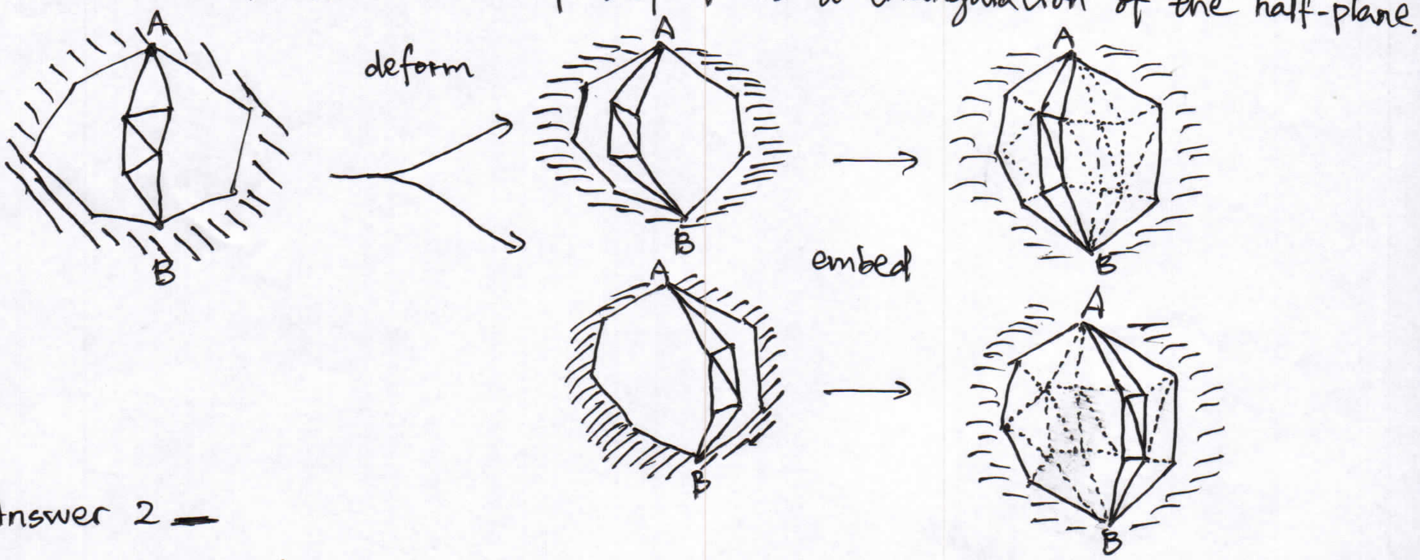
Solutions for exercise sheet IV

— Answer 1 —

Let V be a half-plane triangulation with boundary.
We give a proof only for the easier implication, namely

$$V \text{ is rigid} \Rightarrow \left| \begin{array}{l} \text{the union of the faces of } V \text{ together with the} \\ \text{lower half-plane is 3-connected.} \end{array} \right.$$

Assume that the union on the RHS is not 3-connected, then we can find two points A and B whose removal disconnects the union.
Consider the finite component which is disconnected by the removal of A and B.
It can be deformed to 2 non-overlapping positions, this shows that we can embed two distinct copies of V is a triangulation of the half-plane.



— Answer 2 —

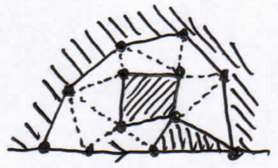
Here $E(V, n_1, \dots, n_K)$ is a subset of all the triangulations with a (finite or infinite) simple boundary. First, we seek to describe this set when the length of the boundary is assumed to be m.
We claim that there is the following bijection

$$g: \begin{array}{ccc} M_{n_0, m+\delta} \times M_{n_1, m_1} \times \cdots \times M_{n_K, m_n} & \longrightarrow & E(V, n_1 \cdots n_K) \cap M_{n,m} \\ (T_0, T_1, \cdots, T_K) & \longmapsto & T = g(T_0 \cdots T_K) \end{array}$$

