# Scattering off of Black Holes, Isomonodromy and Painlevé

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# Importance of Scattering Theory

- Astrophysical phenomena: detection of gravitational waves
- Stability criteria of gravitational solutions
- AdS/CFT applications: quark-gluon plasma and condensed matter systems
- Quantum description of black holes

# Scalar Field Perturbation

- Non-minimally coupled massless scalar field $\phi(x)$

$$(\nabla^2 + \xi R)\phi(x) = 0, \quad \nabla^2\phi \equiv \frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}g^{ab}\partial_b\phi)$$

- Radial and Angular equations for Kerr-NUT-(A)dS metric

$$\partial_r(P_r(r)\partial_r\phi_{\omega\ell m}) - Q_r(r)\phi_{\omega\ell m} = 0$$

$$\partial_\theta(P_\theta(\theta)\partial_\theta S_{\omega\ell m}) - Q_\theta(\theta)S_{\omega\ell m} = 0$$

- Angular eigenvalues from (A)dS-spheroidal harmonics

# Complex ODEs and Monodromy

- Radial equation

$$\partial_z(U(z)\partial_z\phi(z)) - V(z)\phi(z) = 0, \quad z \in \mathbb{CP}^1$$

- Ingoing and outgoing solutions

$$\phi_i^{\pm}(z) = (z - z_i)^{\pm\theta_i/2}\big(1 + \mathcal{O}(z - z_i)\big)$$

- Singular points = Branch points ⇒ Monodromy

$$\phi_i^{\pm}(ze^{2\pi i}) = e^{\pm i\pi\theta_i}\phi_i^{\pm}(z)$$

# Monodromies and Gauge Connection

- Gauge connection formulation

$$(\partial_z - A(z))\Phi(z) = 0 \,,$$

$$A(z) = \begin{pmatrix} 0 & U^{-1} \\ V & 0 \end{pmatrix} \quad , \quad \Phi(z) = \begin{pmatrix} \phi_1 & \phi_2 \\ U\partial_z\phi_1 & U\partial_z\phi_2 \end{pmatrix}$$

- Monodromy matrix

$$\Phi_\gamma(z) = \mathcal{P}\exp\left(\oint_\gamma A\right)\Phi(z) =: \Phi(z)M_\gamma$$

# Monodromies and Frobenius solutions

- Loop around only one pole $z = z_i \quad \Rightarrow \quad \Phi_{\gamma_i} = \Phi M_i$
- Loop enclosing all poles gives monodromy identity

$$M_1 M_2 ... M_n = \mathbb{1}$$

- Monodromy matrix in arbitrary basis

$$M_i = g_i^{-1} \begin{pmatrix} e^{i\pi\theta_i} & 0 \\ 0 & e^{-i\pi\theta_i} \end{pmatrix} g_i$$

# Scattering Amplitudes and Connection Matrix

- Change of basis matrix = Connection matrix

$$\mathcal{M}_{i\to j} = \Phi_i^{-1}\Phi_j = g_i\, g_j^{-1}$$

- For purely imaginary $\theta_i \notin i\mathbb{Z}$

$$\mathcal{M}_{i\to j} = \begin{pmatrix} \frac{1}{\mathcal{T}} & \frac{\mathcal{R}}{\mathcal{T}} \\ \frac{\mathcal{R}^*}{\mathcal{T}^*} & \frac{1}{\mathcal{T}^*} \end{pmatrix}, \qquad |\mathcal{R}|^2 + |\mathcal{T}|^2 = 1$$

Castro *et al* arxiv:1304.3781

# Transmission between Two Regular Singular Points

Let $g_i \in \mathrm{SL}(2,\mathbb{C})$ and one of the $M_i$ be diagonal. If we write

$$m_{ij} = \mathrm{Tr}\, M_i M_j = 2\cos\pi\sigma_{ij}$$

then

$$|\mathcal{T}|^2 = \frac{\sin\pi\theta_i \sin\pi\theta_j}{\sin\frac{\pi}{2}(\sigma_{ij}+\theta_i-\theta_j)\sin\frac{\pi}{2}(\sigma_{ij}-\theta_i+\theta_j)}$$

