Spectral dimensions from the spectral action

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Is there spontaneous Dimensional Reduction in Short-Distance Quantum Gravity?

Important question in Quantum Gravity: Structure of space-time at very short distances?

When lowering the distance scale / increasing the energy scale
- number of dimensions increases (as e.g. in the ADD model)?
- number of dimensions decreases (as e.g. in QEG or CDT)?
- volumes and areas become quantised (as e.g. in QLG)?
- space-time becomes discrete (as in the work presented here)?
Fractal-like properties of a space(-time):

- **Haussdorff dimension**
  - Determined by $\# N$ of balls necessary to cover a set of points:
    \[ N(R) \propto \frac{1}{R^{D}} \]
  - Real line: $N(R) \propto 1/R$ ($D = 1$) but coast of England: $D \approx 1.2$

- **Spectral dimension $D_s$**
  - Consider diffusion of scalar test particle on some manifold
  - Average return probability $P$ “feels” space-time dimension
    \[ P \propto T^{-D_s/2} \]
    with (fictitious) diffusion time $T$
  - $T \leftrightarrow$ resolution scale $\mu \propto 1/\sqrt{T}$
  - Can be calculated from (classical or quantum) propagator

- **Spectral dimension = Hausdorff dim. of momentum space**
Heuristic picture for the concept of a spectral dimension:

- Ping-pong ball sees three dimensions 😊
- Table-football ball sees two dimensions 😐?

⇒ Interactions change the spectral dimension 😊!
Spectral dimension near and beyond Planck scale:

Several ansätze for Quantum Gravity suggest $D_s = 2$!

Seen in

Causal Dynamical Triangulation & Asympt. Safe Gravity,
arguments given for

Loop Quantum Gravity, Hořava-Lifshitz gravity & ... 


Is the value $D_s = 2$ in the UV generic for
all approaches to Quantum Gravity?

Here:
Calculate the (classical) spectral dimension for
Connes’ non-commutative (resp., almost commutative) geometry
at short distances.
Connes’ “Almost-commutative geometry” implies spectral action for bosons:

\[ S_{\chi,\Lambda} = \text{Tr}(\chi(D^2/\Lambda^2)) \]

1. \( \chi \) positive otherwise arbitrary function
2. \( D \) Dirac operator: contains spin 0, 1 & 2 fields
   \[ D^2 = -(\nabla^2 - i\gamma^\mu \gamma_5 \nabla_\mu \phi - \phi^2 + \frac{i}{4} [\gamma^\mu, \gamma^\nu] F_{\mu\nu} - \frac{1}{4} R) \]
3. \( \Lambda \) physical scale, e.g., Planck mass or GUT Scale

Conjecture:
“High energy bosons do not propagate”
... based on qualitative arguments ...
Expand Spectral Action for $\chi(z) = e^{-z}$ to 2nd order in fields:

$$S^{(2)} \propto \int d^4x \left( \phi G_0(-t\partial^2)\phi + A_\mu G_1(-t\partial^2)A_\mu + h_{\mu\nu} G_2(-t\partial^2)h_{\mu\nu} \right)$$

These functions are non-polynomial in $z = tp^2 = p^2/\Lambda^2$!

$G_0(z = tp^2)$ scalar,
$G_1(z)$ gauge field,
$G_2(z)$ graviton.

$$\lim_{z \to \infty} G_{0,1} = 0$$
$$\lim_{z \to \infty} G_2 = -3/2$$

Note that for scalar, i.e., Higgs, $F_0(0) = m^2 = -\Lambda^2/(4\pi)^2 < 0. \implies \text{SSB}$!
$F_0$ scalar,  
$F_1$ gauge field,  
$F_2$ graviton.

IR ($T \to \infty$):

$D_S \to 4$

UV ($T \to 0$):

$D_S(0) = \frac{4}{N_{\text{max}}}$

determined by highest power of truncation!

$D_S(0) \to 0$ for $N_{\text{max}} \to \infty$
Include full momentum dependence of inverse propagators:

- Regularise mom. integrals $\mathcal{P}(T; \Lambda_{UV}) = \int_{\Lambda_{UV}}^{\Lambda_{UV}} \frac{d^4 p}{(2\pi)^4} e^{-TF(p^2)}$
- $\Lambda_{UV} \to \infty$ (NB: $\Lambda$ fixed!)
- “Late-time” expansion of functions: one can analytically show

$$D_S(T) = \lim_{\Lambda_{UV} \to \infty} D_S(T; \Lambda_{UV}) = 0.$$ 

Consequence:

Spectral dimensions vanish in UV / for very small distances!
Space-time fractures into non-communicating points!
Almost-commutative geometry:

- Calculation of the scale-dependent spectral dimensions of bosons from a spectral action.
- **Spectral dimensions vanish beyond Planck scale!**
  ("High-energy bosons do not propagate!")
- **Space-time fractures into non-communicating points!**

¿ Different than in all other approaches to Quantum Gravity!
¿ **Spectral dimensions if quantum propagators are used?**
Connes’ non-commutative / almost commutative geometry:

- Spectral triple \( \{A, \mathcal{H}, D\} \): Algebra, Hilbert space, Dirac operator
- Continuous spectral triple \( \otimes \) discrete spectral triple
  \( (\equiv \text{Riemannian manifold}) \) (non-comm. part \( \rightarrow \) gauge sym.)

1. Generalisation of Riemannian geometry
2. Universal formula for the action of elementary fields
3. Very high energies: Framework for unification of SM & gravity
Effective Action for a generic function $\chi$ in $S_{\chi,\Lambda} = \text{Tr}(\chi(D^2/\Lambda^2))$, part quadratic in fields:

$$S_{\chi,\Lambda}^{(2)} = \frac{\Lambda^2}{(4\pi)^2} \int d^4x \left[ \phi \mathcal{F}_{0,\chi} \left(-\partial^2/\Lambda^2\right) \phi + A_\mu \mathcal{F}_{1,\chi} \left(-\partial^2/\Lambda^2\right) A_\mu 
+ h_{\mu\nu} \mathcal{F}_{2,\chi} \left(-\partial^2/\Lambda^2\right) h_{\mu\nu} \right].$$

$\mathcal{F}_{0,\chi}(z) = -Q_1 + Q_0 \frac{z}{2} - Q_{-1} \frac{z^2}{12} + Q_{-2} \frac{z^3}{120} + \ldots$

$\mathcal{F}_{1,\chi}(z) = Q_0 \frac{4z}{3} - Q_{-1} \frac{4z^2}{15} + Q_{-2} \frac{z^3}{35} + \ldots$

$\mathcal{F}_{2,\chi}(z) = -Q_2 + Q_1 \frac{z}{12} - Q_0 \frac{z^2}{40} + Q_{-1} \frac{z^3}{336} + \ldots$

$\chi \rightarrow Q_n$ (unique, $Q_n = 1$ for $\chi = e^z$)

$Q_n$’s cannot be adjusted independently!