Spectral dimensions from the spectral action

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Motivation

Is there spontaneous Dimensional Reduction in Short-Distance Quantum Gravity?

Important question in Quantum Gravity:
Structure of space-time at very short distances?

When lowering the distance scale / increasing the energy scale

- number of dimensions increases (as *e.g.* in the ADD model)?
- number of dimensions decreases (as *e.g.* in QEG or CDT)?
- volumes and areas become quantised (as *e.g.* in QLG)?
- space-time becomes discrete (as in the work presented here)?

Motivation

Fractal-like properties of a space(-time):

- Haussdorff dimension
  - Determined by # N of balls necessary to cover a set of points:
    $$N(R) \propto 1/R^D$$
  - Real line: $N(R) \propto 1/R$ ($D = 1$) but coast of England: $D \approx 1.2$
- Spectral dimension D_s
  - Consider diffusion of scalar test particle on some manifold
  - Average return probability $\mathcal{P}$ "feels" space-time dimension
  - $\mathcal{P} \propto T^{-D_s/2}$ with (fictitious) diffusion time T
  - $T \leftrightarrow$ resolution scale $\mu \propto 1/\sqrt{T}$
  - Can be calcuated from (classical or quantum) propagator
- Spectral dimension $=$ Hausdorff dim. of momentum space

Motivation

Heuristic picture for the concept of a spectral dimension:

- Ping-pong ball sees three dimensions ☺
- Table-football ball sees two dimensions ?☹?

$\Longrightarrow$ **Interactions change the spectral dimension ☺!**

Motivation

Spectral dimension near and beyond Planck scale:

Several ansätze for Quantum Gravity suggest $\mathbf{D_s = 2}$!

Seen in

Causal Dynamical Triangulation & **Asympt. Safe Gravity**,

arguments given for

Loop Quantum Gravity, Hořava-Lifshitz gravity & ...

[S. Carlip, talk in Petrópolis 2012 [arXiv:1207.4503] and arXiv:0909.3329]

Is the value $D_S = 2$ in the UV generic for all approaches to Quantum Gravity?

Here:

Calculate the (classical) spectral dimension for

Connes' non-commutative (resp., almost commutative) geometry

at short distances.

Spectral action

Connes' "Almost-commutative geometry" implies spectral action for bosons:

$$S_{\chi,\Lambda} = \mathrm{Tr}\big(\chi(D^2/\Lambda^2)\big)$$

1. χ positive otherwise arbitrary function
2. D Dirac operator: contains spin 0, 1 & 2 fields
$$D^2 = -(\nabla^2 - i\gamma^\mu\gamma_5\nabla_\mu\phi - \phi^2 + \tfrac{i}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu} - \tfrac{1}{4}R)$$
3. Λ physical scale, *e.g.*, Planck mass or GUT Scale

Conjecture:

[M. A. Kurkov, F. Lizzi, D. Vassilevich, Phys. Lett. B **731** (2014) 311.]

"High energy bosons do not propagate"

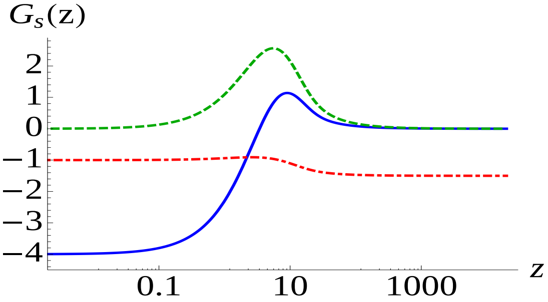
. . . based on qualitative arguments . . .

Results

Expand Spectral Action for $\chi(z) = e^{-z}$ to 2nd order in fields:

$$S^{(2)} \propto \int d^4x \left(\phi G_0(-t\partial^2)\phi + A_\mu G_1(-t\partial^2) A_\mu + \mathbf{h}_{\mu\nu} G_2(-t\partial^2)\mathbf{h}_{\mu\nu} \right)$$

These functions are non-polynomial in $z = tp^2 = p^2/\Lambda^2$!

