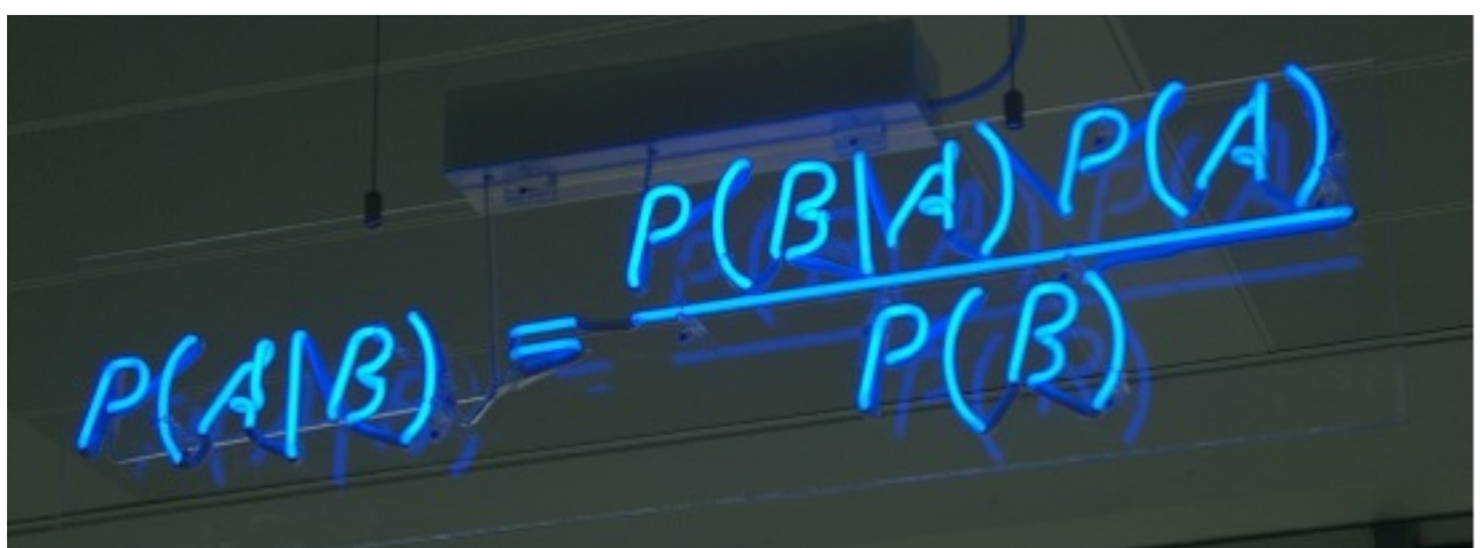
Models ⇔Theories

Lecture 2

Joe Zuntz



Overview

- Notes on Gaussians
- Building Priors
- Building Likelihoods
- Distance measures in cosmology
- Cepheid Likelihoods (fitting a straight line)
- Type 1A Supernova Likelihoods (data modelling)
- What is perturbation theory
- Weak Lensing Likelihoods (handling systematic errors)

Notes on Gaussians

Gaussians: Properties

- **Central limit theorem:**
  Given a collection of random variables X_i:

$$\frac{1}{s_n}\sum_{i=1}^{n}(X_i - \mu_i) \rightarrow \mathcal{N}(0,1)$$

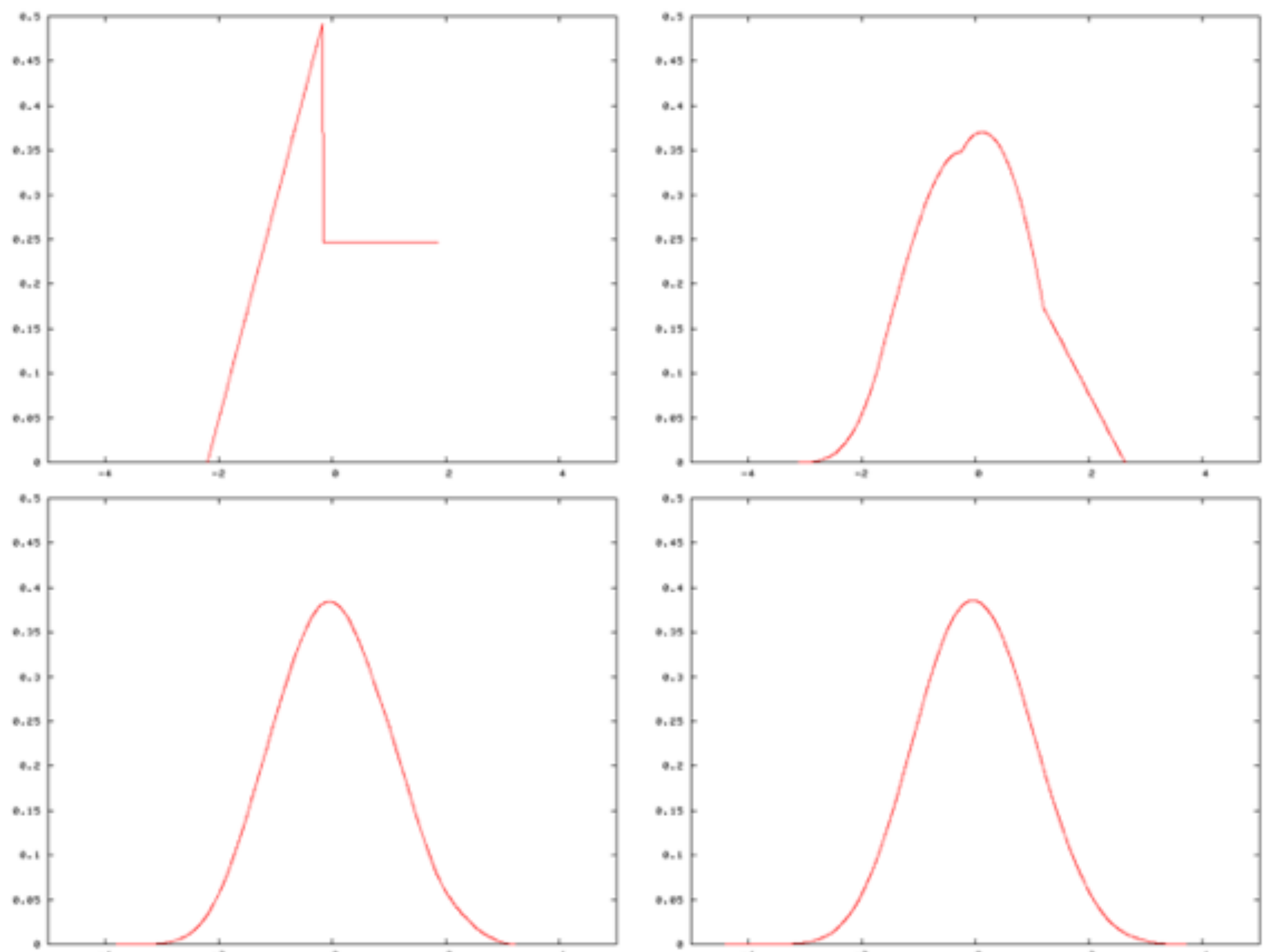
$$s_n^2 = \sum_{i=1}^{n}\sigma_i^2$$

- Provided that:

$$\frac{1}{s_n^2}\sum E\left[(X - \mu_i)^2\right] \rightarrow 0$$

Gaussians: Properties

- **Central limit theorem:**



Single distribution

Mean of 2

Mean of 3

Mean of 4

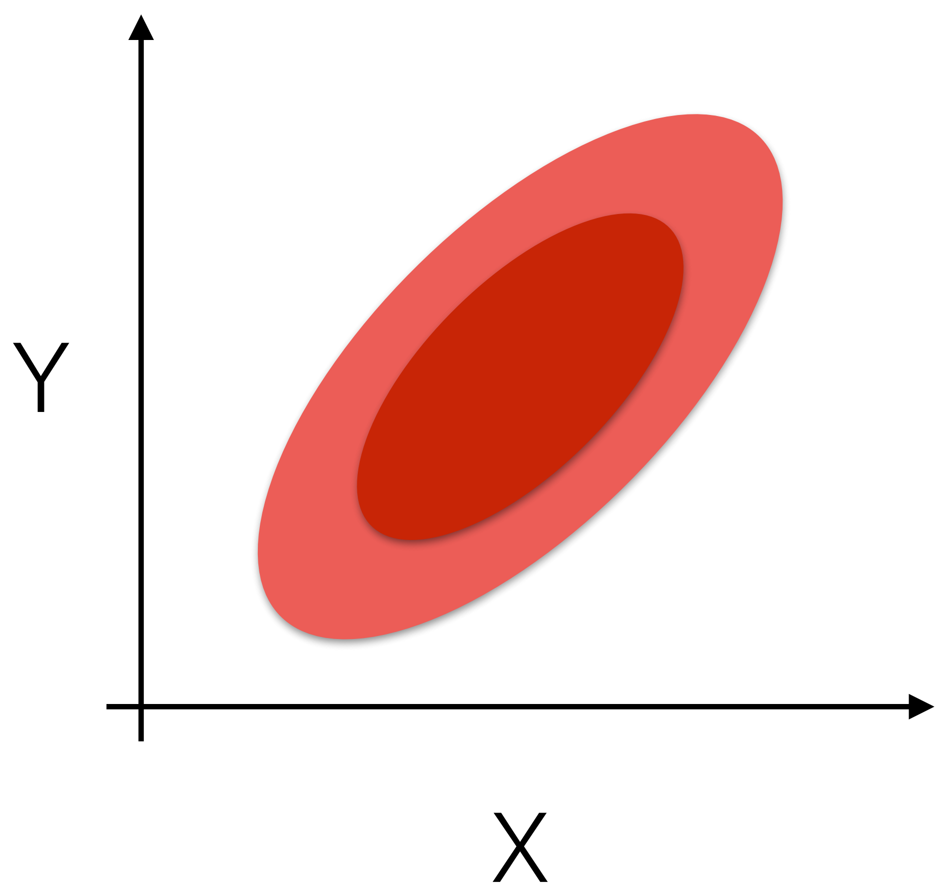
Gaussians: Multivariate

$$P(\underline{\boldsymbol{x}}; \underline{\boldsymbol{\mu}}, C) = \frac{1}{(2\pi)^{\frac{n}{2}} |C|} \exp\left[-\frac{1}{2}(\underline{\boldsymbol{x}} - \underline{\boldsymbol{\mu}})^T C^{-1} (\underline{\boldsymbol{x}} - \underline{\boldsymbol{\mu}})\right]$$

