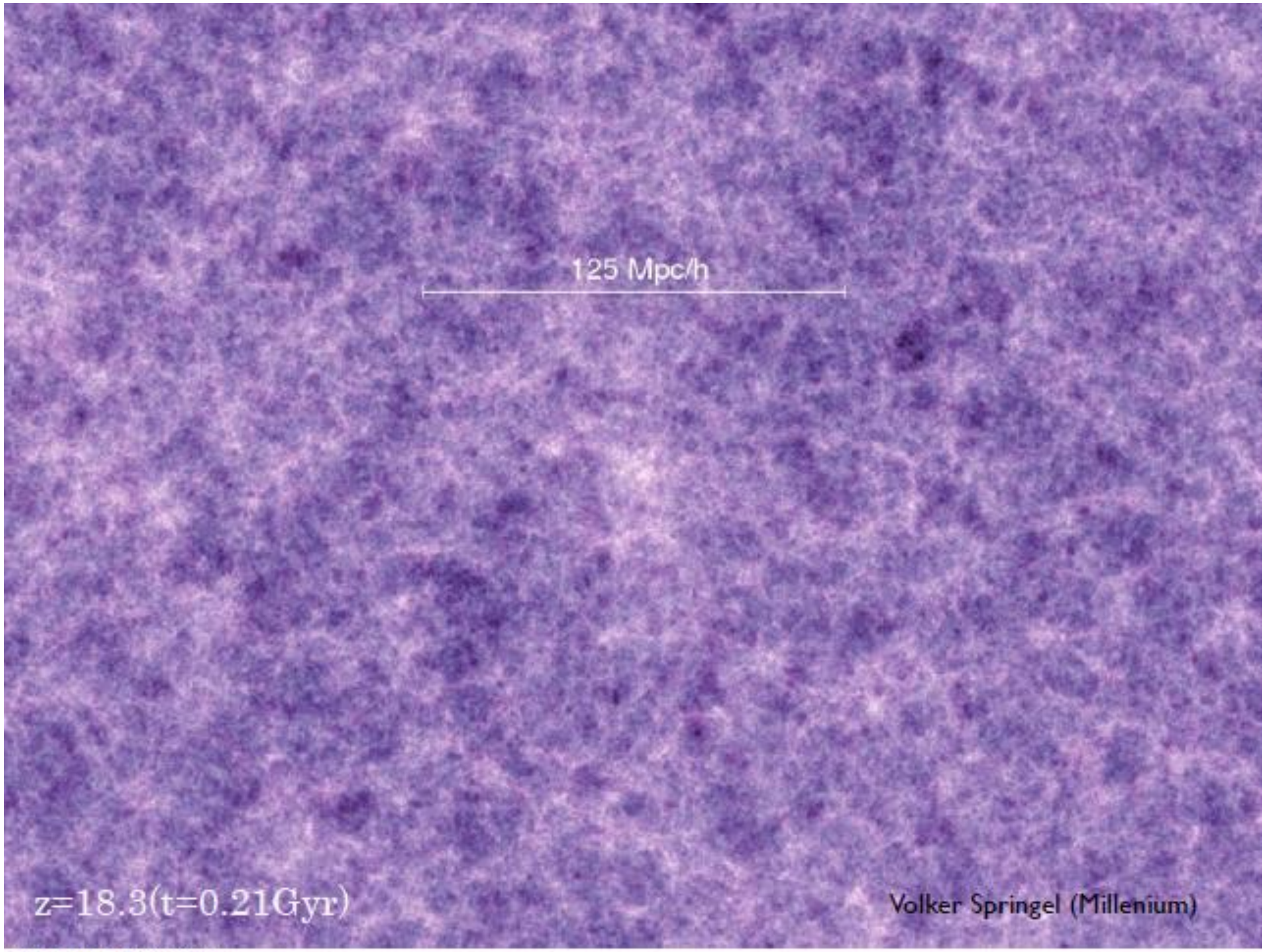
Dark Matter and Structure Formation

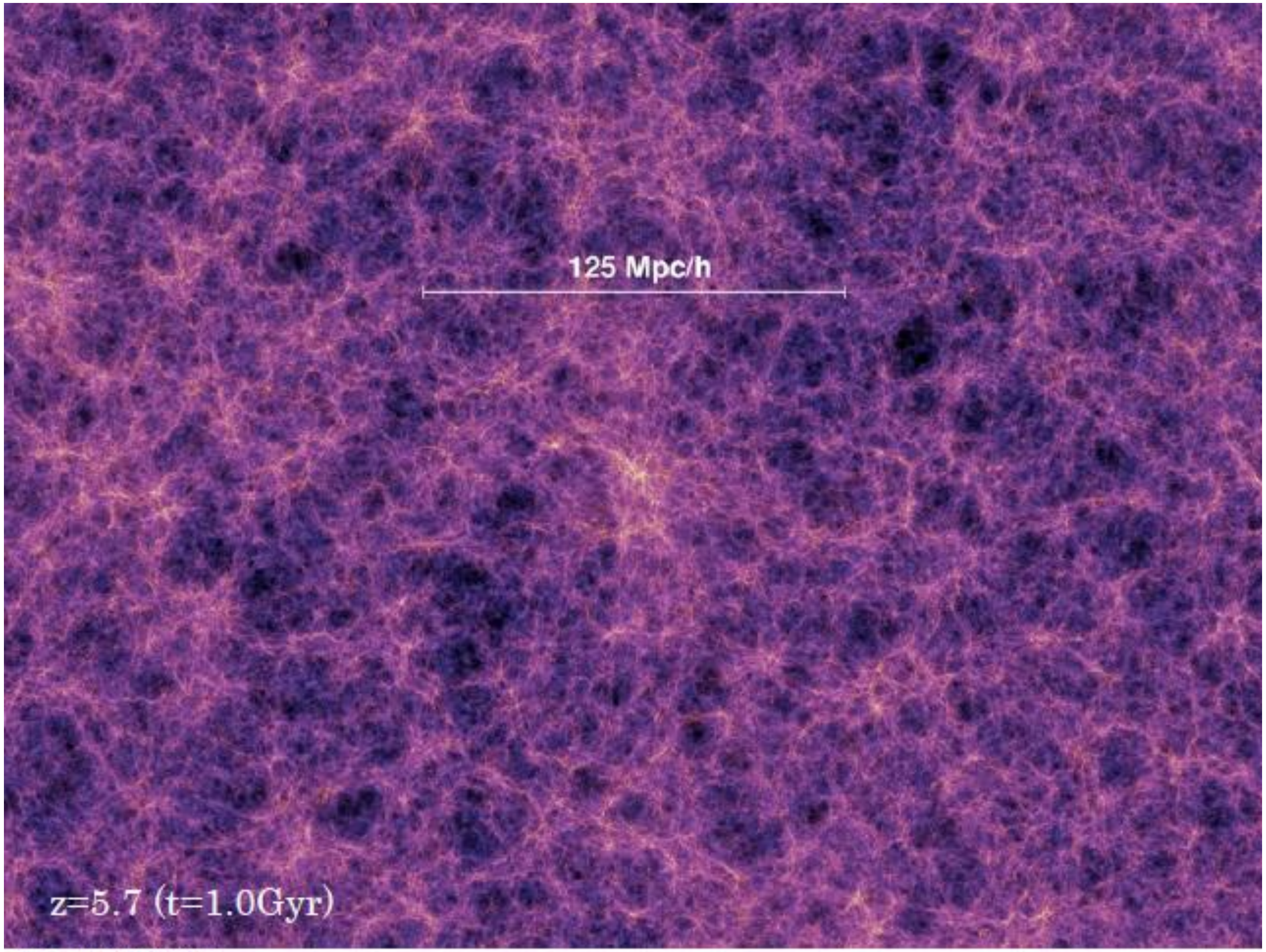
Beyond linear theory

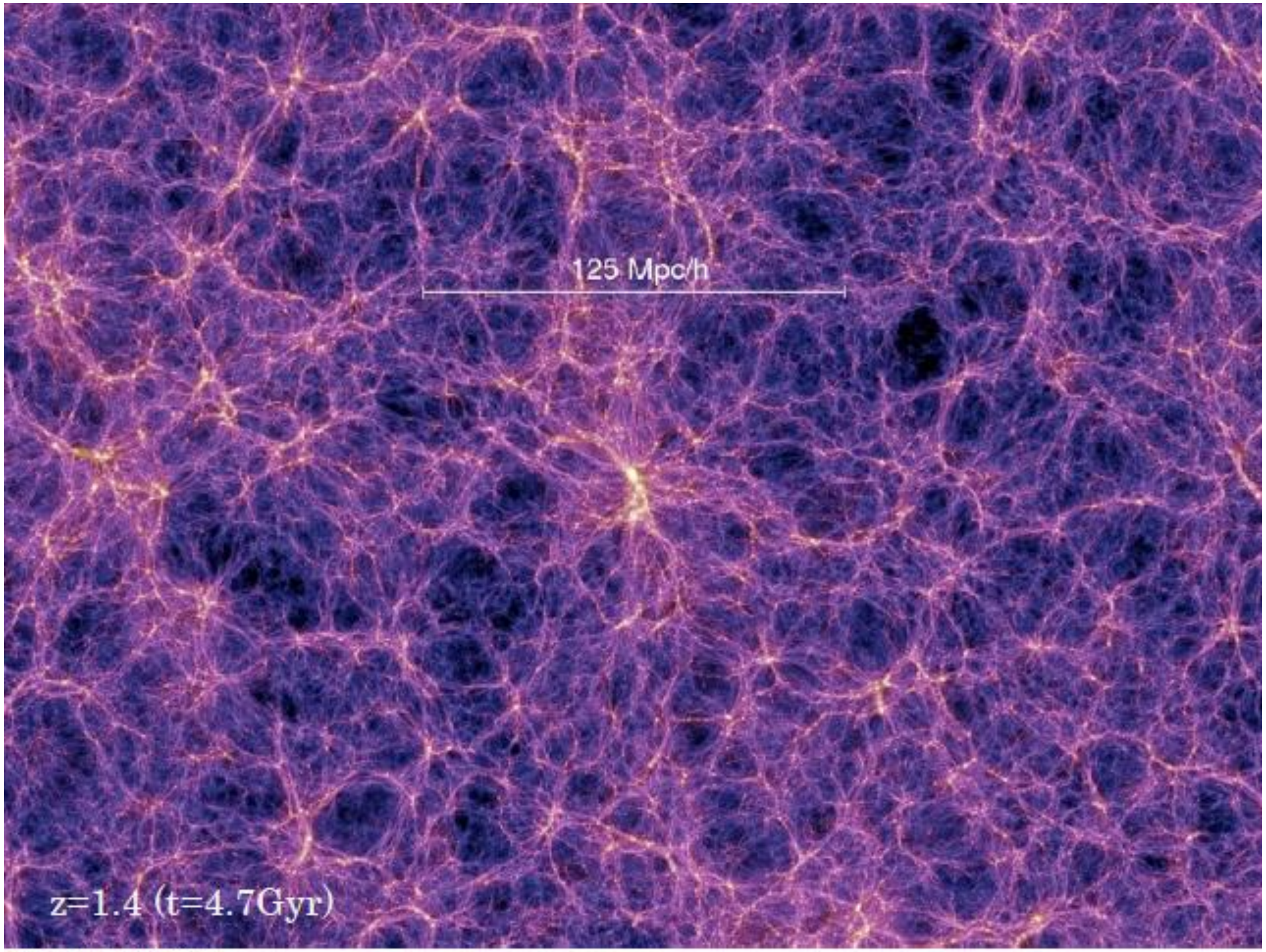
Lagrangian PT: Zeldovich approximation

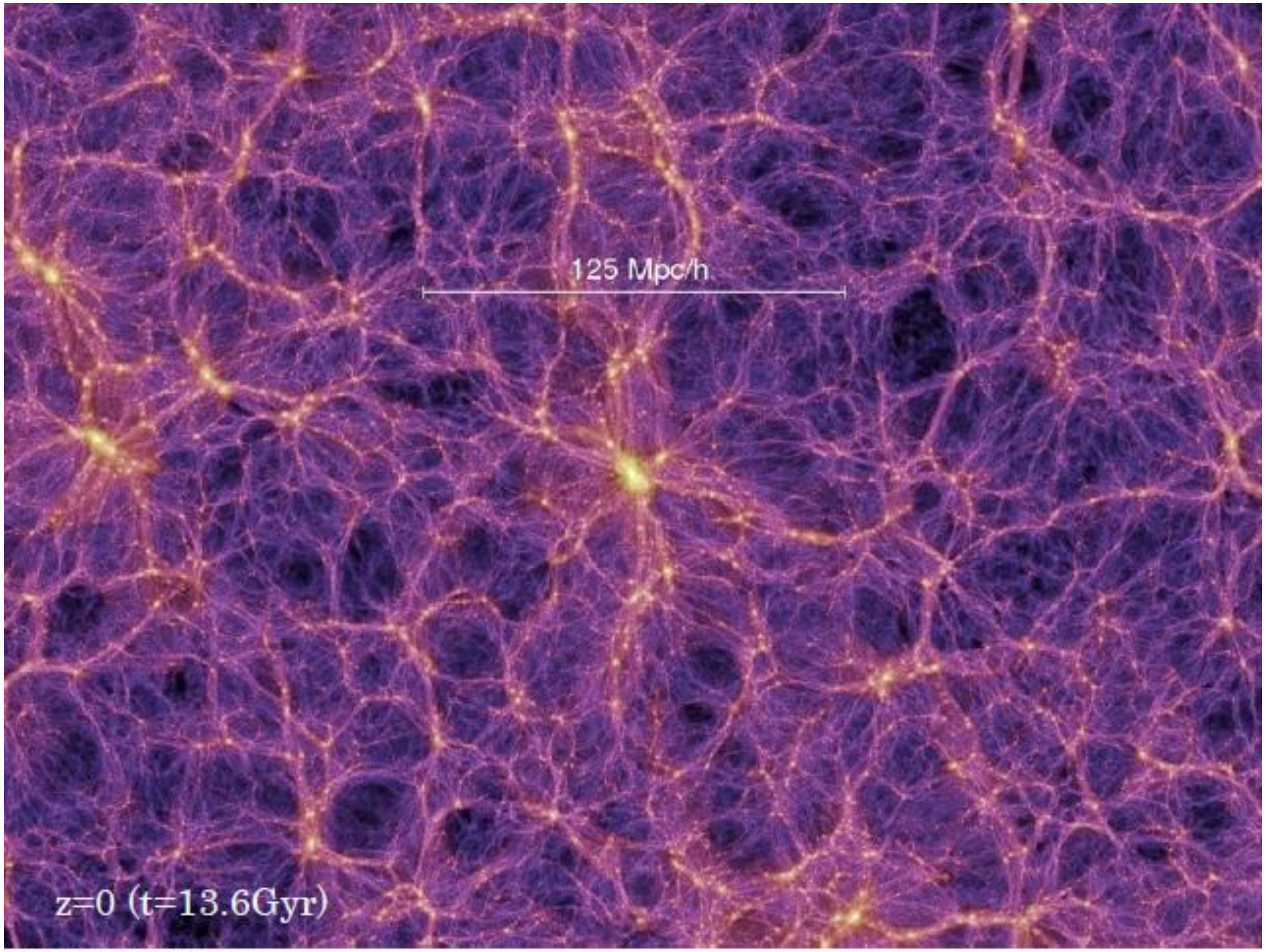
Eulerian PT: Spherical collapse

Redshift space distortions


125 Mpc/h
z=18.3(t=0.21Gyr)
Volker Springel (Millenium)


125 Mpc/h
z=5.7 (t=1.0Gyr)


125 Mpc/h
z=1.4 (t=4.7Gyr)


125 Mpc/h
z=0 (t=13.6Gyr)

Recall linear theory:

- When radiation dominated ($H = 1/2t$):

$$(d^2\delta/dt^2) + 2H\,(d\delta/dt) = (d^2\delta/dt^2) + (d\delta/dt)/t = 0$$

$$\delta(t) = C_1 + C_2 \ln(t) \quad \text{(weak growth)}$$

- In distant future (H = constant):

$$(d^2\delta/dt^2) + 2H_\Lambda(d\delta/dt) = 0$$

$$\delta(t) = C_1 + C_2 \exp(-2H_\Lambda t)$$

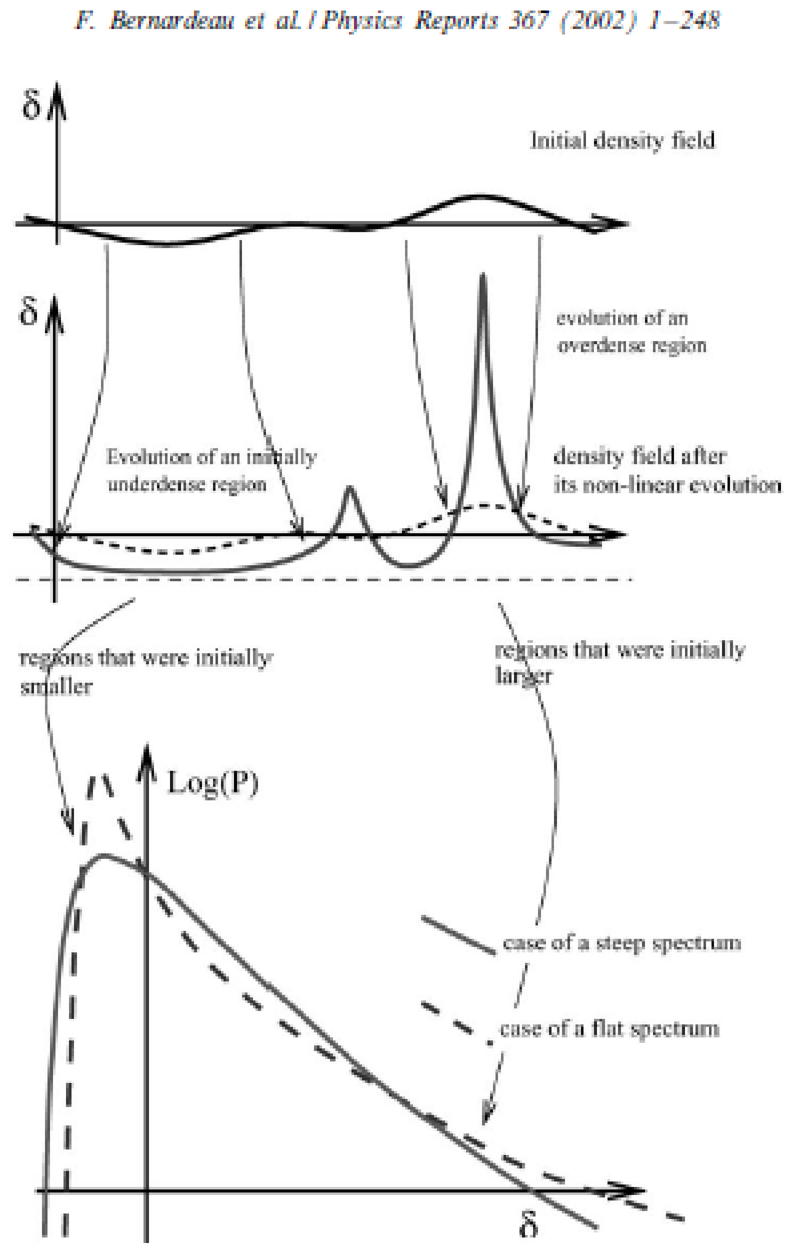
- If flat matter dominated ($H = 2/3t$):

$$\delta(t) = \delta_+\, t^{2/3} + \delta_-\, t^{-1} \propto a(t) \quad \text{at late times}$$

- Linear growth just multiplicative factor, so if initial conditions Gaussian, linearly evolved field is too

Initially Gaussian fluctuation field becomes very non-Gaussian

Linear growth just multiplicative factor, so cannot explain non-Gaussianity at late times



N-body simulations of

gravitational clustering

in an expanding universe

It’s a capitalist’s life…

- Most of the action is in the big cities
- Newcomers to the city are rapidly stripped of (almost!) all they have
- Encounters generally too high-speed to lead to long-lasting mergers
- Repeated ‘harassment’ can lead to change
- Real interactions take place in the outskirts
- A network exists to channel resources from the fields to feed the cities

Nonlinear evolution

Assume a spherical cow ….

Spherical evolution model

$$d^2R/dt^2 = -\,GM/R^2 + \Lambda\, R$$
$$= -\,\rho\,(4\pi G/3H^2)\,H^2 R + \Lambda\, R$$
$$= -\,½\,\Omega(t)H(t)^2\,R \quad + \Lambda\, R$$

- Note: currently fashionable to modify gravity. Should we care that only $1/R^2$ or R give stable circular orbits?

Spherical evolution model

- Initially, $E_i = -GM/R_i + (H_iR_i)^2/2$
- Shells remain concentric as object evolves; if denser than background, object pulls itself together as background expands around it
- At 'turnaround': $E = -GM/r_{\text{max}} = E_i$
- So $-GM/r_{\text{max}} = -GM/R_i + (H_iR_i)^2/2$
- Hence $(R_i/r) = 1 - H_i^2R_i^3/2GM$

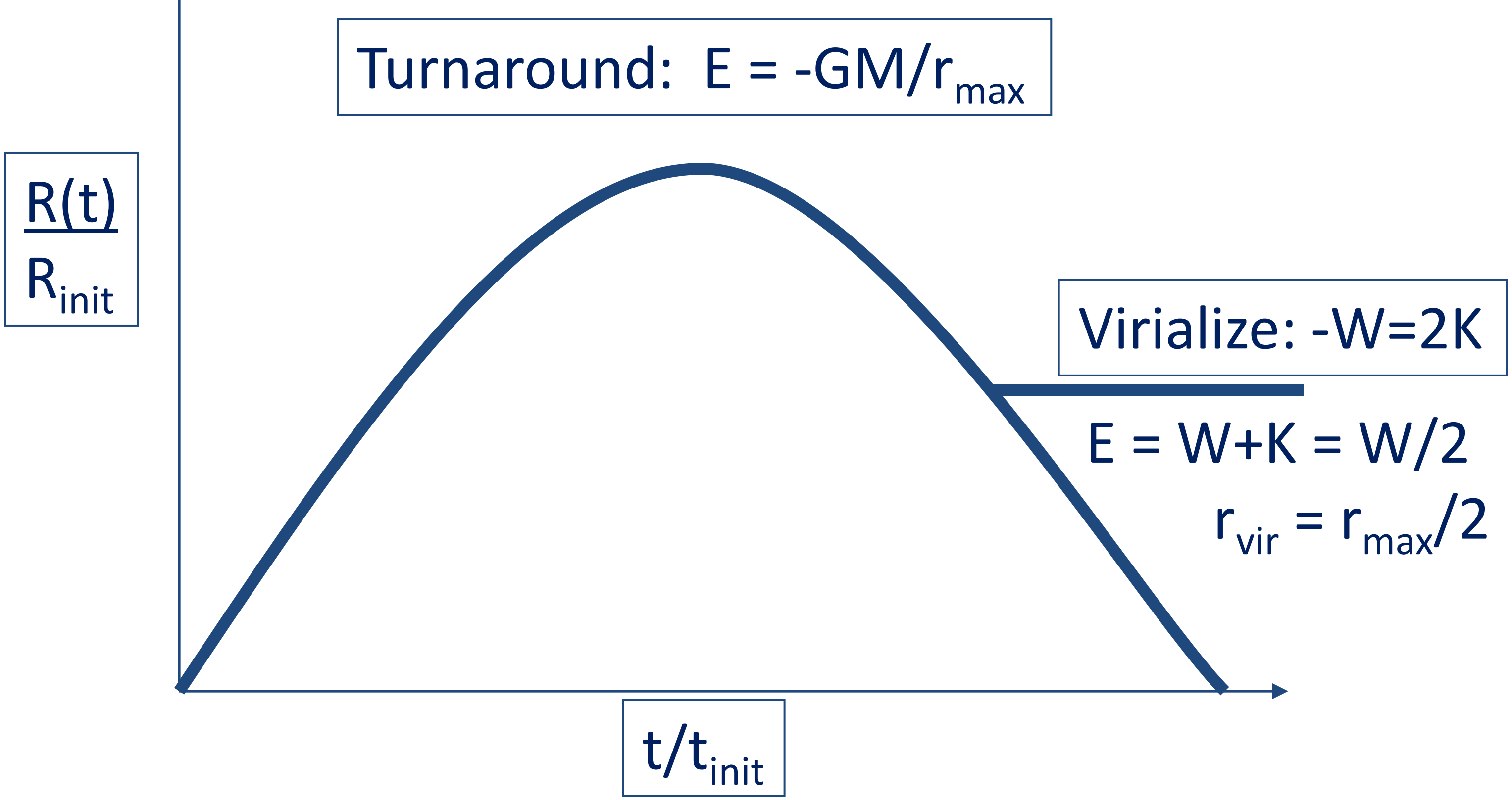
$$= 1 - (3H_i^2/8\pi G)\,(4\pi R_i^3/3)/M$$

$$= 1 - 1/(1+\Delta_i) = \Delta_i/(1+\Delta_i) \approx \Delta_i$$

Virialization

- Final object virializes: $-W = 2K$
- $E_{vir} = W+K = W/2 = -GM/2r_{vir} = -GM/r_{max}$
  - so $r_{vir} = r_{max}/2$:
- Ratio of initial to final size = (density)$^{1/3}$
  - final density determined by initial overdensity
- To form an object at present time, must have had a critical over-density initially
- Critical density same for all objects!
- To form objects at high redshift, must have been even more over-dense initially