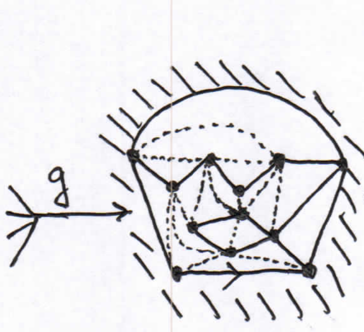
where $n_0 = n - N - \sum_{i=1}^{k} n_i$,

and T is the triangulation of the m-gon obtained by "glueing" $T_1 \cdots T_K$ into the K holes of V, and glueing T_0 to the exterior of V

$\boxed{\text{Example}}$

$V =$  $\delta = 3$, $N = 8$

$T_0 =$  $n_0 = 0$



$= T$, $n=8$, $m=4$

$T_1 =$  $n_1 = 0$

$T_2 =$  $n_2 = 0$

The surjectivity of g is obvious from the construction.

→ g is injective thanks to the rigidity of V, since $T = g(T_0, T_1 \cdots T_k)$ gives an embedding of V into T. By rigidity, this embedding, as well as the hole components $T_0, T_1 \cdots T_k$, are unique.

Recall that μ_m is supported by the set $\bigcup_{n\geq 0} M_{n,m}$ (i.e. $\mu_m(\bigcup_{n\geq 0} M_{n,m}) = 1$) and $\theta_{n,m}$ by the set $M_{n,m}$.

Therefore

$$\mu_m(E(V, n_1 \cdots n_k)) = \sum_{n_0 \geq 0} \underbrace{\sum_{T_0, T_1 \cdots T_k}}_{\text{sum over } M_{n_0, m+\delta} \times \prod_{i=1}^{k} M_{n_i, m_i}} \underbrace{\mu_m(g(T_0, T_1 \cdots T_k))}_{= \frac{\rho^{-n}}{Z_m} \text{ by definition of } \mu_m}$$

$$= \sum_{n_0 \geq 0} \phi_{n_0, m+\delta} \cdot \prod_{i=1}^{k} \phi_{n_i, m_i} \cdot \frac{\rho^{-n_0 - N - \sum_{i=1}^{k} n_i}}{Z_m}$$

$$= \frac{1}{Z_m} \underbrace{\left(\sum_{n_0 \geq 0} \phi_{n_0, m+\delta} \rho^{-n_0} \right)}_{Z_{m+\delta}} \prod_{i=1}^{k} \rho^{-n_i} \phi_{n_i, m_i}$$

$$\theta_{n,m}(E(V, n_1, \ldots, n_k)) = \sum_{T_0 \cdots T_k} \mu(g(T_0 \cdots T_k))$$

$$= \frac{\phi_{n_0, m+\delta} \prod_{i=1}^{k} \phi_{n_0, m_i}}{\phi_{n,m}} \qquad \left(n_0 = n - N - \sum_{i=1}^{k} n_i\right)$$

From (3) we get $Z_m \underset{m\to\infty}{\sim} \frac{1}{36\sqrt{\pi}} m^{-5/2} \alpha^m \qquad (\alpha = 9)$

So $$\mu_m(E(V, n_1 \cdots n_k)) \xrightarrow[m\to\infty]{} \alpha^{\delta} \rho^{-N} \prod_{i=1}^{k} \rho^{-n_i} \phi_{n_i, m_i}$$

Similarly, using (1), (2):

$$\theta_{n,m}(E(V, n_1 \cdots n_k)) \xrightarrow[n\to\infty]{} \frac{C_{m+\delta}}{C_m} \rho^{-N - \sum_{i=1}^{k} n_i} \prod_{i=1}^{k} \phi_{n_i, m_i}$$

$$\xrightarrow[m\to\infty]{} \alpha^{\delta} \rho^{-N} \prod_{i=1}^{k} \rho^{-n_i} \phi_{n_i, m_i}$$

Remark: this equality of limits shows the equivalence between the microcanonical ensemble $(\theta_{n,m})$ and the canonical ensemble (μ_m).

