad \boxed{\left(\frac{\dot{a}}{a}\right)^2=\frac{\rho_M+\rho_\psi+\rho_\phi}{3}}$$

- Energy densities and pressures

$$\boxed{\rho_\psi=V} \qquad \boxed{\rho_\phi=\frac{1}{2}\dot{\phi}^2+U} \qquad \boxed{p_\psi=\Psi\frac{dV}{d\Psi}-V} \qquad \boxed{p_\phi=\frac{\dot{\phi}^2}{2}-U}$$

- Existence of Noether symmetry $\Longrightarrow$

$$\boxed{U = U_0(Ae^{\alpha\phi} - Be^{-\alpha\phi})^2} \qquad \boxed{V = V_0\Psi}$$

- Fermionic field behaves as pressureless matter field

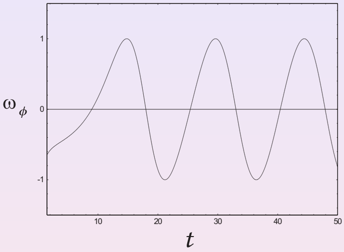
$$\boxed{\rho_\psi = m\Psi} \qquad \boxed{p_\psi = 0}$$

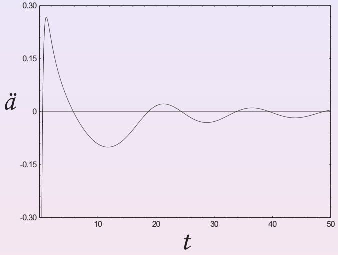
- Solutions of Friedmann and Klein-Gordon equation together with the constant of motion equation $\Longrightarrow$ oscillatory caracter

$$\boxed{a(t) = \left\{\frac{\omega^2}{2U_0}\left[z_1^2t^2 + 2z_1z_2t + z_2^2 - u_0^2\sin^2(\omega t + b_0)\right]\right\}^{\frac{1}{3}}}$$

$$\boxed{\phi(t) = \frac{1}{2\alpha}\ln\frac{B}{A}\left\{\frac{z_1t + z_2 + u_0\sin(\omega t + b_0)}{z_1t + z_2 - u_0\sin(\omega t + b_0)}\right\}}$$

$$\boxed{\psi(t) = \frac{\sqrt{2U_0}}{\omega}\begin{pmatrix}\psi_1^0e^{-\imath mt}\\ \psi_2^0e^{-\imath mt}\\ \psi_3^0e^{\imath mt}\\ \psi_4^0e^{\imath mt}\end{pmatrix} 1/\sqrt{z_1^2t^2 + 2z_1z_2t + z_2^2 - u_0^2\sin^2(\omega t + b_0)}}$$





Left: Barotropic index $\omega_\phi = p_\phi/\rho_\phi$ vs. time
Right: Acceleration vs. time

Fermion, radiation and matter fields:

- Point-like Lagrangian: $V = V(\Psi = \overline{\psi}\gamma^5\psi)$, $F = F(\Psi = \overline{\psi}\gamma^5\psi)$

$$\boxed{\mathcal{L} = 6a\dot{a}^2 F + 6a^2\dot{a}\dot{\Psi}\frac{dF}{d\Psi} + a^3\frac{i}{2}\left(\dot{\overline{\psi}}\gamma^0\psi - \overline{\psi}\gamma^0\dot{\psi}\right) + a^3 V + \rho_m^0 + \frac{\rho_r^0}{a},}$$

- Friedmann equation

$$\boxed{H^2 = \frac{\rho}{6F}} \qquad \boxed{\rho = V - 6H\dot{\Psi}\frac{dF}{d\Psi} + \frac{\rho_m^0}{a^3} + \frac{\rho_r^0}{a^4}}$$

- Noether symmetry: $\Longrightarrow$ F and V arbitrary functions of $\Psi = \overline{\psi}\gamma^5\psi$
- Dirac's equations

$$\boxed{\Psi = \frac{\Psi_0}{a^3}}$$

- Analyzed functions

$$\boxed{V = \Lambda + V_0\Psi = \Lambda + \frac{V_0\Psi_0}{a^3}} \qquad \boxed{F = \frac{1}{2}(1 - \xi\Psi) = \frac{1}{2}\left(1 - \frac{\xi\Psi_0}{a^3}\right)}$$

Fermion, radiation and matter fields:

- For large values of a the non-minimmally coupling dilutes

$$\boxed{F = \frac{1}{2}}$$

$$\boxed{\rho_f = V_0\Psi + \Lambda = \rho_{dm} + \rho_{de}}$$

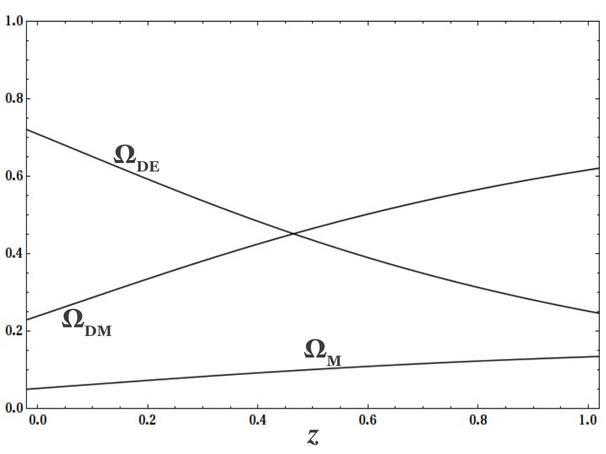
- Fermion field as dark matter and energy
- Numerical solution of the Friedmann equation in terms of the redshift
- Initial conditions

$$\boxed{\Omega_m^0 = 0.0463}$$

$$\boxed{\Omega_{dm}^0 = 0.233}$$

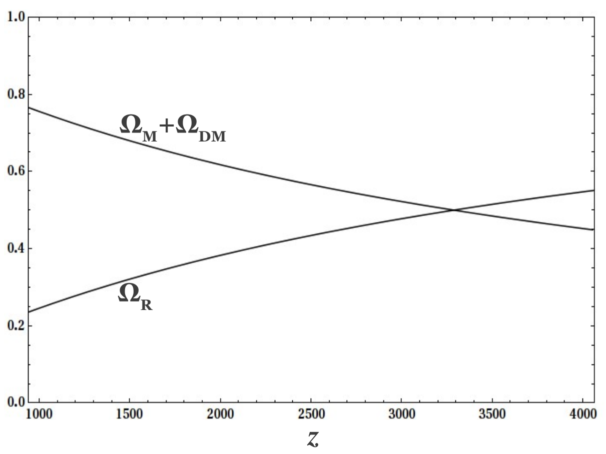
$$\boxed{\Omega_{de}^0 = 0.721}$$

$$\boxed{\Omega_r^0 = 8.5 \times 10^{-5}}$$



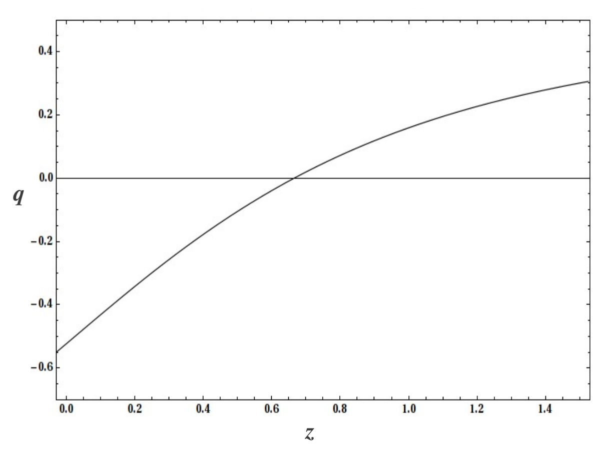
Density parameters for $0 \leq z \leq 1$

Fermion, radiation and matter fields:



Density parameters for $1000 \leq z \leq 4000$

$z_{eq} \approx 3250$ $\quad\left(z_{eq} = 3265^{+106}_{-105}\right)$



Deceleration parameter as a function of the red-shift

$q(0) = -0.55 \quad \left(q(0) = -0,56^{+0,35}_{-0,22}\right) \quad z_T = 0.67 \quad \left(z_T = 0,77^{+0,52}_{-0,57}\right)$

Fermion fields in Einstein-Cartan theory

- Action with $f(R)$

$$\boxed{S = \int \mathsf{e}\, d^4x \left\{ \frac{1}{16\pi G} f(R) - \frac{\imath}{2}\left[\overline{\psi}\gamma^\mu D_\mu\psi - D_\mu\overline{\psi}\gamma^\mu\psi\right] + m\overline{\psi}\psi + V(\overline{\psi}\psi) \right\}}$$

where $\mathsf{e} = \det(\mathsf{e}_a^\mu)$ and the covariant derivatives

$$\boxed{D_\mu\psi = \partial_\mu\psi + \frac{1}{8}\omega_\mu{}^{ab}\left[\gamma_a,\gamma_b\right]\psi, \qquad D_\mu\overline{\psi} = \partial_\mu\overline{\psi} - \frac{1}{8}\omega_\mu{}^{ab}\overline{\psi}\left[\gamma_a,\gamma_b\right]}$$

spin connection $\omega_\mu{}^{ab}$ and Riemmann curvature tensor

$$\boxed{R_{\mu\nu}{}^{ab} = \eta^{cb}\mathsf{e}_\alpha^a\mathsf{e}_c^\beta R^\alpha{}_{\beta\mu\nu} = \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} + \omega_\mu{}^a{}_c\omega_\nu{}^{cb} - \omega_\nu{}^a{}_c\omega_\mu{}^{cb}}$$