Nonlinear evolution: Spherical collapse



Modify gravity → modify collapse

Exact Parametric Solution

(R_i/R) vs. θ and (t/t_i) vs. θ

very well approximated by…

$$(R_{\mathrm{initial}}/R)^3$$

$$= \text{Mass}/(\rho_{\mathrm{com}}\text{Volume})$$

$$= 1 + \delta \approx (1 - D_{\mathrm{Linear}}(t)\ \delta_i/\delta_{sc})^{-\delta sc}$$

Dependence on cosmology from $\delta_{sc}(\Omega,\Lambda)$, but this is rather weak

$$1 + \delta \approx (1 - \delta_{Linear}/\delta_{sc})^{-\delta sc}$$

- As $\delta_{Linear} \rightarrow \delta_{sc}$ ($\approx$ 1.686), $\delta \rightarrow$ infinity
  - This is virialization limit
- As $\delta_{Linear} \rightarrow 0$, $\delta \approx \delta_{Linear}$
- If $\delta_{Linear} = 0$ then $\delta = 0$
  - This does not happen in modified gravity models where $D(t) \rightarrow D(k,t)$
  - Related to loss of Birkhoff's theorem when r^{-2} lost?
- Note $1+\delta \rightarrow 0$ as $\delta_{Linear} \rightarrow -\infty$
  - Why is $\delta_{Linear} < -1$ sensible?

$$1 + \delta \approx (1 - \delta_{\rm Linear}/\delta_{\rm sc})^{-\delta sc}$$
$$\approx 1 + \delta_{\rm Linear} + (1+1/\delta_{\rm sc})\delta_{\rm Linear}^2/2 + \ \ldots$$
$$\approx \sum_j a_j\, \delta_{\rm Linear}^j$$

- Terms like $\delta_{\rm Linear}^2$ being products in real space are convolutions in k-space
- Therefore k-modes of nonlinear δ are coupled, so evolved density field is non-Gaussian
- Spherical evolution not the full story …

Estimate of ‘nonlinear’ scale

- $<\delta^2(t)> = \int dk/k\ 4\pi\ k^3\ P(k,t)\ W^2(kR)$
- If $P(k) = Ak^n$ then $<\delta^2(t)> \sim R^{-(3+n)} \sim M^{-(3+n)/3}$ converges only for $n>-3$.
- Convergence of potential fluctuations only if $n=1$.
- Note: $P(k,t) = D_+^2(t)\ P(k)$, so $<\delta^2(t)> \sim 1$ means nonlinear structure on scales smaller than $R_{nl} \sim D_+^{2/(3+n)} \sim t^{(4/3)/(3+n)}$

Hierarchical structure formation for $-3<n<1$

More generally …

- $\nabla_r^2 \phi = 4\pi G \bar{\rho}\, \delta$ (Poisson equation)
- $\partial\rho/\partial t = -\nabla_r.(\rho v)$ (Continuity equation)
  - Since $\rho = \bar{\rho}_0 (1 + \delta)/a^3$ we have
  - $\partial\rho/\partial t = -3\rho H + \rho\, (\partial\delta/\partial t)/(1 + \delta)$
    $= -\rho\nabla.(Hr) + \rho\, (\partial\delta/\partial t)/(1 + \delta)$
  - $\nabla.(\rho v) = \rho/(1 + \delta)\ \nabla.(1+\delta)(v\text{-}Hr) + \rho/(1 + \delta)\ \nabla.(1+\delta)(Hr)$
    $= \rho/(1 + \delta)\ \nabla.(1+\delta)v_{pec} + \rho\,\nabla.(Hr) + \rho/(1 + \delta)\ Hr\nabla.\delta$
- $\partial\delta/\partial t \approx -\nabla_r.v_{pec} = -\nabla_x.(v_{pec}/a) = -\nabla_x.u$

Fourier transform …

- $\delta(x,t) = \sum_k \delta_k(t)\, e^{-ik.x}$ and $u(x,t) = \sum_k u_k(t)\, e^{-ik.x}$
- $\nabla_r^2 \phi = 4\pi G\, \bar{\rho}\, \delta = (3\Omega_0 H_0^2/2a^3)\, \delta$ (matter domination)

  $\rightarrow$ $- k^2 \phi_k = (3\Omega_0 H_0^2/2a)\, \delta_k$
  - When Ω=1 then $\delta_k \propto a$ so the potential does not evolve!
- $\partial\delta/\partial t = -\nabla_x.u \rightarrow$ $\partial\delta_k/\partial t = -iku_k$
  - $\partial\delta_k/\partial t = \partial \ln D/\partial t\, \delta_k = (\partial \ln D/\partial \ln a)\, H\, \delta_k = fH\, \delta_k$
- So $u_k /(fH) = i\, (\mathbf{k}/k)\, (\delta_k/k)$
  - Note that u/H has units of distance
  - Velocities are more sensitive to small k (large scales), because of the factor of 1/k
- In practice, the expressions above mean that one need simply specify/generate ϕ_k, since then u_k and δ_k are completely determined.

The Zeldovich Approximation I.

The physical displacement of a particle in time dt is

$d\mathbf{r} = \mathbf{v}\, dt$

so comoving displacement is

$d\mathbf{x} = d\mathbf{r}/a = (d\mathbf{r}/dt)/a\, dt = (\mathbf{v}/a)\, dt.$

Hence, in linear theory,

$d\mathbf{x}/dD = (\mathbf{v}/a)\,(dt/dD) \sim \delta_i\, (\mathbf{v}/a)\,(dt/d\delta) \sim \delta_i\, (\mathbf{v}/a)\,(r/\mathbf{v}) \sim \delta_i\, x$

$=$ constant

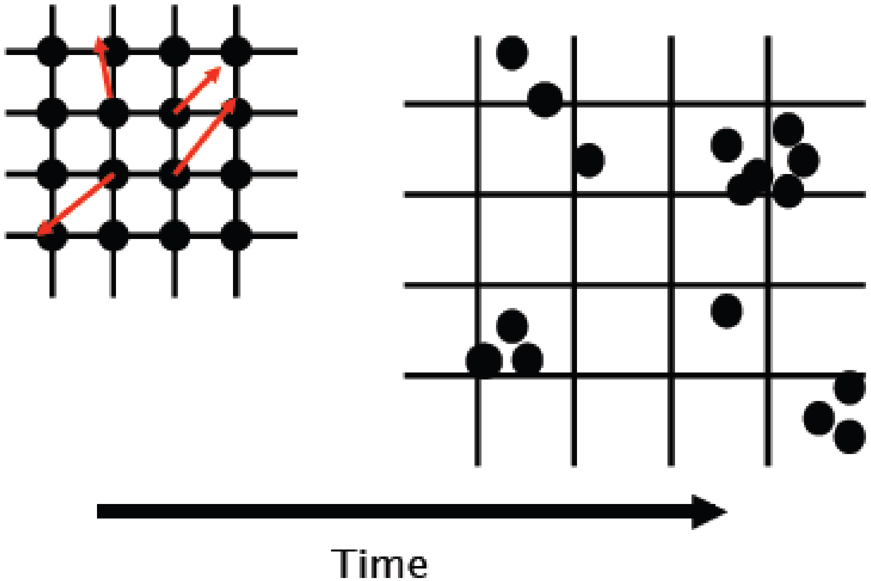
Hence, if initial comoving position was **q**, then comoving position **x** at a later time, when the growth factor is D, is