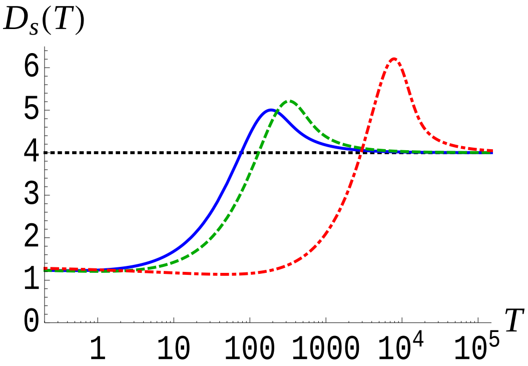


$G_0(z = tp^2)$ scalar,
$G_1(z)$ gauge field,
$G_2(z)$ graviton.

$$\lim_{z\to\infty} G_{0,1} = 0$$
$$\lim_{z\to\infty} G_2 = -3/2$$

Note that for scalar, *i.e.*, Higgs, $F_0(0) = m^2 = -\Lambda^2/(4\pi)^2 < 0. \implies$ SSB!

Results



F_0 scalar,
F_1 gauge field,
F_2 graviton.

IR ($T \to \infty$):

$D_S \to 4$

UV ($T \to 0$):

$D_S(0) = 4/N_{max}$ determined by highest power of truncation!

$D_S(0) \to 0$ **for** $N_{max} \to \infty$

Results

Include full momentum dependence of inverse propagators:

- Regularise mom. integrals $\mathcal{P}(T;\Lambda_{\mathrm{UV}}) = \int^{\Lambda_{\mathrm{UV}}} \frac{d^4p}{(2\pi)^4}\, e^{-TF(p^2)}$
- $\Lambda_{\mathrm{UV}} \to \infty$ (NB: Λ fixed!)
- "Late-time" expansion of functions: one can analytically show

$$D_S(T) = \lim_{\Lambda_{\mathrm{UV}}\to\infty} D_S(T;\Lambda_{\mathrm{UV}}) = 0\,.$$

Consequence:

Spectral dimensions vanish in UV / for very small distances!
Space-time fractures into non-communicating points!

Conclusions

Almost-commutative geometry:

- Calculation of the scale-dependent spectral dimensions of bosons from a spectral action.
- **Spectral dimensions vanish beyond Planck scale!** ("High-energy bosons do not propagate!")
- **Space-time fractures into non-communicating points!**

? Different than in all other approaches to Quantum Gravity!

? Spectral dimensions if **quantum** propagators are used?

Spectral Action

Connes' non-commutative / almost commutative geometry:

[A. Connes, "Non-commutative geometry," Academic Press, 1994;
K. van den Dungen, W.D. van Suijlekom, Rev.Math.Phys.**24**, 1230004 (2012).]

- Spectral triple $\{A, \mathcal{H}, D\}$: Algebra, Hilbert space, Dirac operator
- Continuous spectral triple $\otimes$ discrete spectral triple
  ($\equiv$ Riemannian manifold) (non-comm. part $\rightarrow$ gauge sym.)

1. Generalisation of Riemannian geometry
2. Universal formula for the action of elementary fields
3. Very high energies: Framework for unification of SM & gravity

Results

Effective Action for a generic function χ in $S_{\chi,\Lambda} = \mathrm{Tr}\big(\chi(D^2/\Lambda^2)\big)$, part quadratic in fields:

$$S^{(2)}_{\chi,\Lambda} = \frac{\Lambda^2}{(4\pi)^2} \int d^4x \left[\phi\, \mathcal{F}_{0,\chi}\left(-\partial^2/\Lambda^2\right) \phi + A_\mu\, \mathcal{F}_{1,\chi}\left(-\partial^2/\Lambda^2\right) A_\mu \right.$$
$$\left. + \mathbf{h}_{\mu\nu}\, \mathcal{F}_{2,\chi}\left(-\partial^2/\Lambda^2\right) \mathbf{h}_{\mu\nu} \right].$$

$$\mathcal{F}_{0,\chi}(z) = -Q_1 + Q_0 \tfrac{z}{2} - Q_{-1} \tfrac{z^2}{12} + Q_{-2} \tfrac{z^3}{120} + \ldots$$
$$\mathcal{F}_{1,\chi}(z) = Q_0 \tfrac{4z}{3} - Q_{-1} \tfrac{4z^2}{15} + Q_{-2} \tfrac{z^3}{35} + \ldots$$
$$\mathcal{F}_{2,\chi}(z) = -Q_2 + Q_1 \tfrac{z}{12} - Q_0 \tfrac{z^2}{40} + Q_{-1} \tfrac{z^3}{336} + \ldots$$

- ⋆ $\chi \to Q_n$ (unique, $Q_n = 1$ for $\chi = e^z$)
- ⋆ **Q_n's cannot be adjusted independently!**