# $D = 4$ Kerr-(A)dS Black Hole

$$ds^2 = -\frac{Q(r)}{r^2+p^2}(dt + p^2 d\phi)^2 + \frac{P(p)}{r^2+p^2}(dt - r^2 d\phi)^2 + \frac{r^2+p^2}{Q(r)}dr^2 + \frac{r^2+p^2}{P(p)}dp^2$$

$$P(p) = -\frac{\Lambda}{3}p^4 - \epsilon p^2 + k$$

$$Q(r) = -\frac{\Lambda}{3}r^4 + \epsilon r^2 - 2Mr + k$$

# Separation of Variables in KG equation

- Radial equation (5 regular singular points)

$$\partial_r(Q(r)\partial_r R(r)) + \left(-4\Lambda\xi r^2 + \frac{(\Psi_0 r^2 + \Psi_1)^2}{Q(r)}\right) R(r) = C_\ell R$$

- Conformally Coupled Case $\xi = 1/6$

$$y'' + \left(\frac{1-\theta_0}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\theta_{t_0}}{z-t_0}\right) y' + \left(\frac{1+\theta_\infty}{z(z-1)} - \frac{t_0(t_0-1)K_0}{z(z-1)(z-t_0)}\right) y = 0$$

Heun equation (4 regular singular points)

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Heun equation (4 regular singular points)

How to find $\sigma_{ij}$ ?

# Deformed Heun and Apparent Singularity

- *Deformed* Heun equation with one apparent singularity

$$\partial_z^2 y + \left(\frac{1-\theta_0}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\theta_t}{z-t} - \frac{1}{z-\lambda}\right)\partial_z y$$
$$+ \left(\frac{\kappa}{z(z-1)} - \frac{t(t-1)K}{z(z-1)(z-t)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)}\right) y = 0$$

- Initial condition for our Heun

$$\lambda(t_0) = t_0, \quad \mu_0 = -\frac{K_0}{\theta_t}$$

and $\theta_t \to \theta_t - 1$

# Isomonodromic Hamiltonian System

- $z = \lambda$ is an apparent singularity if

$$K(\lambda, \mu, t) = \frac{1}{t(t-1)}[\lambda(\lambda-1)(\lambda-t)\mu^2 - \{\theta_0(\lambda-1)(\lambda-t) + \theta_1\lambda(\lambda-t) + (\theta_t-1)\lambda(\lambda-1)\}\mu + \kappa(\lambda-t)]$$

- Hamiltonian System

$$\frac{d\lambda}{dt} = \frac{\partial K}{\partial \mu}, \quad \frac{d\mu}{dt} = -\frac{\partial K}{\partial \lambda}$$