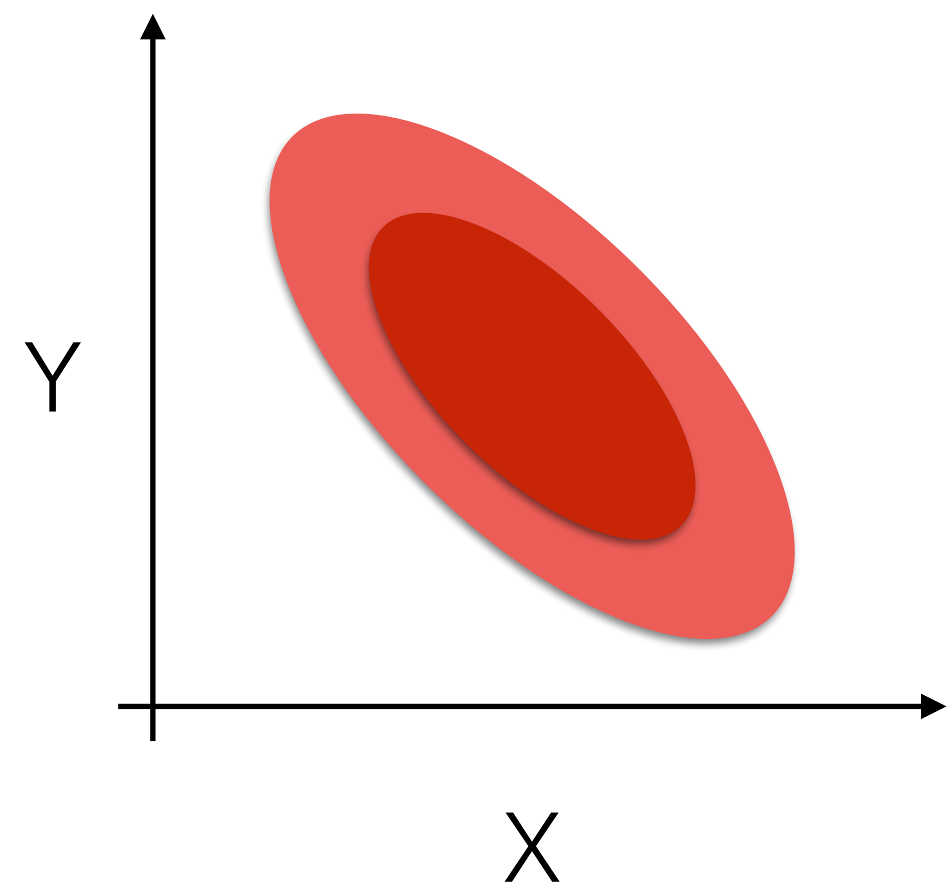
- C is the covariance matrix - describes correlations between quantities
  - For example: data points often have correlated errors

Descriptive Statistics: Covariance

$\sigma_{XY} > 0$



$\sigma_{XY} < 0$



Building Likelihoods: General Rules

- Most of this is physics!
- When you don’t know something, marginalize over it
- Reduce to probabilities you *do* know
- Use the problem logic to understand things
    - What quantities are independent?
- Basic distributions like Poisson, Gaussian, etc., very useful

Building Priors

- Priors encode what you knew about the parameters before you got this new data
- Results of previous experiments!
- Physical limits (e.g. positivity)
- Experiment to check dependence of answers
    - If changing your prior changes the results significantly then new data not very informative

Building Priors

- Lazy: just use flat priors on things
- Remember how probabilities transform:

$$u = f(x)$$
$$P(u) = P(x)/f'(x)$$

Data Sets, Likelihoods, and Limitations

- **Cepheid Variables**
- **Type IA Supernovae**
- **Baryon Acoustic Oscillations**
- **Strong Lensing**
- **Light element abundances**
- **Globular cluster ages**

- **Cosmic Microwave Background**
- **Redshift Space Distortions**
- **Weak Lensing**
- **Large-Scale Structure**
- **Cluster Counts**
- **21cm line structure**
- **Lyman Alpha Forest**

Data Sets, Likelihoods, and Limitations

- **Cepheid Variables**
- **Type IA Supernovae**
- Baryon Acoustic Oscillations
- Strong Lensing
- Light element abundances
- Globular cluster ages
- Cosmic Microwave Background
- Redshift Space Distortions
- **Weak Lensing**
- Large-Scale Structure
- Cluster Counts
- 21cm line structure
- Lyman Alpha Forest

Distance Measures in Cosmology

Different distance measures depending on exactly what you measure. See Hogg (2000).

Co-moving LOS Distance

- Co-moving line of sight distance describes integral of distances scaled to the cosmic expansion

$$D_c(z) = \int_0^z \frac{\mathrm{d}z'}{H(z')}$$

$$H(z) = H_0\sqrt{\Omega_M(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda}$$

Co-moving Transverse Distance

- Co-moving transverse distance accounts for the curvature of space:

$$D_M = \frac{c}{H_0\sqrt{|\Omega_k|}} \sin_k\left(\frac{c}{H_0\sqrt{|\Omega_k|}} D_c\right)$$

$$\sin_k(x) = \begin{cases} \sin x & x > 0 \\ x & x = 0 \\ \sinh x & x < 0 \end{cases}$$

Luminosity Distance

- Luminosity Distance describes relationship between flux emitted from a source and luminosity received.

$$D_L \equiv \sqrt{\frac{L}{4\pi S}} = (1+z)D_M$$

- Fluxes and redshifts are observable quantities: we are getting somewhere useful!

- Measure D_L and z of some objects
$\Longrightarrow$ constraint the H(z) and Ω values

Angular Diameter Distance

- Describes relationship between angle subtended by object and object physical size

$$D_A = \frac{\Delta x}{\Delta \theta} = D_M/(1+z)$$

- If we can measure angle of object with known size then constrain expansion

Small Distances

- These distances equal for close objects

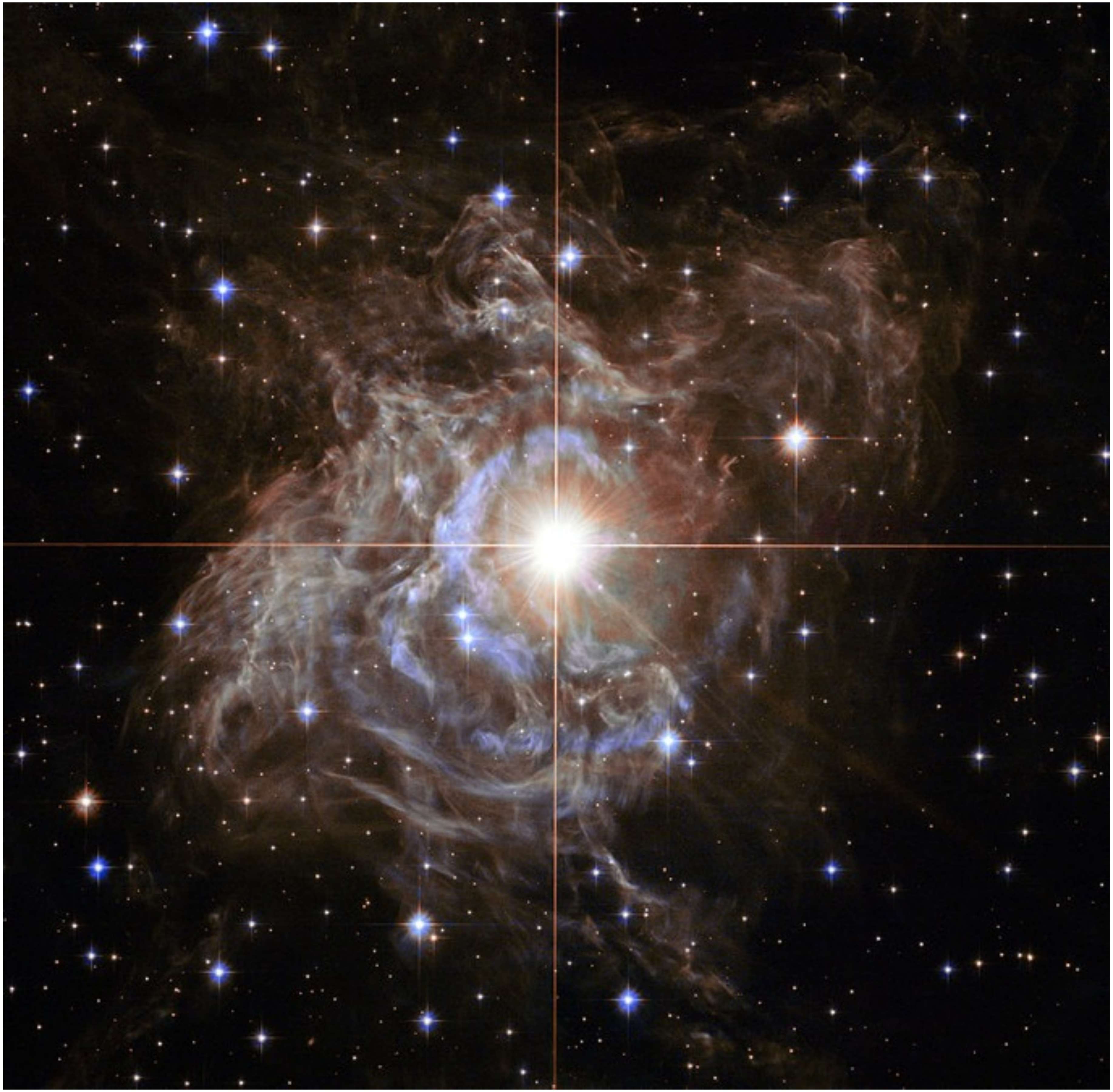
$$v = cz = H_0 D$$

Magnitudes

- Astronomers use logarithmic system of luminosities/ distances
- Apparent Magnitude *m*:
  Log of observed luminosity relative to standard
- Absolute Magnitude *M*:
  What apparent magnitude would be if object were 10 parsecs away
- Distance Modulus: $$\mu \equiv m - M = 5 \log_{10} \left[\frac{D_L}{1\,\mathrm{pc}} \right] - 5$$