- Variation of the action with respect to the tetrad e_a^μ

$$\boxed{f'(R)R_{\mu\nu} - \frac{f(R)}{2}g_{\mu\nu} = 8\pi G\mathcal{T}_{\mu\nu}}$$

prime differentiation with respect to R, energy-momentum tensor

$$\mathcal{T}_{\mu\nu} = \frac{\imath}{2}\left[\overline{\psi}\gamma_\mu D_\nu\psi - D_\nu\overline{\psi}\gamma_\mu\psi - \left(\overline{\psi}\gamma^\sigma D_\sigma\psi - D_\sigma\overline{\psi}\gamma^\sigma\psi\right)g_{\mu\nu}\right] + \left[m\overline{\psi}\psi + V(\overline{\psi}\psi)\right]g_{\mu\nu}$$

- Dirac's equations: variation with respect to $\overline{\psi}$, ψ

$$\boxed{\imath D_\mu \overline{\psi}\gamma^\mu + \frac{\imath}{2} T_\mu \overline{\psi}\gamma^\mu + m\overline{\psi} + \frac{dV(\overline{\psi}\psi)}{d\psi} = 0}$$

$$\boxed{\imath \gamma^\mu D_\mu \psi + \frac{\imath}{2} T_\mu \gamma^\mu \psi - m\psi - \frac{dV(\overline{\psi}\psi)}{d\overline{\psi}} = 0}$$

Torsion tensor $\boxed{T_{\mu\nu}{}^{\sigma} = \Gamma_{\mu\nu}{}^{\sigma} - \Gamma_{\nu\mu}{}^{\sigma}}$ Contraction: $\boxed{T_\mu = T_{\mu\nu}{}^{\nu}}$

- Variation with respect to the spin connection $\omega_\mu{}^{ab}$

$$\boxed{-\frac{1}{2}\left(\delta^\nu_\sigma \delta^\mu_\tau - \delta^\nu_\tau \delta^\mu_\sigma\right) \partial_\nu f'(R) + f'(R)\, T_{\tau\sigma}{}^{\mu} = 4\pi G\, g_{\tau\lambda}\, g_{\sigma\theta}\, \epsilon^{\nu\mu\lambda\theta}\, \overline{\psi}\, \gamma_\nu\, \gamma^5\, \psi}$$

- For $\boxed{f(R) = R}$ $\Longrightarrow$

$$\boxed{T_{\tau\sigma}{}^{\mu} = 4\pi G\, g_{\tau\lambda}\, g_{\sigma\theta}\, \epsilon^{\nu\mu\lambda\theta}\, \overline{\psi}\gamma_\nu\gamma^5\psi} \qquad \boxed{T_\mu = 0}$$

- Massive fermionic field in spatially flat FLRW metric

- Friedmann and acceleration equations

$$\boxed{\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho} \qquad \boxed{\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3p)}$$

- Energy density and pressure

$$\boxed{\rho = m\overline{\psi}\psi - \frac{3\pi G}{2}\sigma^2} \qquad \boxed{p = -\frac{3\pi G}{2}\sigma^2} \qquad \boxed{\sigma^2 = (\overline{\psi}\gamma_5\gamma_d\psi)^2}$$

- Dirac's equations

$$\boxed{\dot\psi + \frac{3}{2}\frac{\dot a}{a}\psi + \imath m\gamma^0\psi = 3\pi G\imath\gamma^0\left(\overline{\psi}\gamma_5\gamma^i\psi\right)(\gamma_5\gamma_i\psi)}$$

$$\boxed{\dot{\overline{\psi}} + \frac{3}{2}\frac{\dot a}{a}\overline{\psi} - \imath m\overline{\psi}\gamma^0 = -3\pi G\imath\left(\overline{\psi}\gamma_5\gamma^i\psi\right)\left(\overline{\psi}\gamma_5\gamma_i\right)\gamma^0}$$

- Numerical solution of the acceleration equation and Dirac equation
- Transition accelerated-decelerated for $\frac{3}{2}\sigma^2 < \frac{m}{\pi G}\overline{\psi}\psi < 6\sigma^2$
- By increasing the fermionic mass only decelerated regime

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