generates isomonodromic flow $(\lambda(t), \mu(t), K(\lambda, \mu, t))$

- Second-order equation for $\lambda(t)$ = Painlevé VI

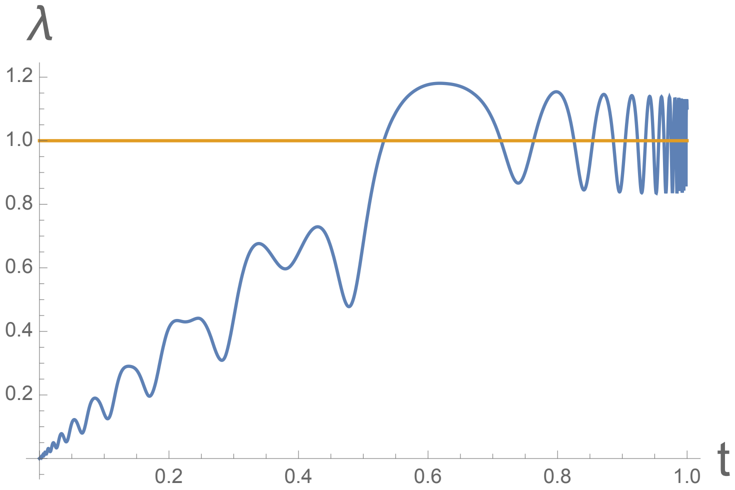
# Painlevé VI Asymptotics

- $P_{VI}$ asymptotics for $0 < \operatorname{Re}\sigma_{ij} < 1$

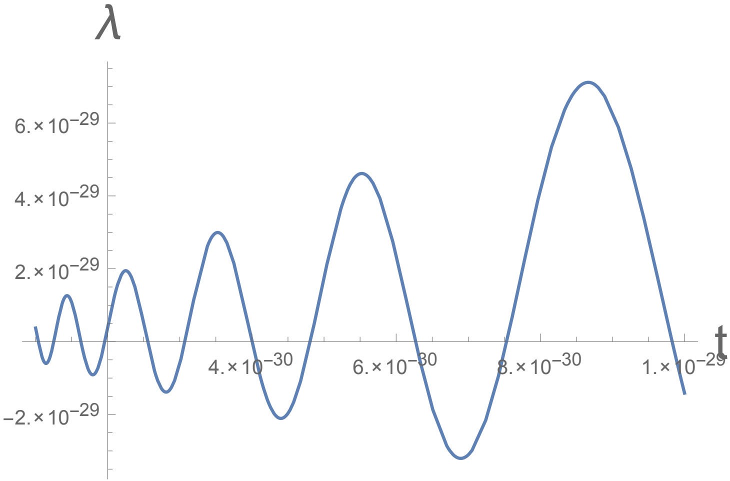
$$\lambda(t) = \begin{cases} a_0 t^{1-\sigma_{0t}}(1 + O(t^\delta)), & |t| < r, \\ 1 + a_1(1-t)^{1-\sigma_{t1}}(1 + O((1-t)^\delta), & |t-1| < r, \\ a_\infty t^{\sigma_{01}}(1 + O(t^{-\delta})), & |1/t| < r, \end{cases}$$

where $r, \delta > 0$ and $a_i$ are functions of monodromy data

# Numerical Integration of $P_{VI}$



# Numerical Integration of $P_{VI}$ near $t = 0$



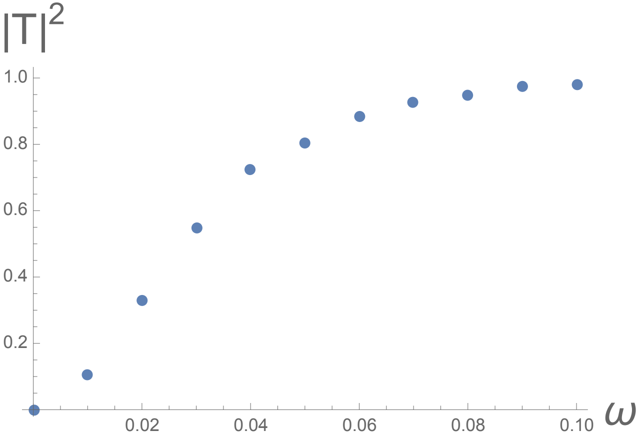
# Kerr-dS Greybody Factor

$$\gamma_\ell(\omega, m) = \frac{\sinh(\frac{\omega - \Omega_H m}{2T_H}) \sinh(\frac{\omega - \Omega_C m}{2T_C})}{\cosh\left(\frac{\omega - \Omega_H m}{2T_H} + \frac{\omega - \Omega_C m}{2T_C}\right) - \cosh(2\pi\nu_{HC})}$$

- $\nu_{HC}$ encodes all global information
- $\Omega_H m < \omega < \Omega_C m \Rightarrow$ *superradiance*

# Kerr-dS Greybody Factor

Preliminary result $(\ell = 0, m = 0)$



# Conclusions

- Monodromy technique is the most powerful way to treat scattering problems
- General formula for scattering amplitudes between two regular singular points
- Conformally coupled case is easier
- Valid for higher-dimensional Kerr-NUT-(A)dS black holes
- Flat case can be recovered by confluence

# Conclusions

- Monodromy technique is the most powerful way to treat scattering problems
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Thank you

# Perspectives

- Higher-spin modes and gravitational stability
- Higher-dimensional Kerr-(A)dS and SUGRA backgrounds
- Recover literature via $\Lambda \to 0$ confluence. Irregular singular points ($P_V$ and $P_{III}$)
- Quasinormal modes and plasma thermalization
- CFT dual of extremal Kerr-(A)dS conformal modes?
- Twistorial and geometrical interpretation of isomonodromic hidden symmetry

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# Hidden Symmetry of Isomonodromic Flow