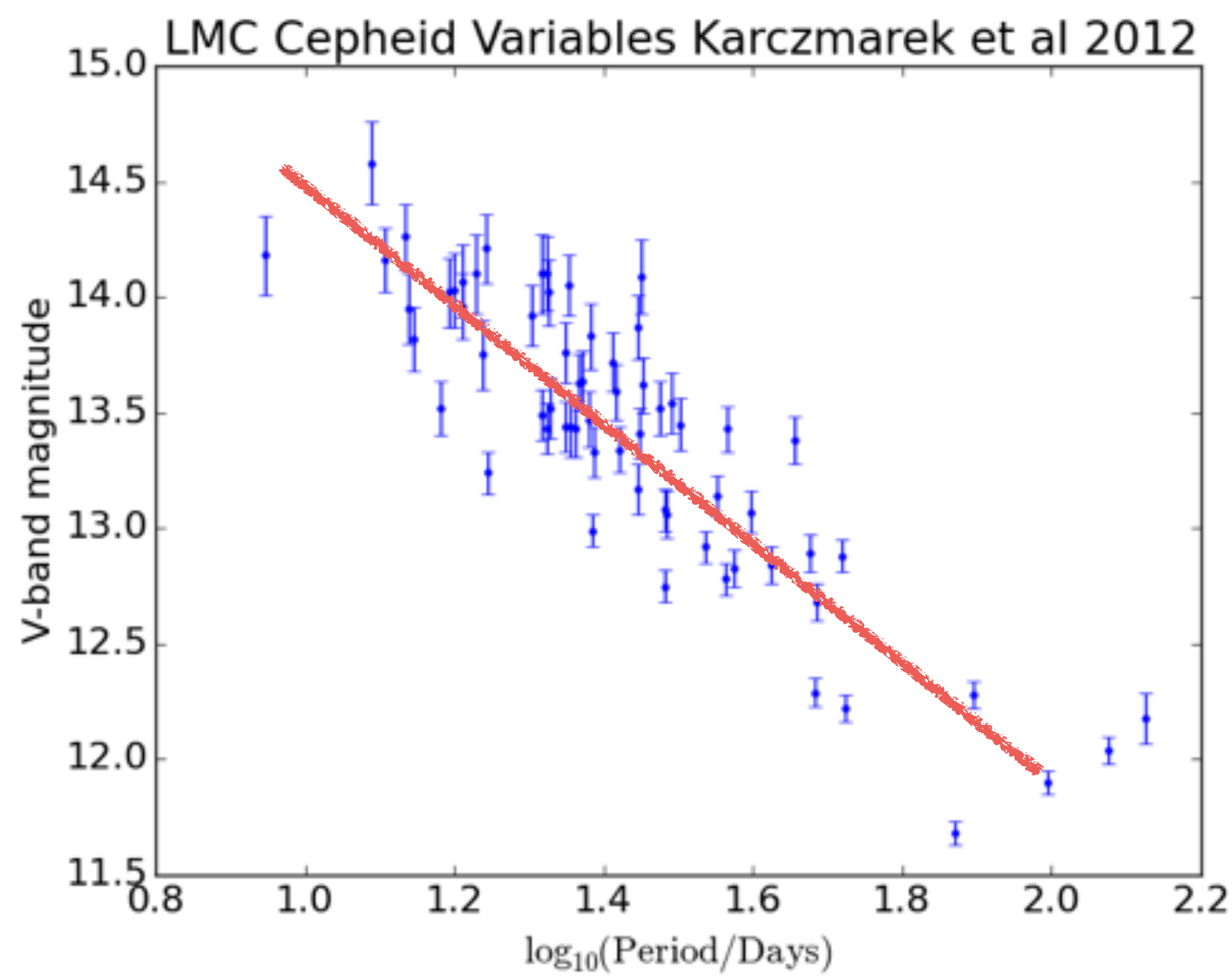
Cepheid Variables

Fitting straight lines with two sources of error



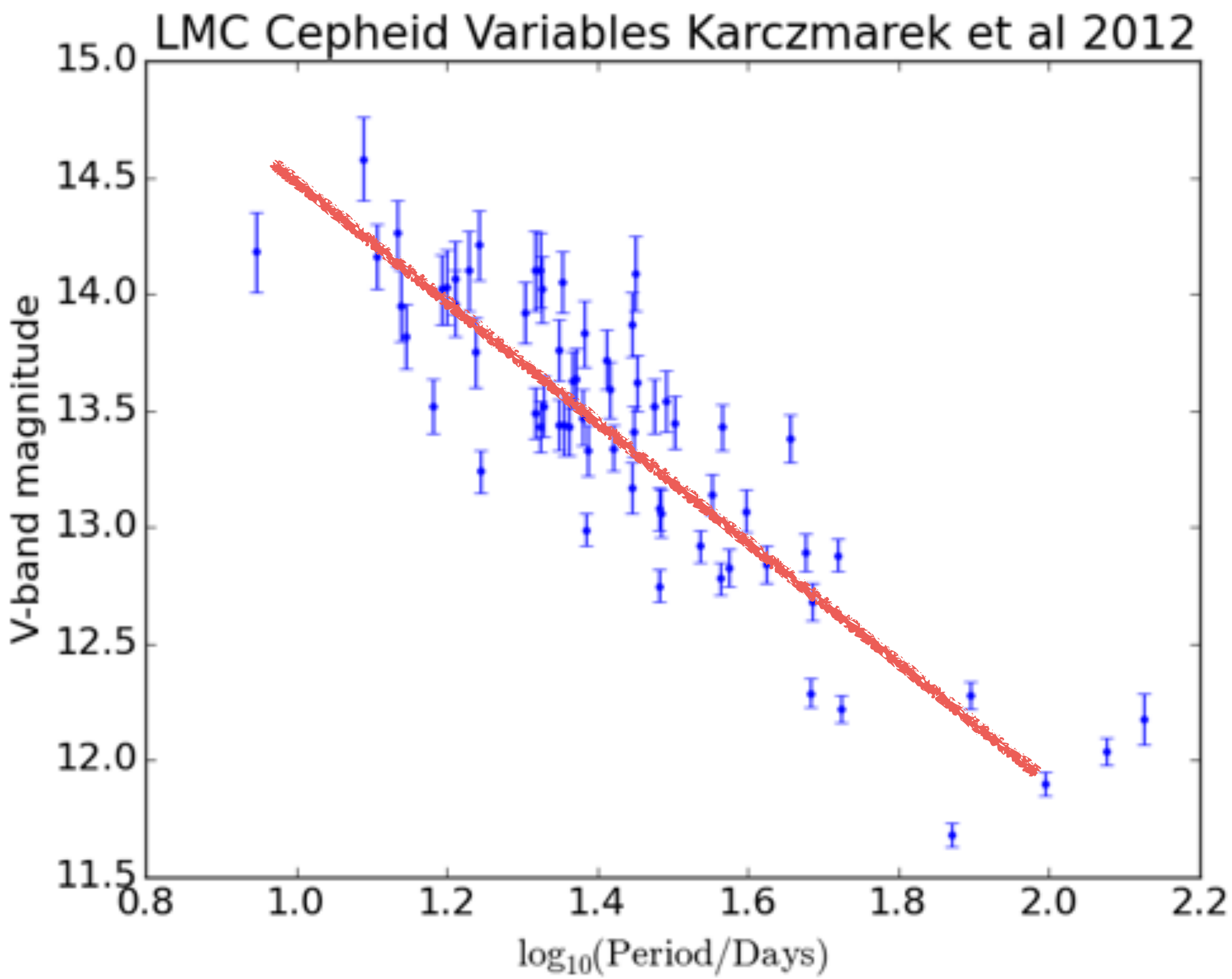
Cepheid Likelihoods: Model

- Linear relation between log period and magnitude
- Scatter clearly not just from noise
  - intrinsic scatter



Cepheid Likelihoods: Standard Inference

- Find extragalactic Cepheids too!
- Use the same linear fit to deduce their luminosity
    - Get redshift-distance relation H_0