$x = q + D(t)\, u(q)/(fH)$ (note that u/H is a distance)

Structure grows because of perturbations in the initial <u>velocity</u> field

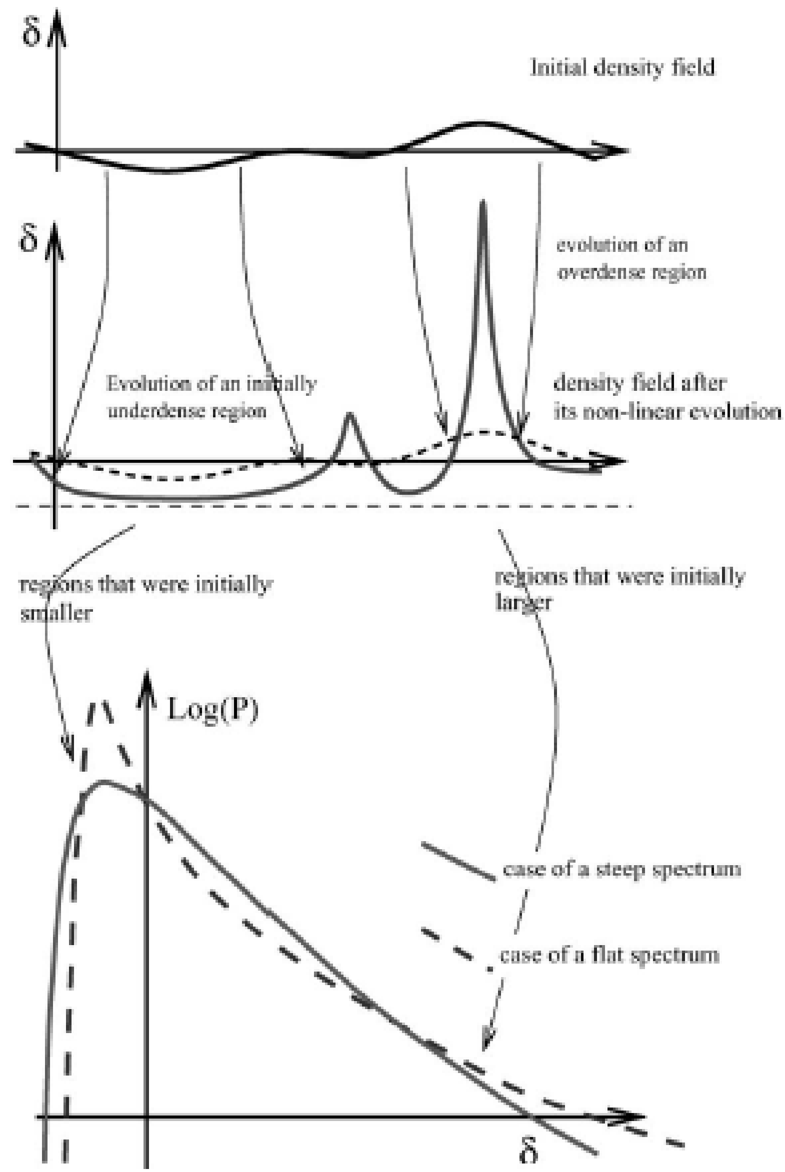
Initially distribution of matter is approximately homogeneous (δ is small)

Final distribution is clustered

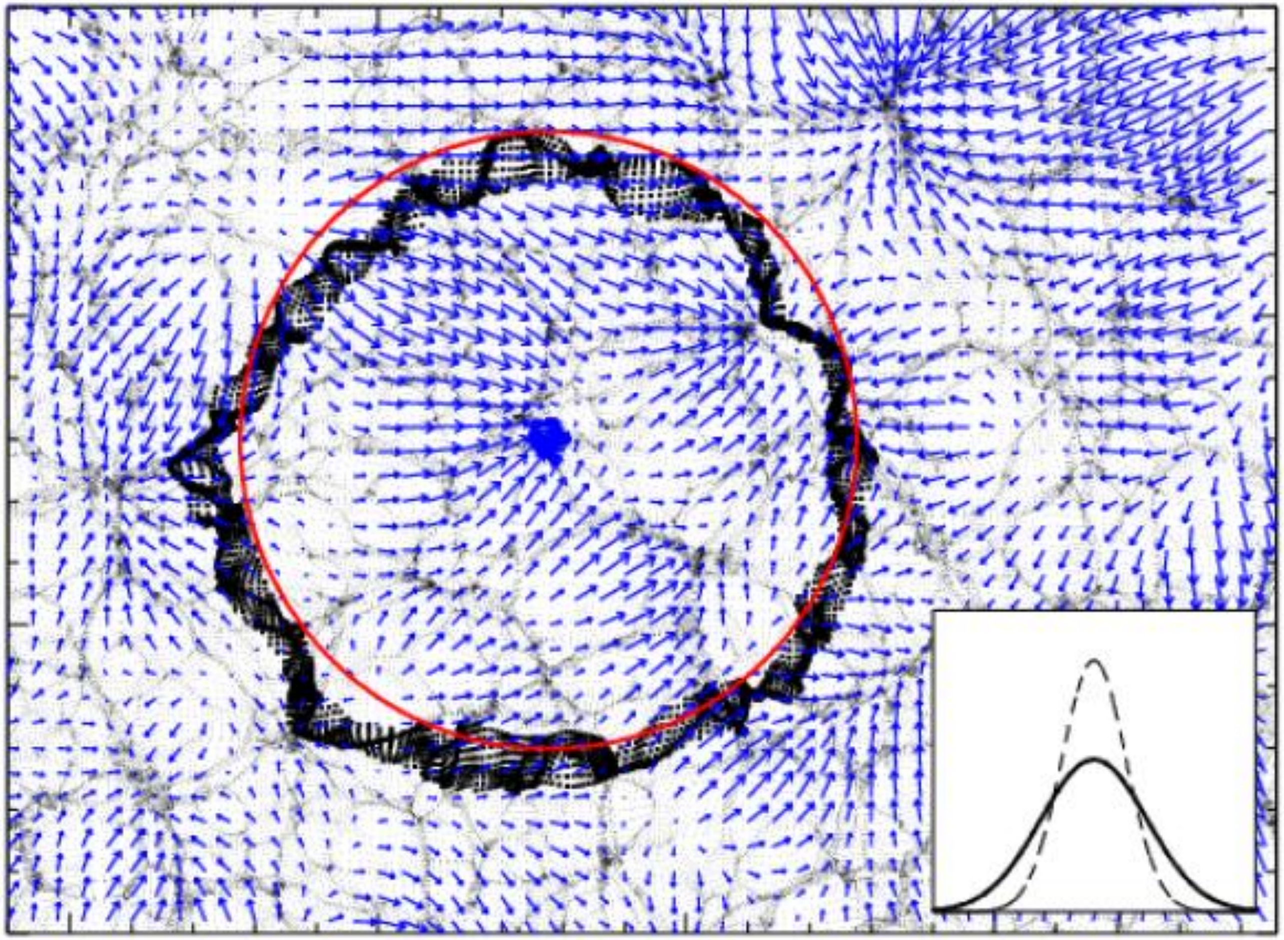


Because of these motions, the fluctuation field can become very non-Gaussian (even though the displacements themselves are Gaussian!)

F. Bernardeau et al. / Physics Reports 367 (2002) 1–248



Zeldovich displacements (further) smear out the BAO spike



The Zeldovich Approximation II.

$$\mathbf{x} = \mathbf{q} + D(t)\,\mathbf{u}(\mathbf{q})/(fH) = \mathbf{q} + D(t)\,\mathbf{S}(\mathbf{q})$$

How are Zeldovich displacements S (for shift) related to density?

$$d\mathbf{x}_i/d\mathbf{q}_j = \delta_{ij} + D(t)\, d\mathbf{S}_i/d\mathbf{q}_j = \delta_{ij} - D(t)\, d[d\Phi/d\mathbf{q}_i]/d\mathbf{q}_j$$

- Evidently, displacements are related to one derivative of potential so Jacobian of x-q transformation involves second derivatives of potential: a 3x3 matrix.
- The 3 eigenvalues of Φ_{ij}, say λ_1, λ_2, λ_3, describe the principal axes of an ellipsoid (not a sphere!): in this respect, Zeldovich is more general than spherical.

