Solutions for exercise session III

— Answer 1 —

Let Q be the set of rooted and pointed planar quadrangulations ~~with n faces.~~
Recall that the image of Q_n by the CVS bijection, denoted $\mathcal{T}$, is the set of well-labeled trees ~~having n edges~~, such that

① all labels are ≥ 1
② there is at least one label $= 1$.

Moreover, the vertices of the tree are exactly the vertices of the non-pointed vertices of the quadrangulation. And these vertices have the same label in the quadrangulation and in the tree: $l(v) = d(v_0, v)$.

Now let $Q_i = \{ q \in Q \mid d(v_0, v_1) \leq i \}$ where v_1 is defined in the statement above Question 1.
Apply the CVS bijection to Q_i.
Since v_1 becomes the root vertex of the tree associated to Q, the image of $Q_{n,i}$ by the CVS bijection is

$$\tilde{\mathcal{T}}_i = \{ (T, l) \in \mathcal{T} \mid l(v_1) \leq i \}$$

We define $\mathcal{T}_i = \{ (T,l) \mid (T,l)$ is a well-labeled tree ~~having n edges~~, rooted at v_1, and s.t $l(v_1) = i$, $\forall v, l(v) \geq 1 \}$

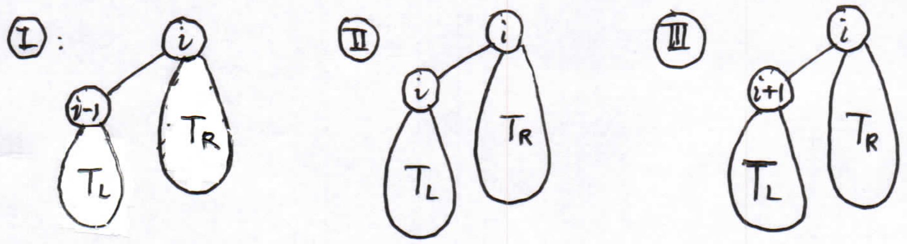
Thanks to condition ② on the labeling of trees in $\mathcal{T}_{n,i}$, the change of labeling $l(v) \leftarrow l(v) + (i - l(v_1))$ defines a bijection from $\tilde{\mathcal{T}}_i$ to $\mathcal{T}_i$.

We denote by $\mathcal{T}_{i,n,m}$ the subset of $\mathcal{T}_i$ of trees with n edges and having m local maxima in its labeling.
Since the CVS bijection turns each face of the quadrangulation into an edge of the tree, and preserves the number of local maxima,
we have
$$T_i(g,h) = \sum_{n \geq 0, m \geq 1} (\# \mathcal{T}_{i,n,m}) \, g^n h^m$$

— Answer 2 —

A natural way to decompose a rooted planar tree consists of cutting the left-most edge from the root vertex. For a labeled tree with root-label i, we can separate 3 cases according to the label of the left-most child of the root.

3 cases :



In each of the 3 cases, denote by n_1 (resp n_2) the number of edges in T_L (resp. T_R), and by m_1 (resp. m_2) the number of locally maximal labels in T_L (resp. T_R).

We have $n_1+n_2=n-1$.

However, there is not a definite relation between m_1+m_2 and m, because we do not know if the root of T_L (or T_R) is a local maximum.

In other words, the decomposition of $\mathcal{T}_{i,n,m}$ yields other classes of trees, and the recursion cannot be closed.

To copple with this situation, we introduce $\underline{\underline{\mathcal{U}_{i,n,m}}}$, the set of well-labeled trees defined in the same way as $\mathcal{T}_{i,n,m}$, except for that m is the number of local maximal labels <u>without counting the root vertex</u>.