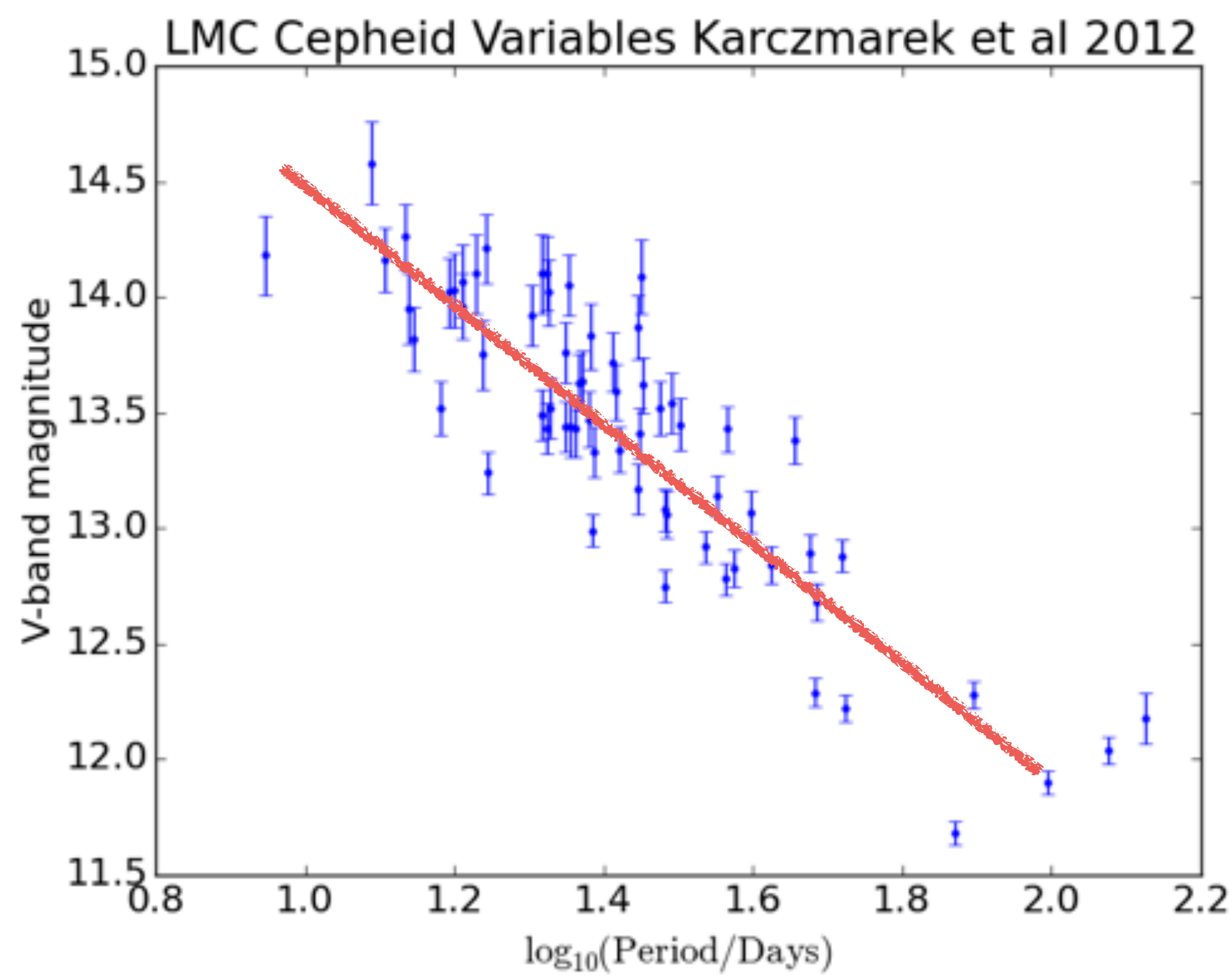


Cepheid Likelihoods: Model

$$\mu_i = \alpha + \beta \log_{10} [P_i/\mathrm{Days}]$$

$$V_i \sim \mathcal{N}(\mu_i, \sigma_{\mathrm{int}}^2)$$

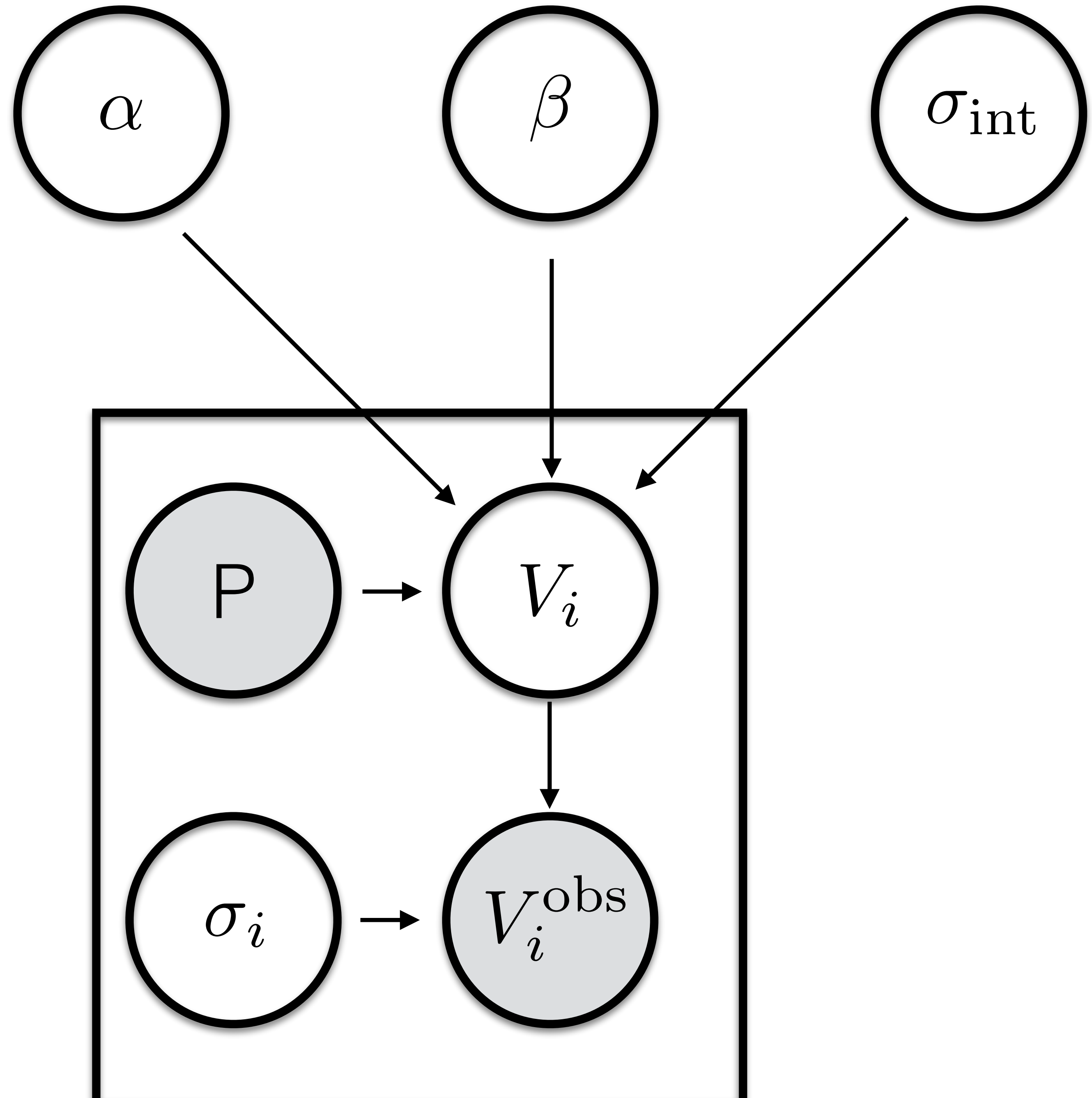
$$V_i^{\mathrm{obs}} \sim \mathcal{N}(V_i, \sigma_i^2)$$



Aside: Bayesian Networks

- Nice way to build/ illustrate Bayesian Networks aka Hierarchical Models

- Can use for inference



Cepheid Likelihoods: Standard Inference

$\mathbf{p} \equiv \{\alpha, \beta, \sigma_{\rm int}\}$

We want $P(\mathbf{p}|V^{\rm obs})$

$$P(V^{\rm obs}|\mathbf{p}) = \prod_i P(V^{\rm obs}|\mathbf{p})$$

Cepheid Likelihoods: Standard Inference

$$
\begin{aligned}
P(V_i^{\rm obs}|\mathbf{p}) &= \int P(V_i^{\rm obs}|\mathbf{p}V_i)P(V_i|\mathbf{p})\ {\rm d}V_i \\
&= \int P(V_i^{\rm obs}|V_i)P(V_i|\mathbf{p})\ {\rm d}V_i \\
&= \int \mathcal{N}(V_i^{\rm obs}; V_i, \sigma_i^2)\ \mathcal{N}(V_i^{\rm obs}; \alpha + \beta\log_{10}P_i, \sigma_{\rm int}^2)\ {\rm d}V_i
\end{aligned}
$$

Building Likelihoods: Example

- Exercise 1
  Show that this is given by:

$$P(V_i^{\rm obs}|\mathbf{p}) \propto \frac{1}{\sigma_{\rm int}^2 + \sigma_i^2} \exp -0.5 \left(\frac{(V_i^{\rm obs} - (\alpha + \beta \log_{10} P_i))^2}{\sigma_{\rm int}^2 + \sigma_i^2} \right)$$

- In one of the exercises you will evaluate and use this likelihood using some simulated data

H_0 Likelihoods:

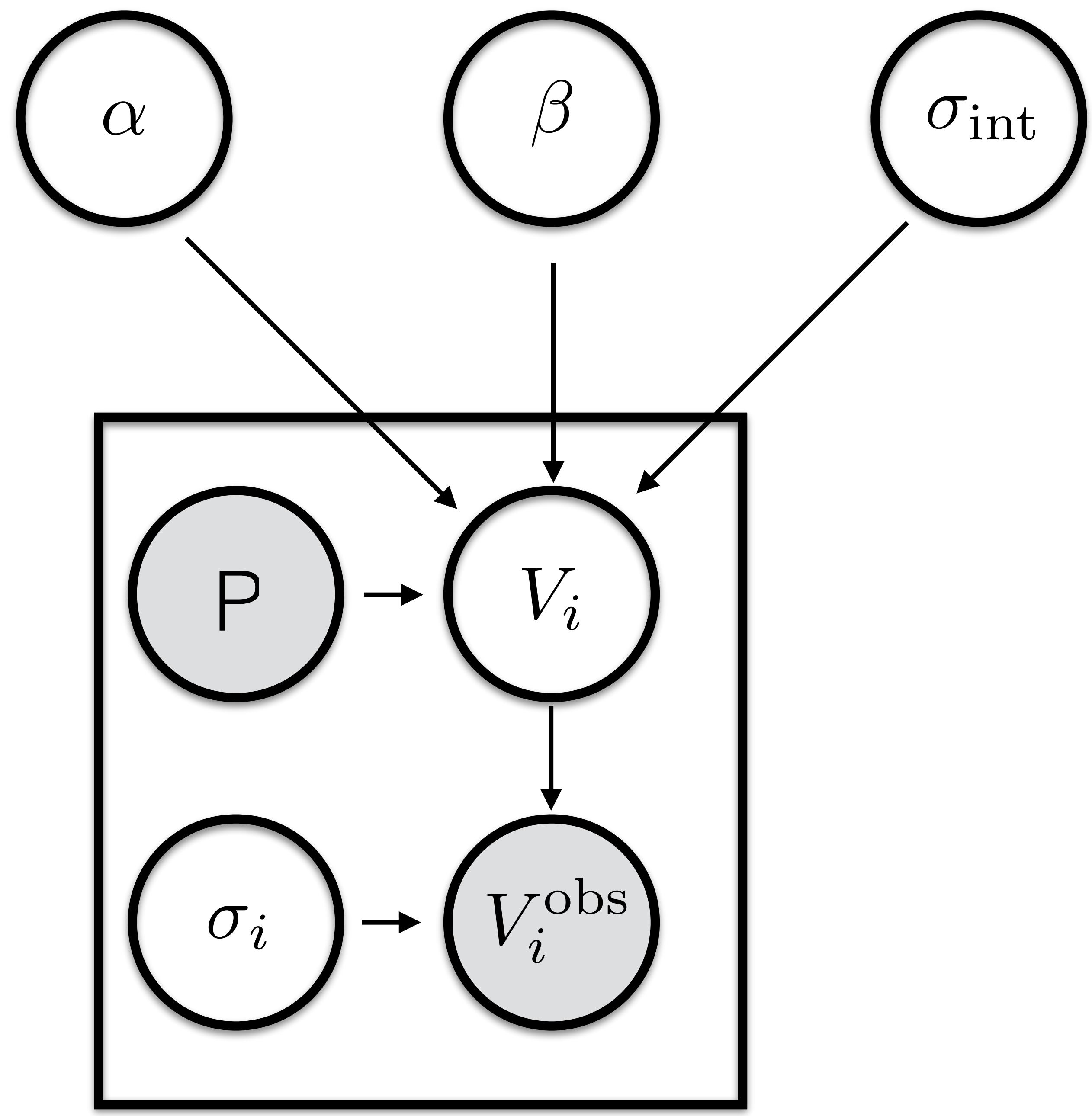
- Standard approach:
    - Use the LMC to find maximum likelihood values of the alpha, beta, and sigma parameters
    - Fix these parameters to analyse cosmological

H_0 Likelihoods
A better way!

- Simultaneously analyse LMC data and cosmological cepheids!

Exercise 2

- Exercise 2: Extend this Bayesian Network diagram to do the simultaneous analysis of LMC and extragalactic Cepheids



Exercise 3

- Code up a function to calculate this likelihood

Type IA Supernovae

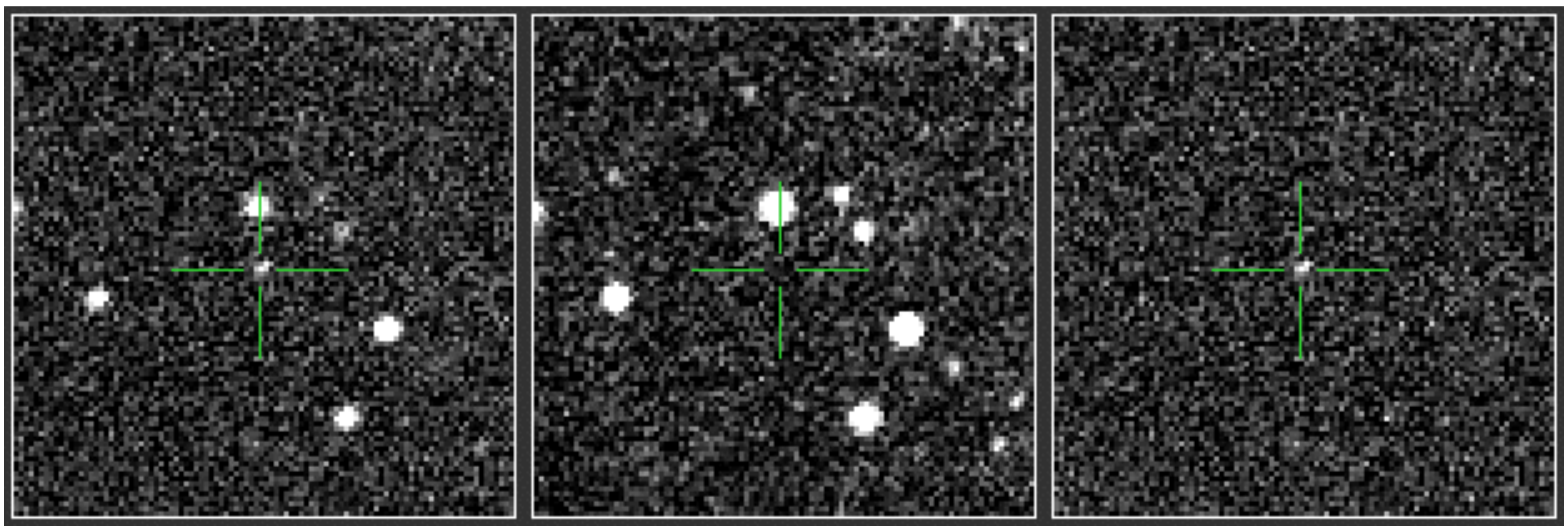
"Standardizing" using modelling

Type IA Supernovae Physics

- White Dwarf accretes matter from binary companion.
- Reaches Chandrasekhar limit 1.39 $M_\odot$
- Core becomes relativistic => equation of state changes => core cannot support mass =>supernova!
- Same mass => same brightness