Zeldovich approximation III.

In principal axis frame:

$$d\mathbf{x}_i/d\mathbf{q}_i = 1 - D(t)\,\lambda_i$$

Thus D(t) λ describes how the axis shrinks (or expands).

Hence, the density is

$$1 + \delta(t) = \Pi_{i=1}^{3}\,(1 - D(t)\lambda_i)^{-1}$$

To lowest order this is

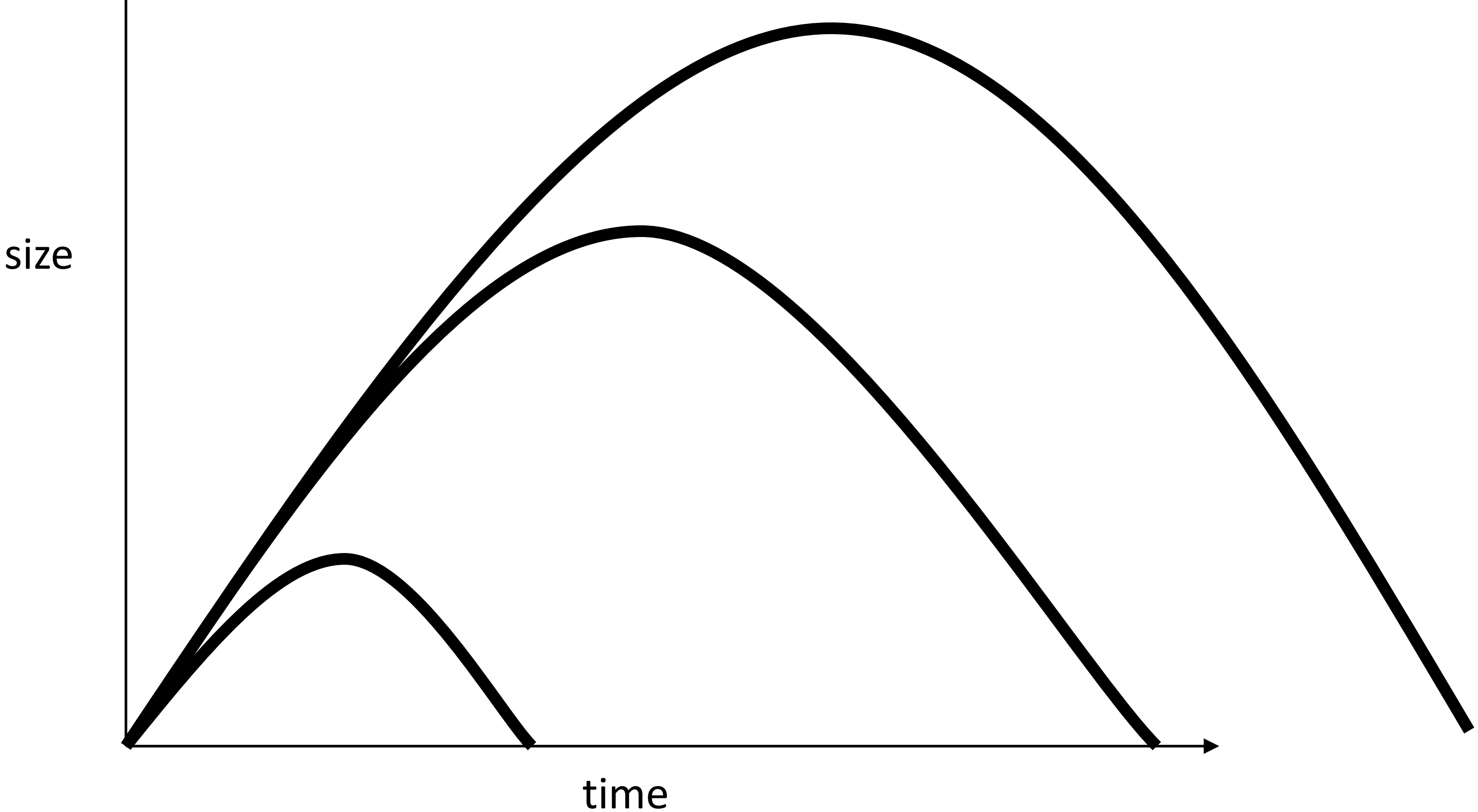
$$1 + \delta(t) = 1 + D(t) \sum \lambda_i + D^2(t)\,(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\,\lambda_3) + \ldots$$
$$= 1 + D(t)\,\delta_{initial} + D^2(t)\,(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\,\lambda_3) + \ldots$$

Evidently, δ_{Linear} is just the trace of Φ_{ij}. This is why it can be arbitrarily negative, and even when it is, the true overdensity is still sensible.

Only very fat cows are spherical....

(Lin, Mestel & Shu 1963; Icke 1973; White & Silk 1978; Bond & Myers 1996; Sheth, Mo & Tormen 2001; Ludlow, Boryazinski, Porciani 2014)

Triaxial collapse: initial sphere evolves because of triaxial shear



Evolution of 2nd axis very similar to spherical model of same initial density

Collapse of 1st axis sooner than in spherical model; collapse of all 3 axes takes longer

Tri-axial (ellipsoidal) collapse

- Evolution determined by properties of initial deformation field, described by 3×3 matrix at each point (Doroshkevich 1970)
- Tri-axial because 3 eigenvalues/invariants; Trace = initial density δ_{in} = quantity which determines spherical model; other two (e,p) describe anisotropic evolution of patch
- Critical density for collapse no longer constant: On average, $\delta_{ec}(\delta_{in},e,p)$ larger for smaller patches $\longrightarrow$ low mass objects

Convenient Approximations

- Zeldovich Approximation (1970):

$$(1+\delta)_{\mathrm{Zel}} = \Pi_{i=1}^{3} (1 - D(t)\lambda_i)^{-1}$$

- Zeldovich Sphere ($\lambda_1 = \lambda_2 = \lambda_3 = \delta_{\mathrm{Linear}}/3$):

$$(1+\delta)_{\mathrm{ZelSph}} = (1 - \delta_{\mathrm{Linear}}/3)^{-3}$$

$$(1+\delta)_{\mathrm{EllColl}} \approx$$

$$(1+\delta)_{\mathrm{SphColl}} \, (1+\delta)_{\mathrm{Zel}}/(1+\delta)_{\mathrm{ZelSph}}$$

Open questions

- Virial density scales with background or critical density?
  - In ΛCDM, critical seems more reasonable
  - Can address by running simulations beyond present epoch!
- Tri-axial collapse from initially spherical or tri-axial patches?
  - How best to incorporate tidal effects? Simulations suggest longest axis initially aligned with direction of largest compression (correlation is reversed by the final time)
  - What is equivalent of virial size?
  - Predicting final axial ratios is tough problem (generically predict larger halos rounder; this is true in initial conditions, but not at final time)

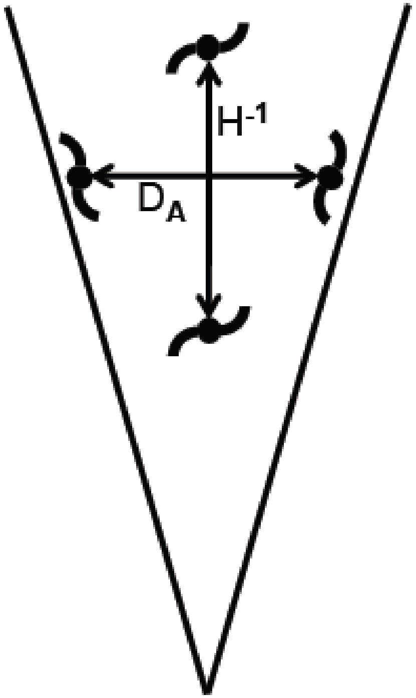
Spherical collapse with DM + DE + νs!