— Answer 3 —

By one-endedness of the UIHPT, the numbers $n_1, \ldots, n_k$ are finite with probability 1 under the measure ν.

Therefore $$\nu(V \subset T) = \sum_{n_1 \geq 0} \cdots \sum_{n_k \geq 0} \nu(E(V, n_1, \ldots, n_k))$$

$$= \alpha^{\delta} \rho^{-N} \prod_{i=1}^{k} \left(\sum_{n_i = 0}^{\infty} \rho^{-n_i} \phi_{n_i, m_i} \right) = \alpha^{\delta} \rho^{-N} \prod_{i=1}^{k} Z_{m_i}$$

— Answer 4+5 —

The assertion in these 2 questions can be proved by computing explicitly the probabilities of some events under the law ν, and observe that these probabilities are products of the law of free triangulations of the m_i-gon and the law of a UIHPT.

However, the computation is more straight-forward for the law μ_m.

Consider the event $\tilde{E}(V, T_0, T_1 \cdots T_k)$ of having $V \subset T$, and observing the triangulations $T_0, T_1, \dots, T_k$ in the $k+1$ connected components of $T \setminus V$ (1 outer component + k holes).

Then $$\mu_m(\tilde{E}(V, T_0, T_1 \cdots T_k)) = \frac{\rho^{-n}}{Z_m} \quad \text{for } n = n_0 + N + \sum_{i=1}^{k} n_i$$
$$m + \delta = m_0$$

By summing over all possible choices of $(T_0, \cdots T_k)$, we get

$$\begin{aligned}
\mu_m(V \subset T) &= \sum_{n_0 \cdots n_k \geq 0} \sum_{T_0 \cdots T_k} \mu_m(\tilde{E}(V, T_0 \cdots T_k)) \\
&= \sum_{n_0 \cdots n_k \geq 0} \phi_{n_0, m+\delta} \phi_{n_1 m_1} \cdots \phi_{n_k, m_k} \cdot \frac{\rho^{-n_0 - N - \sum_{i=1}^{k} n_i}}{Z_m} \\
&= \frac{\rho^{-N}}{Z_m} \prod_{i=1}^{k} \left(\sum_{n_i \geq 0} \rho^{-n_i} \phi_{n_i, m_i} \right) \cdot \sum_{n_0 \geq 0} \rho^{-n_0} \phi_{n_0, m+\delta} \\
&= \frac{\rho^{-N}}{Z_m} Z_{m+\delta} \cdot \prod_{i=1}^{k} Z_{m_i}
\end{aligned}$$

So the conditional probability to observe $T_0 \cdots T_k$, knowing that $V \subset T$ is

$$\begin{aligned}
&\mu_m(\tilde{E}(V, T_0 \cdots T_k) \mid V \subset T) \\
&= \frac{\mu_m(\tilde{E}(V, T_0 \cdots T_k))}{\mu_m(V \subset T)} = \frac{\rho^{-n_0}}{Z_{m+\delta}} \cdot \prod_{i=1}^{k} \frac{\rho^{-n_i}}{Z_{m_i}}
\end{aligned}$$

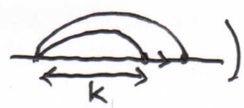
Therefore, under the measure μ_m and conditional to $\{V \subset T\}$, the components $(T_0, T_1 \cdots T_k)$ are independent free triangulations of the $(m+\delta)$-gon for the outer components, and of the m_i-gon for other components.

As $m \to \infty$, they remain independent, and the law of the outer component converges weakly to the law of a UIHPT (ν).