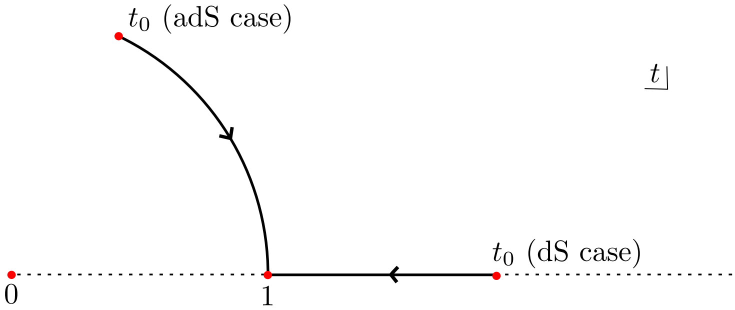
- Limit $t \to 1$ of Garnier ODE gives hypergeometric equation

$$\partial_z^2 y + \left( \frac{1-\theta_0}{z} + \frac{1-\theta_1-\tilde{\theta}_t}{z-1} \right) \partial_z y + \left( \frac{\kappa}{z(z-1)} + \frac{L_1}{z(z-1)^2} \right) y = 0$$

- Fixed point of isomonodromic flow corresponds to some (near-horizon) extremal black hole

# Hidden Symmetry of Isomonodromic Flow

- This suggests that scattering data of non-extremal black holes is equivalent in some sense to extremal black hole scattering



# Relation with Fuchsian Equation

- Fuchsian ODE normal form with $n$ finite singular points

$$\psi''(z) + T(z)\psi(z) = 0, \quad T(z) = \sum_{i=1}^{n} \left( \frac{\delta_i}{(z - z_i)^2} + \frac{c_i}{z - z_i} \right),$$

$$\sum_{i=1}^{n} c_i = 0\,, \quad \sum_{i=1}^{n} (c_i z_i + \delta_i) = 0\,, \quad \sum_{i=1}^{n} (c_i z_i^2 + 2\delta_i z_i) = 0$$

- Local monodromies: $\delta_i = (1 - \theta_i^2)/4$
- Accessory parameters $c_i$ have global properties
- $2(n-3)$ independent parameters: $(c_i, z_i)$

# Symplectic Structure of Flat $\mathrm{SL}(2,\mathbb{C})$ Connections

- Moduli space of flat connections $A \sim$ moduli space of monodromy group
- Atiyah-Bott symplectic structure

$$\Omega = \sum_{i=1}^{n-3} dc_i \wedge dz_i = \sum_{i=1}^{n-3} d\nu_i \wedge d\mu_i$$

where $(\nu_i, \mu_i)$ are trace coordinates (Nekrasov et al 2011)

- Canonical transformation connects both set of coordinates
- Suggests analytical approach to find composite monodromies
- Relation with classical conformal blocks of 2D CFT

# Recurrence Relations

- Taylor solution $y(z) = \sum_{n=0}^{\infty} g_n z^{n/2}$, $|z| < 1$

$$-(Q_0 + q)g_0 + R_0 g_1 = 0,$$

$$P_n g_{n-1} - (Q_n + q)g_n + R_n g_{n+1} = 0, \quad (n > 0)$$

$$P_n = (n - 1 + \alpha_+)(n - 1 + \alpha_-),$$
$$Q_n = n((t + 1)(n - 1 + \gamma) + t\delta + \epsilon),$$
$$R_n = t(n + 1)(n + \gamma)$$

- Solved using Leaver's continued-fraction method (Leaver 1985, Berti, Cardoso and Will (2006))

# Continued-fraction Method

- Augmented convergence for $|z| \geq 1$ if

$$\lim_{n\to\infty}\left|\frac{g_{n+1}}{g_n}\right| = |t|^{-1} = \hat{a}^2 \;\Rightarrow\; a < L$$

- Recurrence relation in terms of $v_n = g_{n+1}/g_n$

$$v_{n-1} = \frac{P_n}{(Q_n + q) - R_n v_n}$$

- Equivalent to continued-fraction

$$(Q_0 + q) - \frac{R_0 P_1}{(Q_1 + q)-}\;\frac{R_1 P_2}{(Q_2 + q)-}\ldots = 0$$

- Solve numerically with $v_N = \hat{a}^2$ for some large integer $N$

# Schlesinger System Asymptotics

- Near $t = 0$

$$A_0 \approx t^{\Lambda} A_0^0 t^{-\Lambda} \quad \text{and} \quad A_t \approx t^{\Lambda} A_t^0 t^{-\Lambda}, \;\text{where}\; \Lambda = A_0^0 + A_t^0$$

- Schlesinger system degenerates to two hypergeometric connections

$$\frac{dY_0}{dz} = \left( \frac{\Lambda}{z} + \frac{A_1^0}{z-1} \right) Y_0, \quad \frac{dY_1}{dz} = \left( \frac{A_0^0}{z} + \frac{A_t^0}{z-1} \right) Y_1$$