Then, we can check the following bijection

$$\forall n \geq 1 \quad \mathcal{T}_{i,n,m} = \{\text{I}\} \quad \sqcup \quad \{\text{II}\} \quad \sqcup \quad \{\text{III}\}$$

$$\{\text{I}\} \leftrightarrows \left(\bigsqcup_{\substack{n_1+n_2=n-1 \\ m_1+m_2=m}} \mathcal{U}_{i-1,n_1,m_1} \times \mathcal{T}_{i,n_2,m_2} \right), \qquad \{\text{II}\} \,\|\, \left(\bigsqcup_{\substack{n_1+n_2=n-1 \\ m_1+m_2=m}} \mathcal{T}_{i,n_1,m_1} \times \mathcal{T}_{i,n_2,m_2} \right), \qquad \{\text{III}\} \leftrightarrows \left(\bigsqcup_{\substack{n_1+n_2=n-1 \\ m_1+m_2=m}} \mathcal{T}_{i+1,n_1,m_1} \times \mathcal{U}_{i,n_2,m_2} \right)$$

With the boundary cases

$\mathcal{T}_{i,0,1} = \{ (i) \}$ (tree reduced to the root)

$\forall m \geq 2 \quad \mathcal{T}_{i,0,m} = \emptyset$

we get

$$\# \mathcal{T}_{i,n,m} = \delta_{n,0}\,\delta_{m,1} + \sum_{\substack{n_1+n_2=n-1 \\ m_1+m_2=m}} \Big[(\# \mathcal{U}_{i-1,n_1,m_1}) \times (\# \mathcal{T}_{i,n_2,m_2}) + (\# \mathcal{T}_{i+1,n_1,m_1}) \times (\# \mathcal{T}_{i,n_2,m_2}) + (\# \mathcal{T}_{i+1,n_1,m_1}) \times (\# \mathcal{U}_{i,n_2,m_2}) \Big]$$

$$\xRightarrow{\sum_{n,m} g^n h^m \times} \quad T_i = h + g\left(U_{i-1} T_i + T_i^2 + T_{i+1} U_i \right)$$

$$\text{where} \quad U_i(g,h) = \sum_{n \geq 0,\, m \geq 1} (\# \mathcal{U}_{i,n,m})\, g^n h^m$$

Applying the same decomposition to $\mathcal{U}_{i,n,m}$, we get the other equation in (1).

— Answer 4 —

Recall that the generating functions R_i $(i \geq 1)$, defined by Eq. (1) of the exercise sheet II, counts the number of rooted and pointed planar quadrangulations in which the pointed vertex is at distance $\leq i-1$ from the end point of the root edge.

(R$_i$ are defined using the notion of slices in Sheet II. The above definition is equivalent to the slice definition thanks to the result of Question 2 of Sheet II).

Denote by v_0 the pointed vertex, and by v_1 the end point of the root edge.

Let $\mathcal{Q}_{n,i} = \{$ rooted and pointed quadrangulation Q with n faces s.t. $d(v_0, v_1) \leq i-1 \}$

then $R_i(g) = \sum_{n \geq 0} (\# \mathcal{Q}_{n,i})\, g^n$

For all $Q \in \mathcal{Q}_{n,1}$, we have $v_0 = v_1$. So $\mathcal{Q}_{n,1}$ is just the set of rooted quadrangulations with n faces. (The pointed vertex is determined by the root).

Then $\mathcal{Q}_{n,i}$ can be thought of the set of couples $(\tilde{Q}, \tilde{v}_0)$ where $\tilde{Q} \in \mathcal{Q}_{n,1}$ and $\tilde{v}_0$ is a vertex in the ball of radius $i-1$ around the root.

Therefore $\dfrac{[t^n]R_i(t)}{[t^n]R_1(t)} = \dfrac{\# \mathcal{Q}_{n,i}}{\# \mathcal{Q}_{n,1}} =$ the expected number of vertices in the ball of radius $i-1$ around the root in a uniform rooted planar quadrangula° with n faces.

in short, the expected volume of a ball in the uniform quadrangulation of size n.