Type IA Supernovae Observations

- Detection

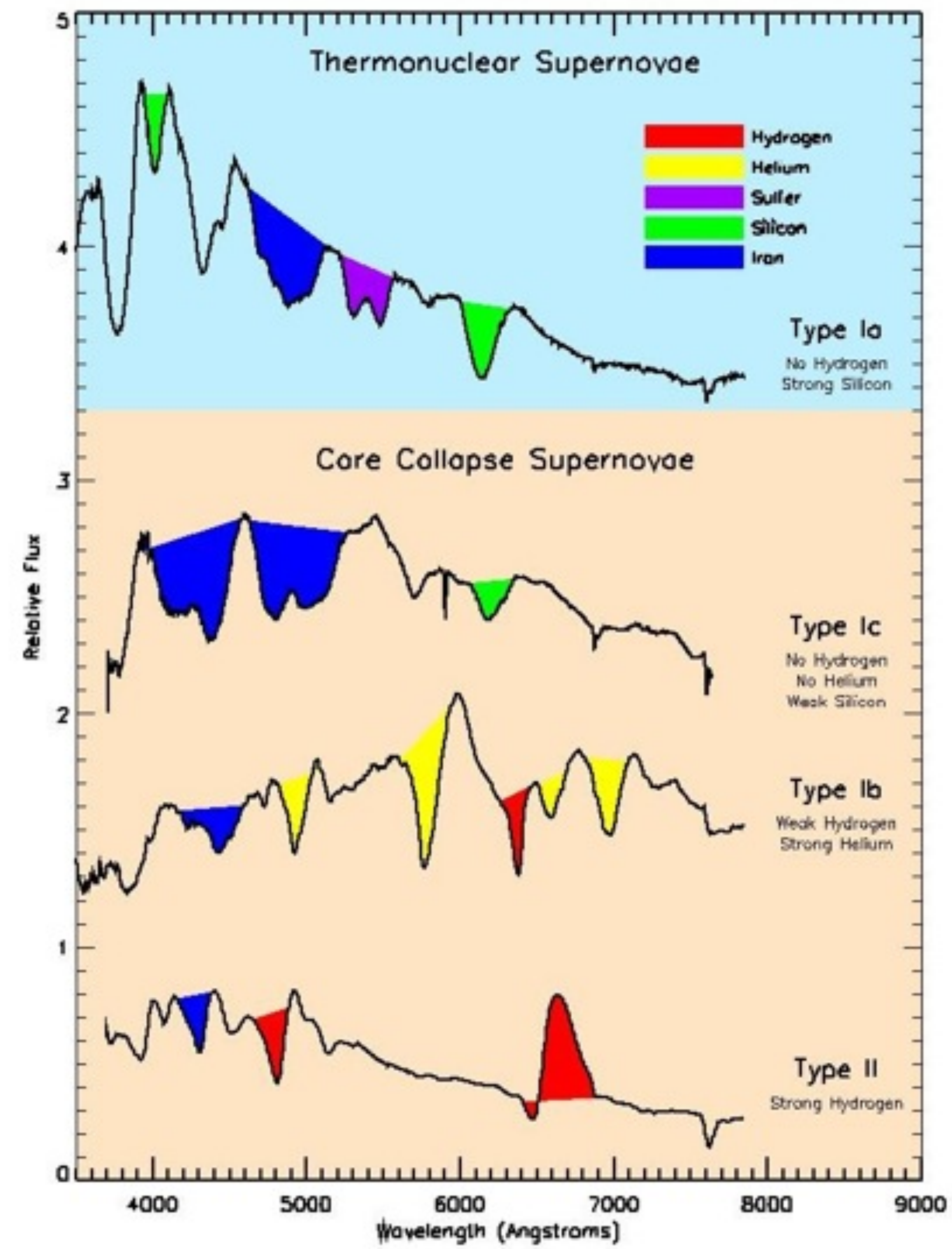


New image Old image Difference

http://supernova.galaxyzoo.org/discoveries

Type IA Supernovae Observations

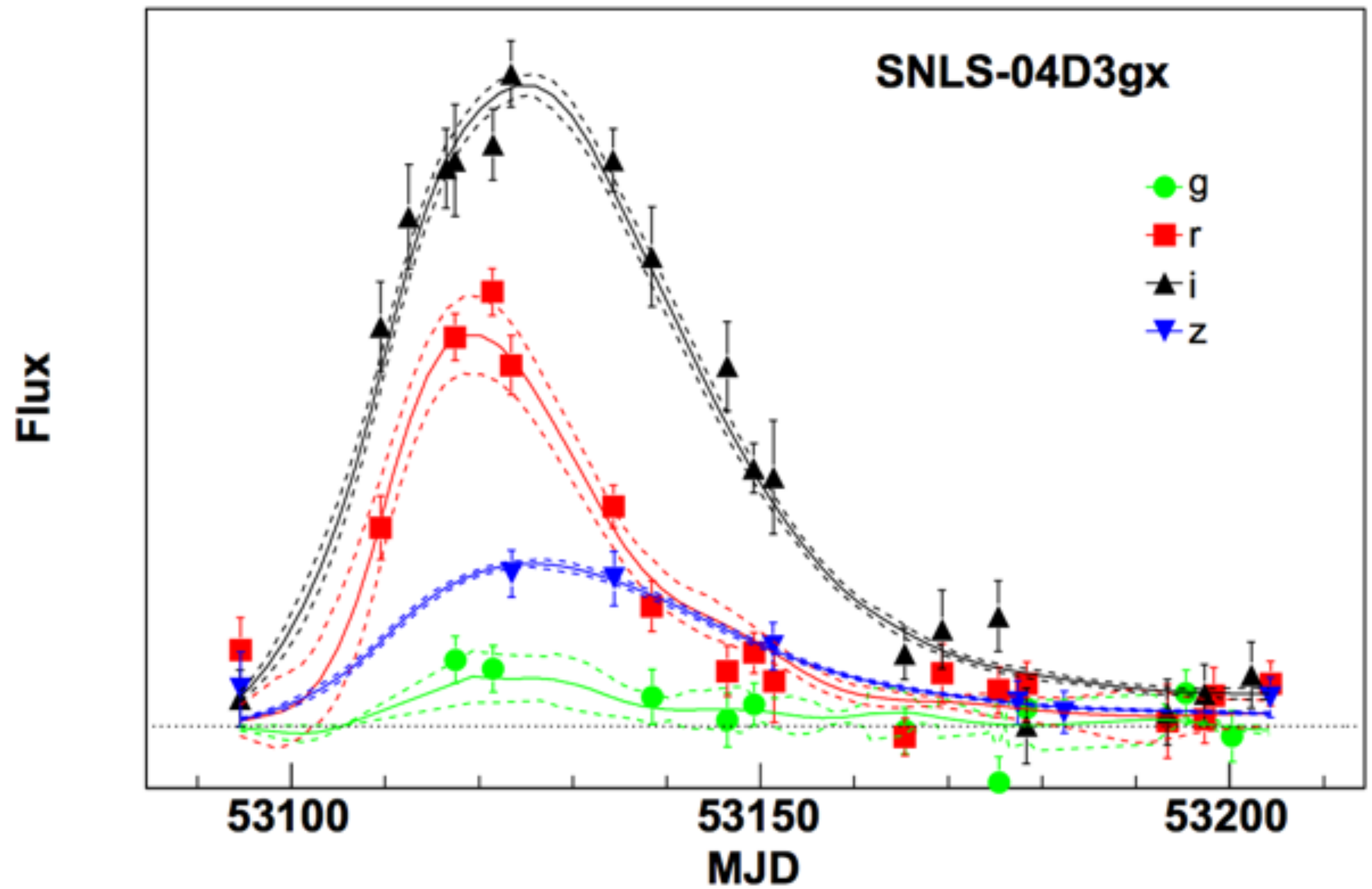
- Spectroscopic follow up
  - redshift
  - type



http://supernova.lbl.gov/

Type IA Supernovae Observations

- Photometric follow up
  - Light curve



http://supernova.lbl.gov/

Type IA Supernovae Observations

- Standard**izable** Candle
- Calculate luminosity from curve shape
    - Various empirical methods
      SALT2: color and stretch parameters

Type IA Supernovae Inference

- Measure distance modulus (log luminosity)

$$\mu = 5\log_{10}\left(\frac{D_L}{1\ \mathrm{Mpc}}\right) + 25$$

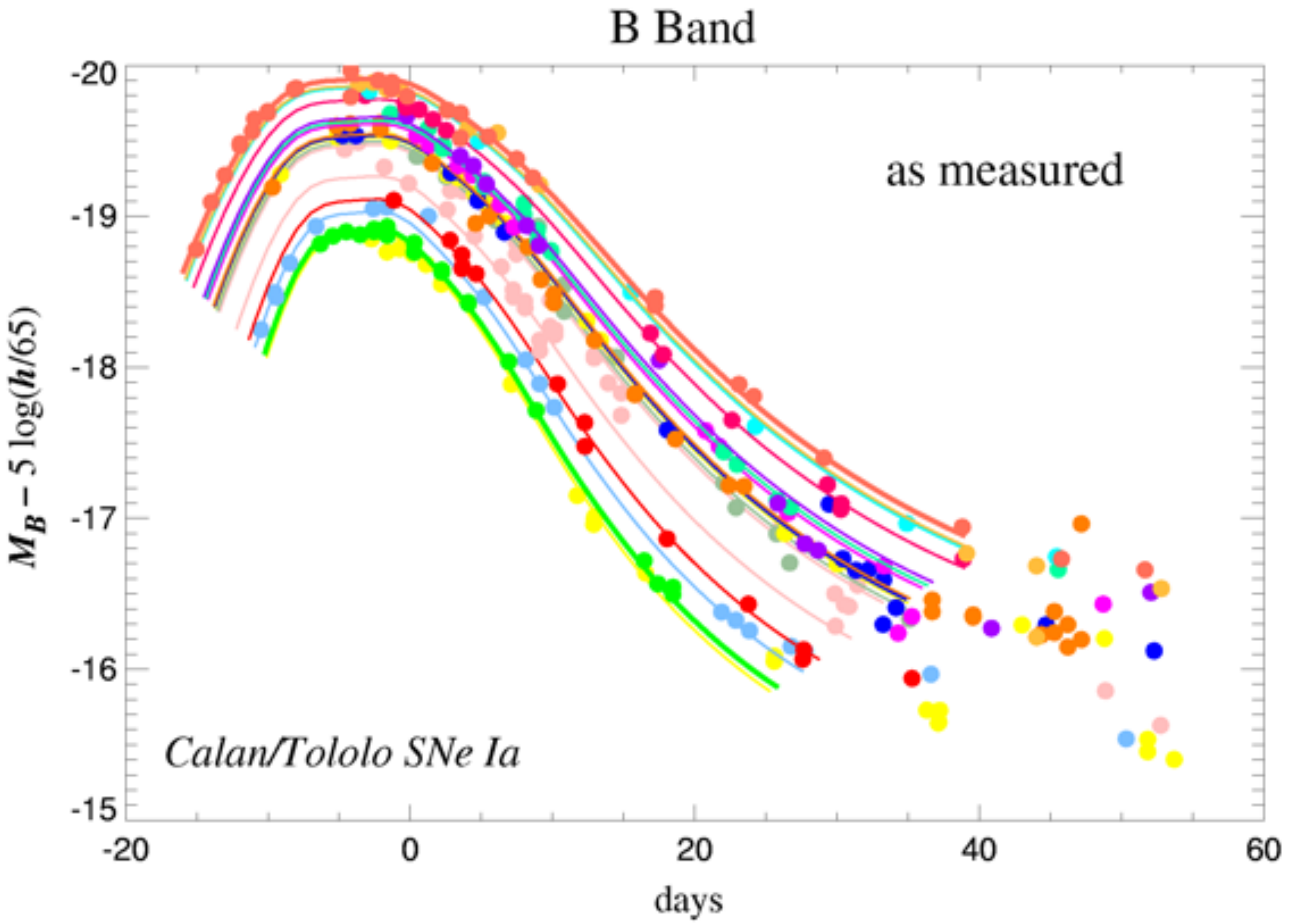
$$\mu = m - M$$

Apparent Magnitude (measured)

Absolute Magnitude (modelled)

Type IA Supernovae Inference

- Need a model for the absolute magnitude given light curve shape
- First, model the shape of the light curve
  - color and stretch