Alcock-Paczynski

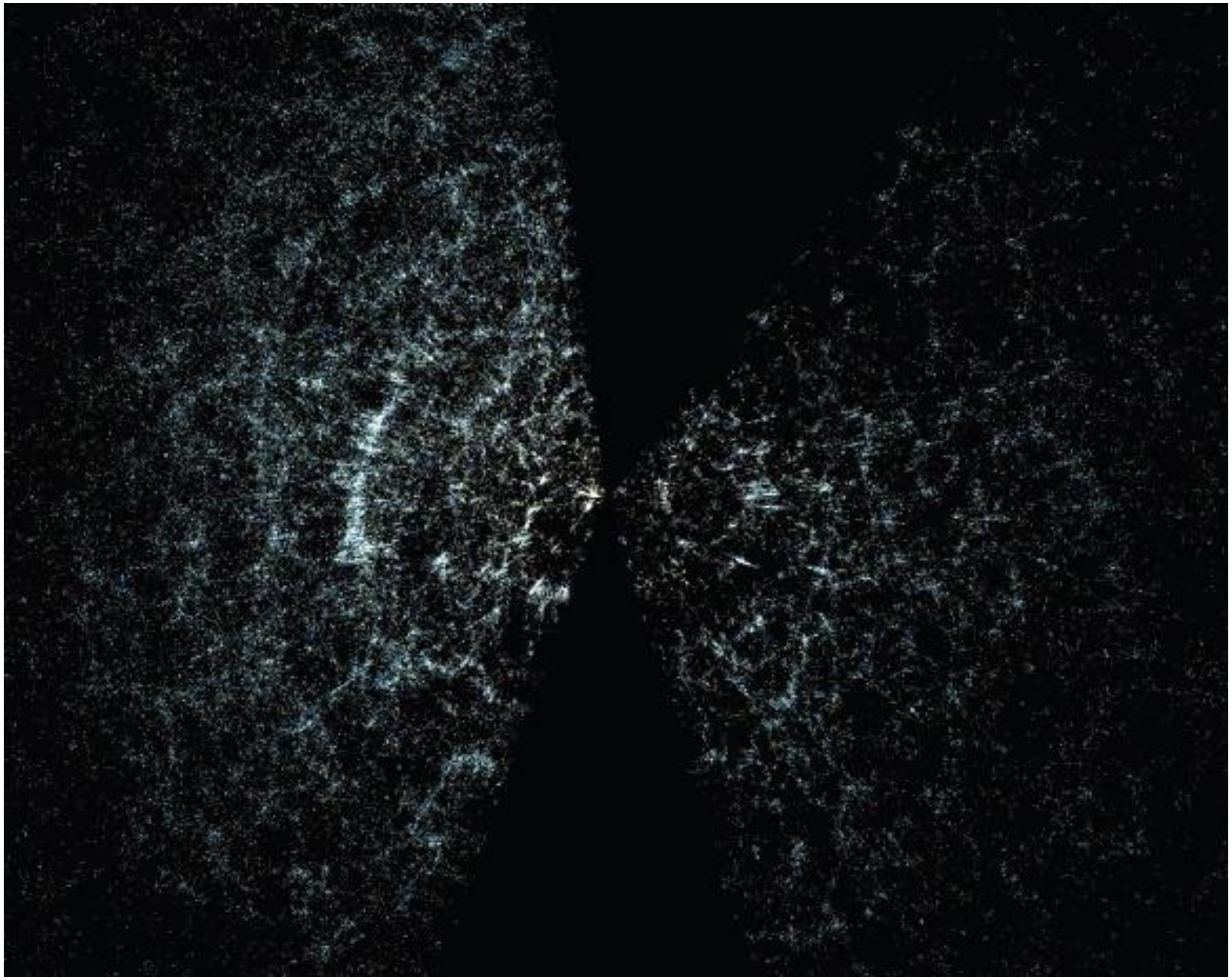
- If the Universe is isotropic, clustering is same radial & tangential
- Stretching at a single redshift slice (for galaxies expanding with Universe) depends on

  $H^{-1}(z)$ (radial)

  $D_A(z)$ (angular)
- Analyze with wrong model -> see anisotropy
- AP effect measures $D_A(z)H(z)$
- RSD limits test to scales where can be modeled



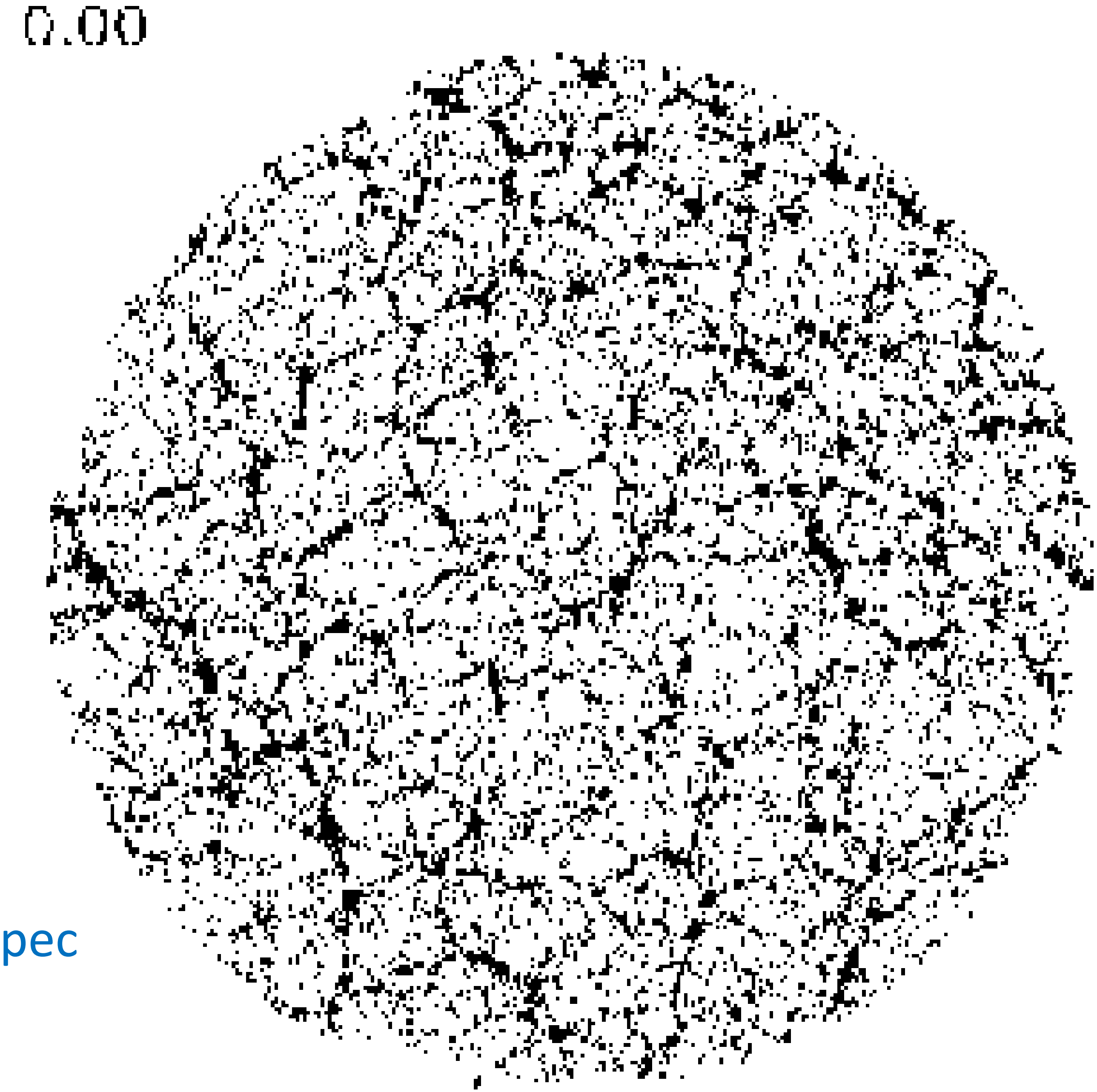
Redshift space distortions



Redshift space distortions:

peculiar velocities driven by gravity

$cz_{obs} = Hd + v_{pec}$



Linear redshift space distortions

- The same velocities which lead to Zeldovich displacements make redshift space position different from real space position.
- $x_s = x + [v(x).d_{los}/|d_{los}|]/H$

  $= q + v(q)/(afH) + [v(q).d_{los}/|d_{los}|]/H$
- Hence (Kaiser 1987)

$$\delta_s = (1 + f\mu^2)\,\delta$$

Virial Motions (within 'halos')