— Answer 5 —

Recall the "transfer theorem" for equivalence relations:

Thm [Section VI.3, "the book" of combinatorics, Flajolet and Sedgewick]

Let $f(z)$ be a power series in z of radius of convergence $\xi > 0$

Assume that $z_c = \xi$ is the only singularity of $f(z)$ on $\underbrace{\{|z| = \xi\}}_{\text{the circle}}$.

and that $f(z) \underset{z \to \xi}{\sim} \left(1 - \frac{z}{\xi}\right)^{\alpha}$ for some $\alpha \in \mathbb{R} \setminus \{0, 1, 2, \dots\}$

then, under some technical conditions (the analyticity of f on some domain larger than the disk $\{|z| < \xi\}$)

we have $[z^n] f(z) \underset{n \to \infty}{\sim} \dfrac{1}{\Gamma(-\alpha)} n^{-\alpha-1} \xi^{-n}$

To obtain the asymptotics of $[t^n] R_i(t)$ with this theorem, we need to compute the first non-polynomial term in the expansion of $R_i(t)$ around its singularity.

We have: $$R_i = R\,\frac{(1-x^i)(1-x^{i+3})}{(1-x^{i+1})(1-x^{i+2})} \qquad (3)$$

where R and x are determined by $$R = 1+3tR^2 \qquad (4.1)$$ $$x + \frac{1}{x} = \frac{4-R}{R-1} \qquad (4.2)$$

((4.1), (4.2) are equivalent to "$R = \frac{1}{6t}(1-\sqrt{1-12t})$" and (4))

Step 1. determine the "critical values":

↳ The discriminant of (4.1) $\Delta = 1-12t \Rightarrow t_c = \frac{1}{12}$

↳ $t \uparrow t_c \xRightarrow[(4.1)]{} R \uparrow R_c = 2 \xRightarrow[(4.2)]{} x \to x_c = 1.$

↳ $R \to 2$, $x \to 1 \Rightarrow R_i \to R_{i,c} = 2\,\frac{(i+3)i}{(i+1)(i+2)}$

Step 2. Let $\delta t = t_c - t = \frac{1}{12} - t$. And replace $\begin{cases} t \\ R \\ x \end{cases}$ by $\begin{cases} \delta t \\ \delta R \\ \delta x \end{cases}$ in

$\delta R = R_c - R = 2 - R$

$\delta x = x_c - x = 1 - x$

equations (4.1) and (4.2).

Solve (4.1) for $\delta R(\delta t)$ and solve (4.2) for $\delta x(\delta R)$, and compute the expansion of the solutions, we get with the help of a computer algebra system:

$$\delta R = 4\sqrt{3}\cdot\sqrt{\delta t} - 24\cdot\delta t + \cdots + o(\delta t^2)$$

$$\delta x = \sqrt{3}\cdot\sqrt{\delta R} - \frac{3}{2}\cdot\delta R + \cdots + o(\delta R^4)$$

Step 3: We expand R_i as a power series in δx and δR, then substitute δx and then δR by their expressions shown above. Again with a software, we get

$$R_i(t) = R_i(\tfrac{1}{12}) + R_i'(\tfrac{1}{12})\cdot\delta t + C(i)\cdot(\delta t)^{3/2} + O(\delta t^2)$$

where $$C(i) = \frac{24}{35}\sqrt{3}\cdot\frac{i(i+3)}{(i+1)(i+2)}\cdot(5i^4 + 30i^3 + 59i^2 + 42i + 4)$$

Now we apply the theorem with $\alpha = 3/2$, $\xi = \frac{1}{12}$

$$\Rightarrow \frac{[t^n]R_i(t)}{[t^n]R_1(t)} \xrightarrow[n\to\infty]{} \frac{C(i)}{C(1)} = \frac{3}{280}\,\frac{i(i+3)}{(i+1)(i+2)}(5i^4+30i^3+59i^2+42i+4)$$

$$\underset{i\to\infty}{\sim} \frac{3}{56} i^4$$