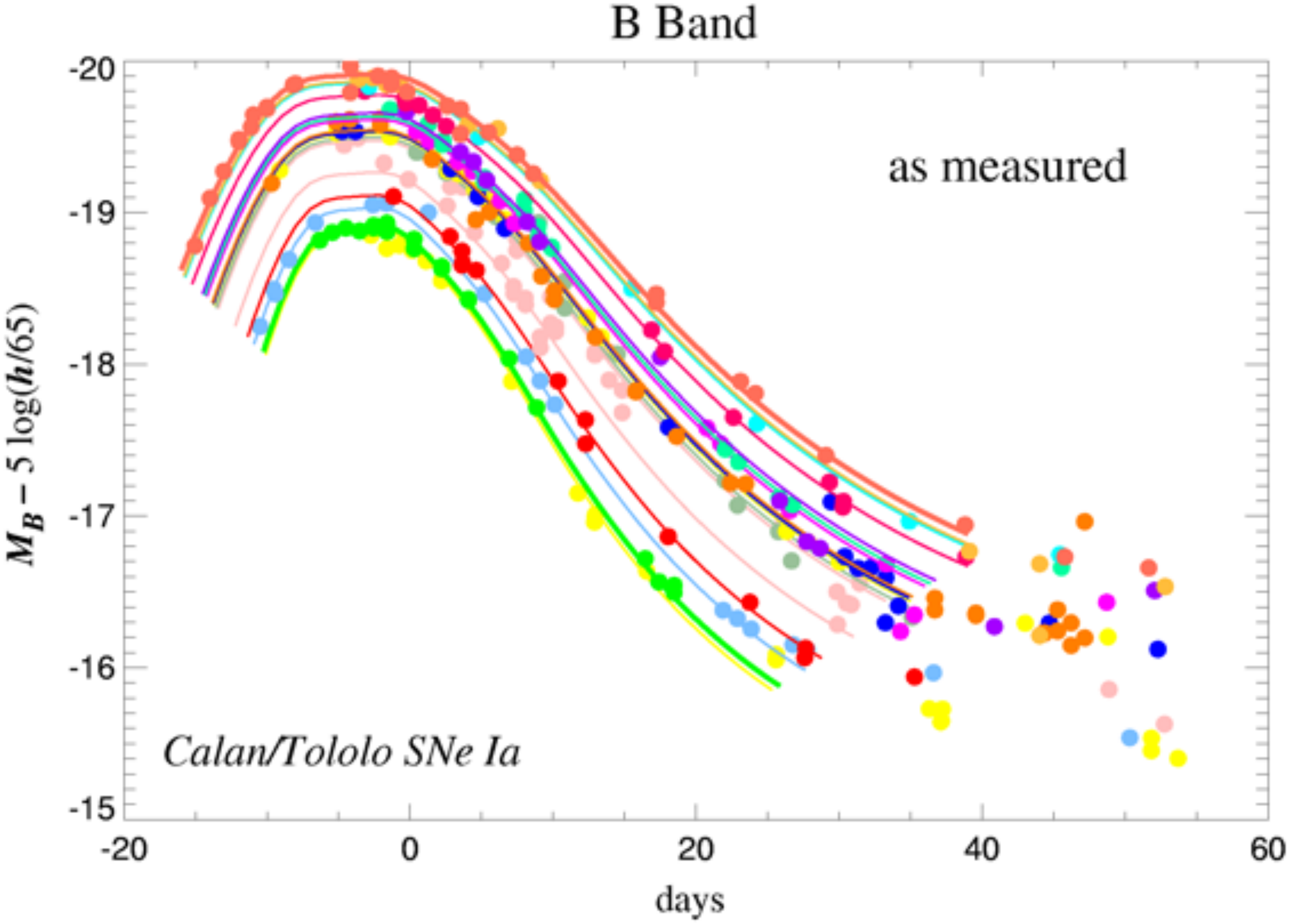


Light Curve Model

$$c = (B - V)_{\max} - \langle B - V \rangle$$

$$M_0(t, \lambda) = \langle M(t, \lambda) \rangle$$

$$M_1(t, \lambda) = \text{First principal component of M}$$



$$M(\lambda, t) = x_0 \left[M_0(t, \lambda) + x_1 M_1(t, \lambda) + \ldots \right] \mathrm{e}^{[cf(\lambda)]}$$

Abs Magnitude Model

- Find empirically from nearby SNe with known Cepheid distance that:

$$M_i = M - \alpha x_{1i} + \beta c_i$$

Type IA Supernovae Inference

- Nuisance parameters included in model:

Measured from light curves

$$\mu = m^* - M_0 + \alpha x_1 - \beta c$$

Fitted nuisance parameters

Type IA Supernovae Inference

- So overall parameters = $\{\Omega_m, \Omega_\Lambda, H_0, M_0, \alpha, \beta\}$
- Steps:
  - Theory: $$D_L^{\text{theory}}(z_{\text{obs}}) = D_L(\Omega_M, \Omega_\Lambda)$$

$$\mu^{\text{theory}} = 5\log_{10}\left(\frac{D_L^{\text{theory}}}{1\ \text{Mpc}}\right) + 25$$

  - "Observable":

$$\mu^{\text{obs}} = m^{*\text{obs}} - M_0 + \alpha x_1^{\text{obs}} - \beta c^{\text{obs}}$$

Type IA Supernovae Inference

- Covariance = (Messy Combination of noise and nuisance params)!

- Likelihood multivariate Gaussian

$$\mathcal{L} \propto |C|^{-1} \exp -\frac{1}{2}(\mu^{\mathrm{obs}} - \mu^{\mathrm{theory}})^T C^{-1} (\mu^{\mathrm{obs}} - \mu^{\mathrm{theory}})$$

- Caution! The priors do matter here! But by coincidence the maximum-likelihood nuisance params work okay. See arxiv:1207.3705

Perturbations

Perturbations

- Can probe fluctuations on top of the background mean cosmology
- Need relativistic perturbation theory to predict
    - Will very briefly discuss extreme basics now!
    - Discuss codes that solve these for you, and approximations

Perturbation Theory

$$\mathrm{d}s^2 = a^2\left(\mathrm{d}\tau^2 + \mathrm{d}\mathbf{x}^2\right)$$

$$\rho(t)$$

perturb

$$\mathrm{d}s^2 = a^2\left(-(1+2\Psi)\mathrm{d}\tau^2 + (1-2\Phi)\mathrm{d}\mathbf{x}^2\right)$$

$$\rho(t) + \delta\rho(\mathbf{x}, t)$$

Perturbation Theory

- Phi and Psi are linear order perturbations to metric
  Have similar perturbations to densities, velocities, and moments.
- All quantities in Fourier space
- Insert perturbed forms into Einstein and geodesic equations

Perturbation Theory

$$\begin{aligned}
k^2\phi + 3\frac{\dot{a}}{a}\left(\dot{\phi} + \frac{\dot{a}}{a}\psi\right) &= 4\pi G a^2 \delta T^0{}_0(\mathrm{Con})\,, \\
k^2\left(\dot{\phi} + \frac{\dot{a}}{a}\psi\right) &= 4\pi G a^2 (\bar{\rho} + \bar{P})\theta(\mathrm{Con})\,, \\
\ddot{\phi} + \frac{\dot{a}}{a}(\dot{\psi} + 2\dot{\phi}) + \left(2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right)\psi + \frac{k^2}{3}(\phi - \psi) &= \frac{4\pi}{3} G a^2 \delta T^i{}_i(\mathrm{Con})\,, \\
k^2(\phi - \psi) &= 12\pi G a^2 (\bar{\rho} + \bar{P})\sigma(\mathrm{Con})\,,
\end{aligned}$$

Conformal Newtonian Gauge

$$\begin{aligned}
\dot{\delta} &= -(1+w)\left(\theta - 3\dot{\phi}\right) - 3\frac{\dot{a}}{a}\left(\frac{\delta P}{\delta\rho} - w\right)\delta\,, \\
\dot{\theta} &= -\frac{\dot{a}}{a}(1-3w)\theta - \frac{\dot{w}}{1+w}\theta + \frac{\delta P/\delta\rho}{1+w}k^2\delta - k^2\sigma + k^2\psi\,.
\end{aligned}$$