— Answer 6 —

Let the revealed triangle play the role of V (note that it is rigid), and apply the result from Question 3, we get

$\nu($  $) = \frac{\alpha}{\rho} = \frac{2}{3}$

$\forall k \geq 1 \quad \nu($  $) = \nu($  $) = \frac{Z_{k+1}}{\alpha^k}$

Note that k cannot be 0 because we have type-II triangulations (no loops).

— Answer 7 —

We have $\mathbb{P}(R=0) = \mathbb{P}\Big($  $\Big) + \sum_{k\geq 1} \mathbb{P}\Big($  $\Big) = \frac{1}{2}\left(1+\frac{\alpha}{\rho}\right) = \frac{5}{6}$

and $\forall k \geq 1 \quad \mathbb{P}(R=k) = \frac{Z_{k+1}}{\alpha^k} \underset{k\to\infty}{\sim} \frac{1}{4\sqrt{\pi}} k^{-5/2}$

(Use Stirling formula or $\binom{2k}{k} \underset{k\to\infty}{\sim} \frac{1}{\sqrt{\pi k}} 4^k$)

To compute $\mathbb{E}[R]$, we use the expansion $(1-4z)^{-1/2} = \sum_{k=0}^{\infty} \binom{2k}{k} z^k \quad (|z|<\frac{1}{4})$

$$\begin{aligned}
\mathbb{E}[R] &= \sum_{k=1}^{\infty} k\, \mathbb{P}(R=k) \\
&= \sum_{k=1}^{\infty} k \cdot \frac{1}{(k+1)k} \binom{2(k-1)}{k-1} \frac{1}{4^{k-1}} = \sum_{k=0}^{\infty} \frac{1}{k+2} \binom{2k}{k} 4^{-k} \\
&= 4 \cdot \left[\sum_{k=0}^{\infty} \binom{2k}{k} \int_0^{\frac{1}{4}} z^{k+1} dz \right] \\
&= 4 \int_0^{\frac{1}{4}} z \cdot (1-4z)^{-1/2} dz \qquad u = 1-4z \\
&= \int_0^1 \frac{1-u}{4} u^{-1/2} du \\
&= \frac{1}{4} \left[2u^{1/2} - \frac{2}{3} u^{3/2} \right]_0^1 = \frac{1}{3}
\end{aligned}$$

Remark: The asymptotics $\mathbb{P}(R=k) \underset{k\to\infty}{\sim} \frac{1}{4\sqrt{\pi}} k^{-5/2}$, or $\mathbb{P}(R\geq k) \underset{k\to\infty}{\sim} \frac{1}{6\sqrt{\pi}} k^{-3/2}$, shows that the (heavy-tailed) law of R is in the domain of attraction of the totally asymmetric $\frac{3}{2}$-stable distribution i.e. for an i.i.d. sequence (R_i) of the same law as R,

we have $\quad \frac{1}{n^{2/3}} \sum_{i=1}^{n} R_i \xrightarrow[n\to\infty]{\text{(distribution)}} S$

where S is the random variable s.t. $\mathbb{E}(e^{\lambda S}) = e^{\lambda^{3/2}}$ $\forall \lambda \geq 0$.

This asymptotic is important for the study of critical exponents of the percolations on the UIHPT.

— Answer 8 —

We proceed by induction on i. The assertion is obvious for $i=0$.

Assume it true for i. Let $\tilde{T}_i$ be the triangulation obtained by re-rooting T_i at the edge $a_i \to a_i+1$.

Since the law of a UIHPT is independent under ~~re~~ rooting (as is the case for μ_m), $\tilde{T}_i$ is also a UIHPT.

Now apply the result of Question 4 and 5 to $\tilde{T}_i$, we conclude that

$T_{i+1} = (\text{peeling of } \tilde{T}_i \text{ at the root}) = \mathrm{Peel}(T_i, a_i)$ $\boxed{\text{consequence of the induction hypothesis}}$

is a UIHPT independent from "$T_i \setminus T_{i+1}$"

Since "$P_{i+1} = P_i \cup (T_{i+1} \setminus T_i)$" and T_{i+1} is indep from P_i, P_{i+1} and T_{i+1} are indep.