- $(R_i/r_{\rm vir}) \sim f(\Delta_i)$: ratio of initial and final sizes depends on initial overdensity
- Mass $M \sim R_i^3$ (since initial overdensity « 1)
- So final virial density $\sim M/r_{\rm vir}^3 \sim (R_i/r_{\rm vir})^3 \sim$ function of critical density: Hence, all virialized objects have the same density, $\Delta_{\rm vir}\,\rho_{\rm crit}(z)$, whatever their mass
- $V^2 \sim GM/r_{\rm vir} \sim (Hr_{\rm vir})^2 \Delta_{\rm vir} \sim (HGM/V^2)^2\,\Delta_{\rm vir} \sim (HM)^{2/3}$: massive objects have larger internal velocities or temperatures; H decreases with time, so, for a given mass, virial motions (or temperature) higher at high z

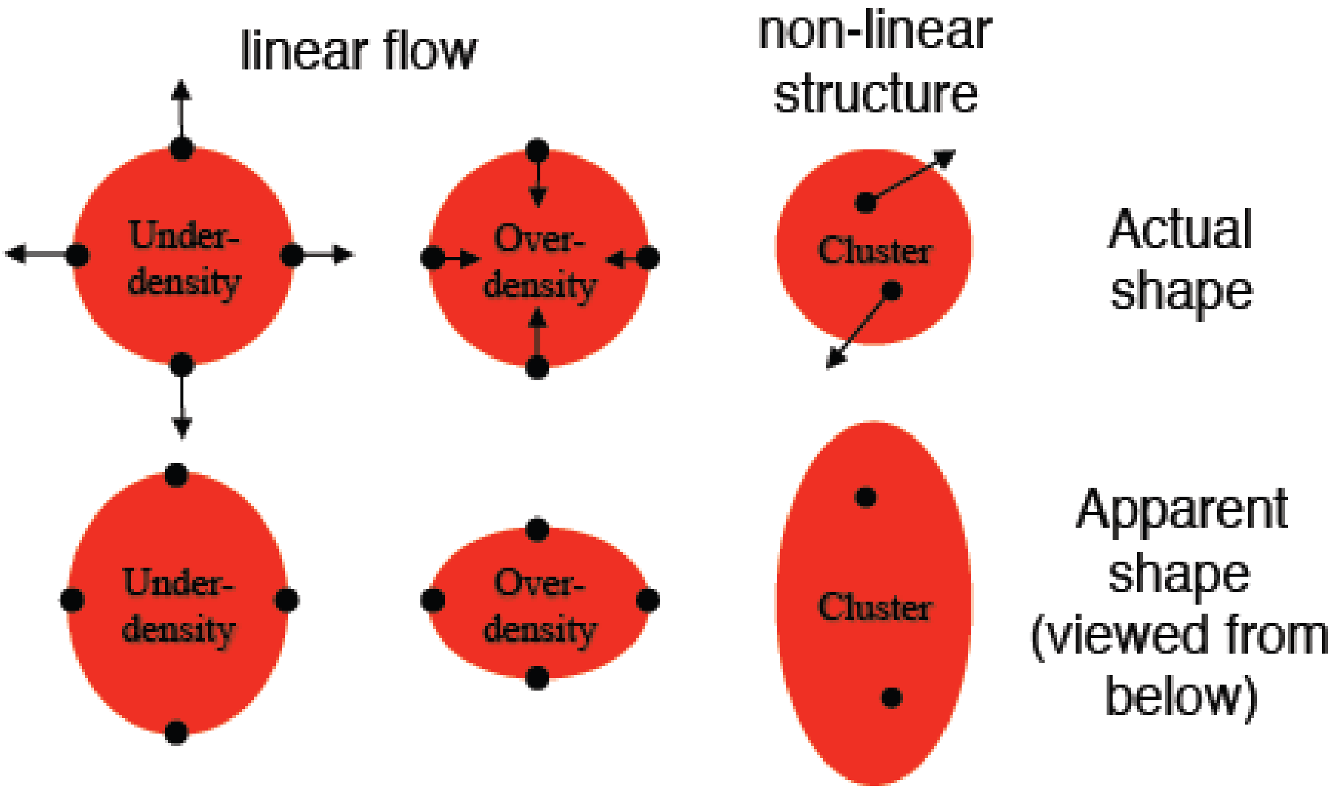
Nonlinear Fingers-of-God

- Virial equilibrium:
- $V^2 = GM/r = GM/(3M/4\pi 200\rho)^{1/3}$
- Since halos have same density, massive halos have larger random internal velocities: $V^2 \sim M^{2/3}$
- $V^2 = GM/r = (G/H^2)\,(M/r^3)\,(Hr)^2$

  $= (8\pi G/3H^2)\,(3M/4\pi r^3)\,(Hr)^2/2$

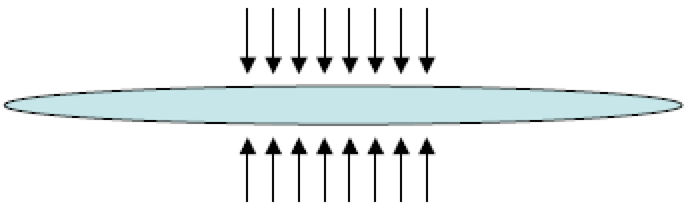
  $= 200\,\rho/\rho_c\,(Hr)^2/2 = \Omega\,(10\,Hr)^2$
- Halos should appear ~ten times longer along line of sight than perpendicular to it: 'Fingers-of-God'
- Think of V^2 as Temperature; then Pressure ~ $V^2\rho$

Two redshift space distortions: Linear + nonlinear

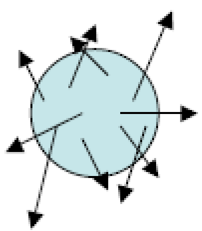


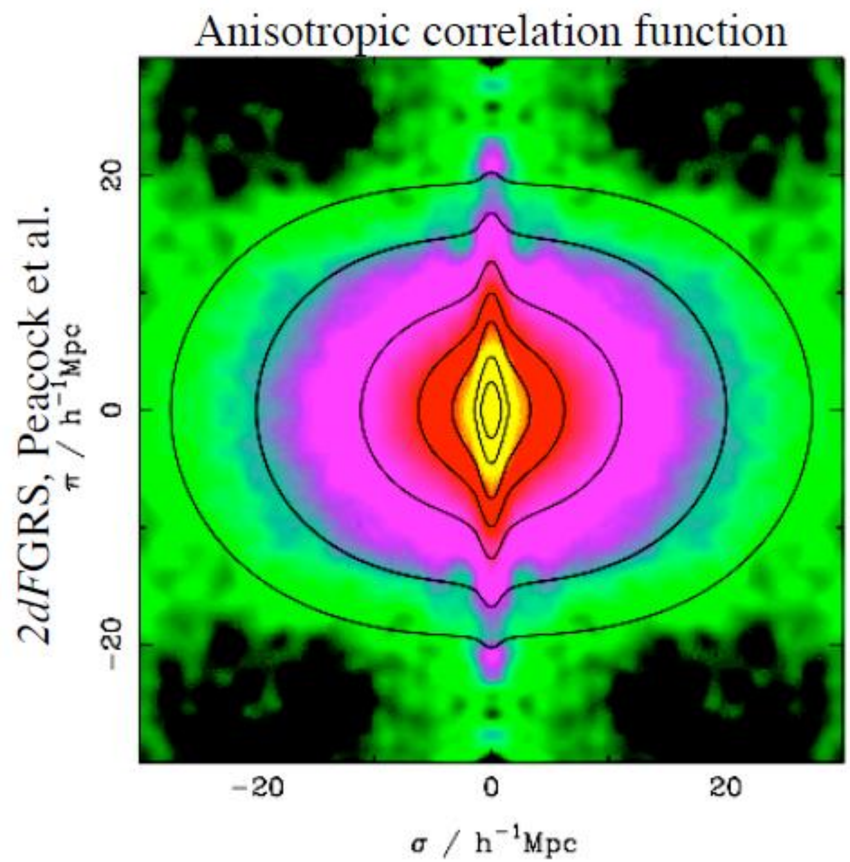
Redshift space distortions

Coherent infall



Random (thermal) motion





$$1 + \xi_s(s_\parallel, s_\perp) = \int_{-\infty}^{\infty} dr_\parallel \; [1 + \xi(r)] \; \mathcal{P}(\underbrace{r_\parallel - s_\parallel}_{\mathbf{v}_p}, \mathbf{r})$$