# Schlesinger System Asymptotics

- Using that $\det A_i^0 = -\theta_i^2/4$ and $\det \Lambda = -\sigma_{0t}^2/4$

$$\Lambda + \frac{1}{2}\sigma\mathbb{1} = \frac{1}{4\theta_\infty}\begin{pmatrix} (-\theta_\infty-\theta_1+\sigma)(\theta_\infty-\theta_1-\sigma) & (-\theta_\infty-\theta_1+\sigma)(\theta_\infty+\theta_1+\sigma) \\ (\theta_\infty-\theta_1+\sigma)(\theta_\infty-\theta_1-\sigma) & (\theta_\infty-\theta_1+\sigma)(\theta_\infty+\theta_1+\sigma) \end{pmatrix}$$

$$A_1^0 + \frac{1}{2}\theta_1\mathbb{1} = \frac{1}{4\theta_\infty}\begin{pmatrix} -(\theta_\infty-\theta_1)^2+\sigma^2 & (\theta_\infty+\theta_1)^2-\sigma^2 \\ -(\theta_\infty-\theta_1)^2+\sigma^2 & (\theta_\infty+\theta_1)^2-\sigma^2 \end{pmatrix}$$

$$A_0^0 + \frac{1}{2}\theta_0\mathbb{I} = G_1\frac{1}{4\sigma}\begin{pmatrix} (\theta_0-\theta_t+\sigma)(\theta_0+\theta_t+\sigma) & (\theta_0-\theta_t+\sigma)(-\theta_0-\theta_t+\sigma) \\ (\theta_0-\theta_t-\sigma)(\theta_0+\theta_t+\sigma) & (\theta_0-\theta_t-\sigma)(-\theta_0-\theta_t+\sigma) \end{pmatrix} G_1^{-1}$$

$$A_t^0 + \frac{1}{2}\theta_t\mathbb{I} = G_1\frac{1}{4\sigma}\begin{pmatrix} (\theta_t+\sigma)^2-\theta_0 & -(\theta_t-\sigma)^2+\theta_0^2 \\ (\theta_t+\sigma)^2-\theta_0 & -(\theta_t-\sigma)^2+\theta_0^2 \end{pmatrix} G_1^{-1}.$$

# Quasinormal Modes

- Modes purely ingoing at $r_H$ *and* purely outgoing at $r_C$
- Possible only for complex $\omega$
- In this case,

$$\mathcal{M}_{C\to H} = \begin{pmatrix} \frac{1}{\mathcal{T}} & \frac{\mathcal{R}}{\mathcal{T}} \\ \frac{\mathcal{R}'}{\mathcal{T}'} & \frac{1}{\mathcal{T}'} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & \frac{1}{\mathcal{T}'} \end{pmatrix}$$

- Poles of transcendental equation

$$\nu_{HC}(\omega,\ell,m) = \frac{\omega-\Omega_H m}{2T_H} + \frac{\omega-\Omega_C m}{2T_C} + 2\pi i n, \qquad n\in\mathbb{Z}$$

# Properties of Greybody Factor

- *Conjectured* scattering regimes

$$\begin{cases} \omega > \Omega_H m \quad \text{or} \quad \Omega_C m > \omega & \text{Normal scattering} \\ \Omega_H m > \omega > \Omega_C m & \text{Superradiant scattering} \end{cases}$$

- Poles of scattering matrix (resonances)

$$\omega = \begin{cases} m\Omega_H - 2\pi i n T_H \\ m\Omega_C + 2\pi i n T_C \end{cases} \qquad (n \in \mathbb{Z}^+)$$

- We expect that

$$\gamma_l(\omega) \to 1, \qquad \text{as} \quad \omega \to \infty$$

$$\gamma_l(\omega) \to 0 \text{ or constant} \qquad \text{as} \quad \omega \to 0$$

# Superradiant Scattering

- Superradiance = wave analog of Penrose process
- In terms of the classical impact parameter $b = \mathcal{L}/\mathcal{E} \sim \ell/\omega$

$$\frac{\omega}{m} = \frac{\omega}{\ell}\frac{\ell}{m} \sim \frac{1}{b}\frac{\mathcal{L}}{\mathcal{L}_z}$$

- Problem: Greybody factor pole even for real $\omega$ in the superradiant range! Meaning?
- Perturbative analysis of $\nu_{HC}$ might shed some light