— Answer 9 —

It suffices to show that R_0 is independent from T_1.

Indeed, since T_1 is a UIHPT, we can apply the same assertion to $T \leftarrow T_1$ to show that R_1 and T_2 are independent.

Then, by induction, we have $\forall n \geq 0$, R_n is independent from T_{n+1}.

But since $(R_{n+1}, R_{n+2}, \dots)$ are constructed from T_{n+1}, we have

$$\forall n \geq 0 \quad R_n \text{ is independent from } (R_{n+1}, R_{n+2}, \dots)$$

This implies that $(R_n)_{n \in \mathbb{N}}$ is an iid sequence.

(The "identically distributed" part is "iid" is obvious since the T_i are all UIHPT)

Now since R_0 is determined by P_0, and P_0 is independent from T_1, R_0 is independent from T_1.

— Answer 10 —

Let i_0 be the first instant at which the boundary of T_i is all black.

We can check that

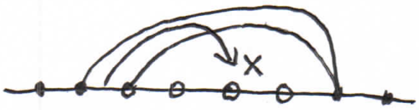
- The black-white-black form of the boundary is preserved by the peeling algorithm up to $i = i_0 - 1$.
- For all $i < i_0$, the choice of a_i is independent from T_i.

This shows that our algorithm defines a peeling process up to time i_0.

5 cases may occur at each step of the peeling process:

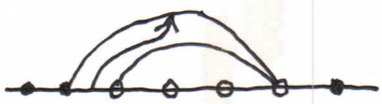
A black internal vertex is revealed

A white internal vertex is revealed



The revealed triangle swallows all the white vertices and terminates the peeling process.

The revealed triangle swallows some black vertices on the left.



The revealed triangle swallows some, but not all the white vertices.

In the 4 cases on the left, the arrow shows how the root edge is transferred between the triangulation of the half-plane before and after peeling.

It is not hard to see that the curve led by this arrow describes the <u>outer boundary of the white cluster containing the origine</u> (root vertex) in T_0.

Though a final portion of the boundary may be hidden in the swallowed component in the last peeling step and remains unexplored, it is still true that the white percolation cluster containing the root vertex is finite if and only if the peeling process stops in finite time i_0.

— Answer 11 + 12 —

By inspection of the 5 cases listed on the previous page, we see that the number of white boundary vertices in T_i is given by

$$S_i = 1 + \sum_{j=1}^{i-1} (\mathbb{1}_{\mathcal{E}_j} - R_i) \qquad \forall i < i_0$$

and that i_0 is the hitting time of the random walk S_i to the set $\{0, -1, -2, \dots\}$.

The fact that $(\mathbb{1}_{\mathcal{E}_i} - R_i)_{i<i_0}$ is an iid sequence can be established in the same way as in Question 9.

— Answer 13 —

We have $\quad \mathbb{P}(\mathcal{E}_i) = \frac{\alpha}{p} \cdot p = \frac{2}{3} p$

so $\quad \mathbb{E}[\mathbb{1}_{\mathcal{E}_i} - R_i] = \mathbb{P}(\mathcal{E}_i) - \mathbb{E}[R_i] = \frac{2}{3}(p - \frac{1}{2})$.

It is well known that a random walk on $\mathbb{Z}$ with iid steps starting from 1 hits $\{0, -1, -2 \dots\}$ with probability 1 if and only if the expectation of its step ≤ 0.

(except for the trivial case where the step is 0 deterministically)

We conclude that

The white cluster containing the root is finite with proba 1

$\Leftrightarrow$ $i_0 < \infty$ with probability 1

$\Leftrightarrow$ $\mathbb{E}[\mathbb{1}_{\mathcal{E}_i} - R_i] = \frac{2}{3}(p - \frac{1}{2}) \leq 0$

$\Leftrightarrow$ $p \leq p_c = \frac{1}{2}$.