Boltzmann Codes

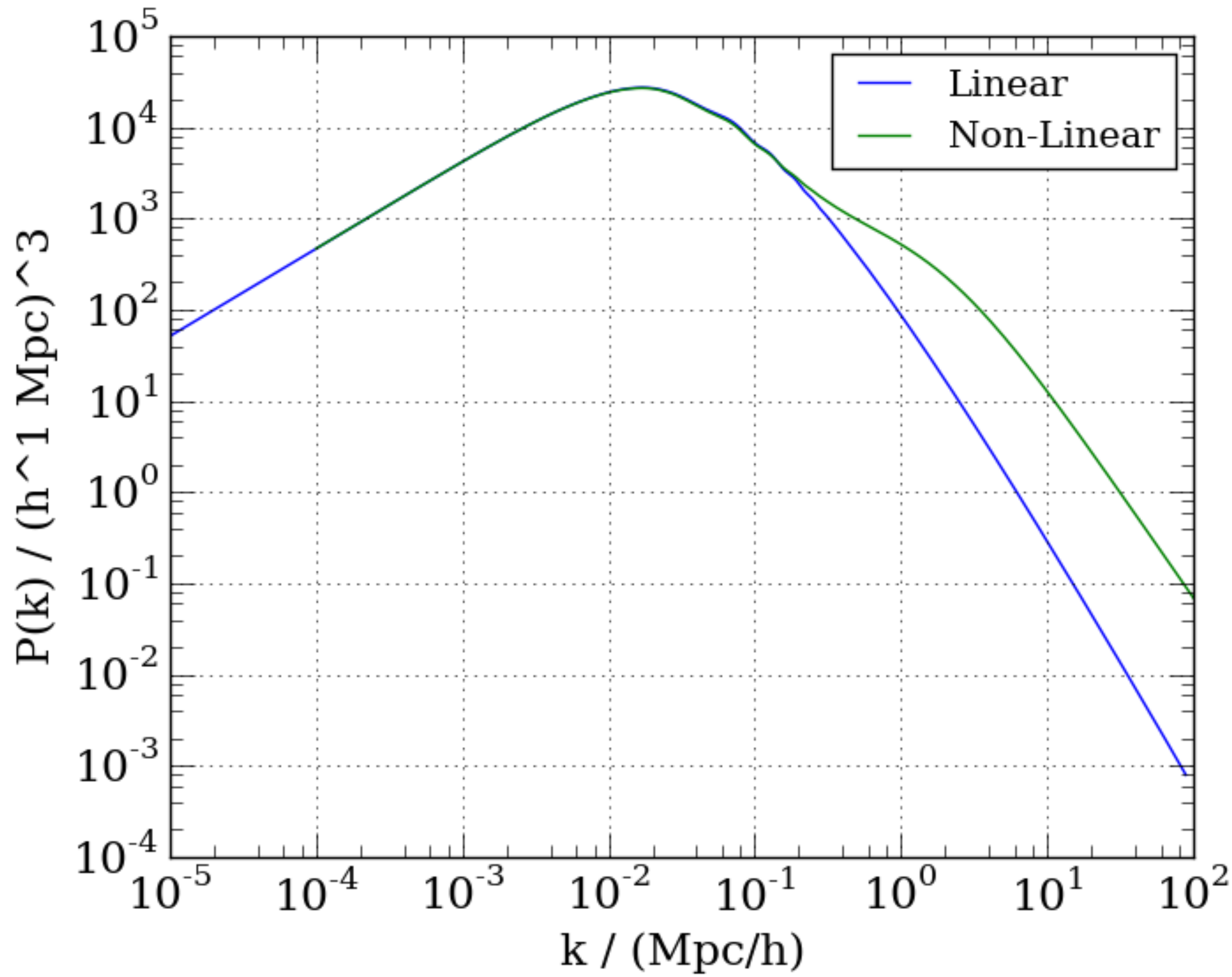
- Convert second order equations into first order by introducing derivatives as new variables:

$$\frac{\partial \mathbf{A}}{\partial \tau} = f(\mathbf{A}) \quad \text{Boltzmann Equation}$$

- Numerically integrate

Perturbation Theory

- Boltzmann Codes (`CAMB`, `CLASS`) solve the Boltzmann equation and evolve perturbations at different k
  - Actually evolve standard deviation for that wavelength - Gaussian fields - see CMB lectures
- This gives *linear* evolution - small scales are *nonlinear*



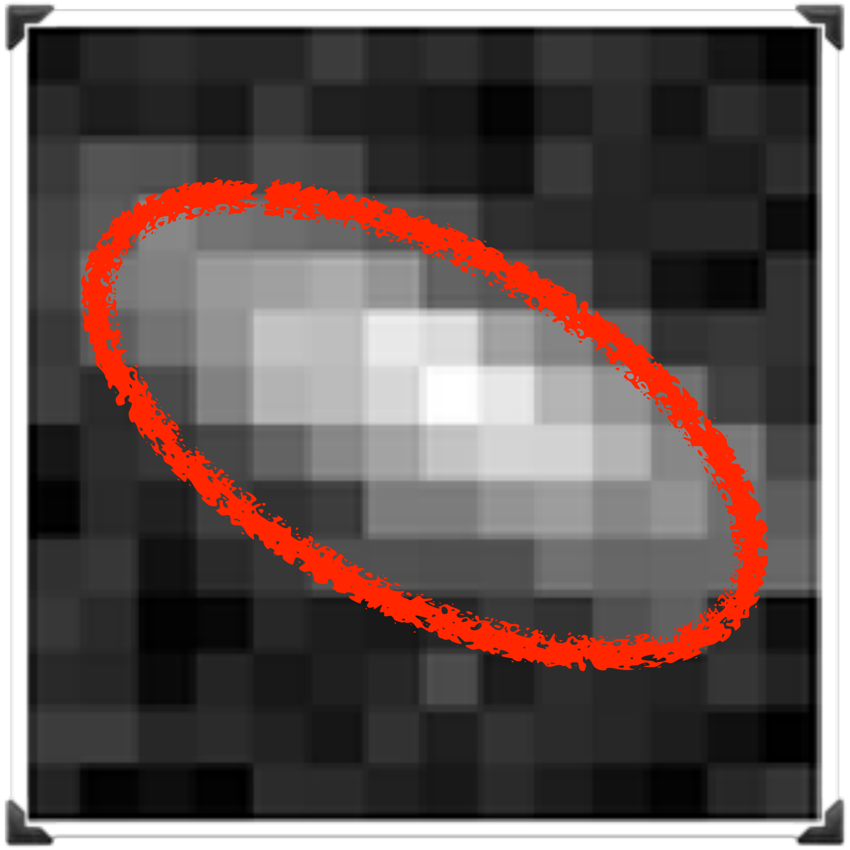
Linear
Non-Linear
P(k) / (h^1 Mpc)^3
k / (Mpc/h)
10^5
10^4
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^0
10^1
10^2

3. Weak Lensing

Modelling systematic errors with new parameters

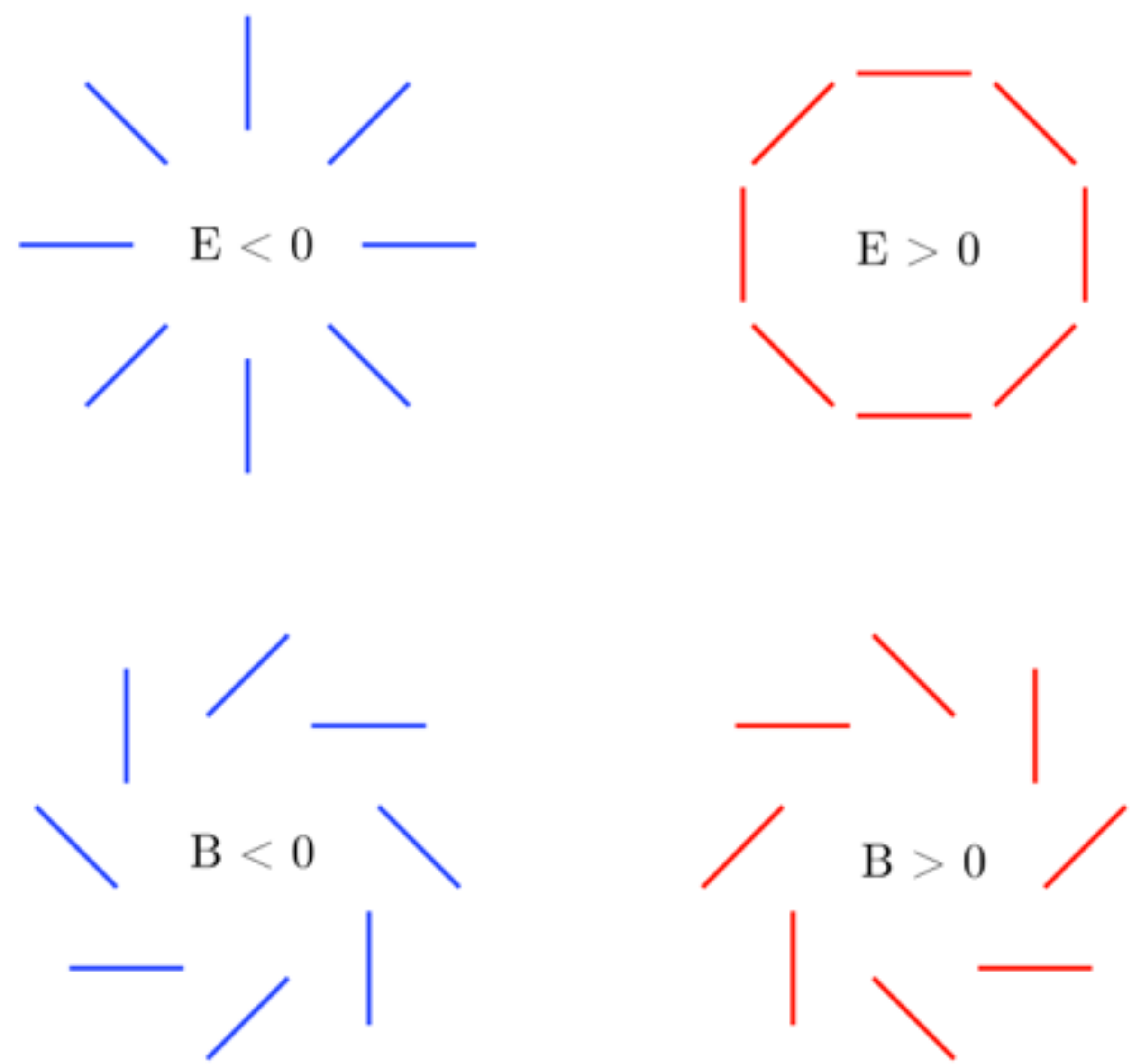
Weak Lensing

- Observation: galaxy ellipticities and magnitudes
- Compute 2D Fourier space correlation function



Weak Lensing

- Lensing is a spin-2 field
- Decompose into E and B modes
- E mode carries cosmological information



Weak Lensing Likelihood

$$P(\hat{C}_\ell^{ij}|p) = \mathcal{N}\left(C_\ell^{ij}(p), \Sigma_\ell^{ij}(p)\right)$$

- Based on simulations
- Note that covariance depends on parameters!

Shear Spectra

Compute with
Boltzmann + NL

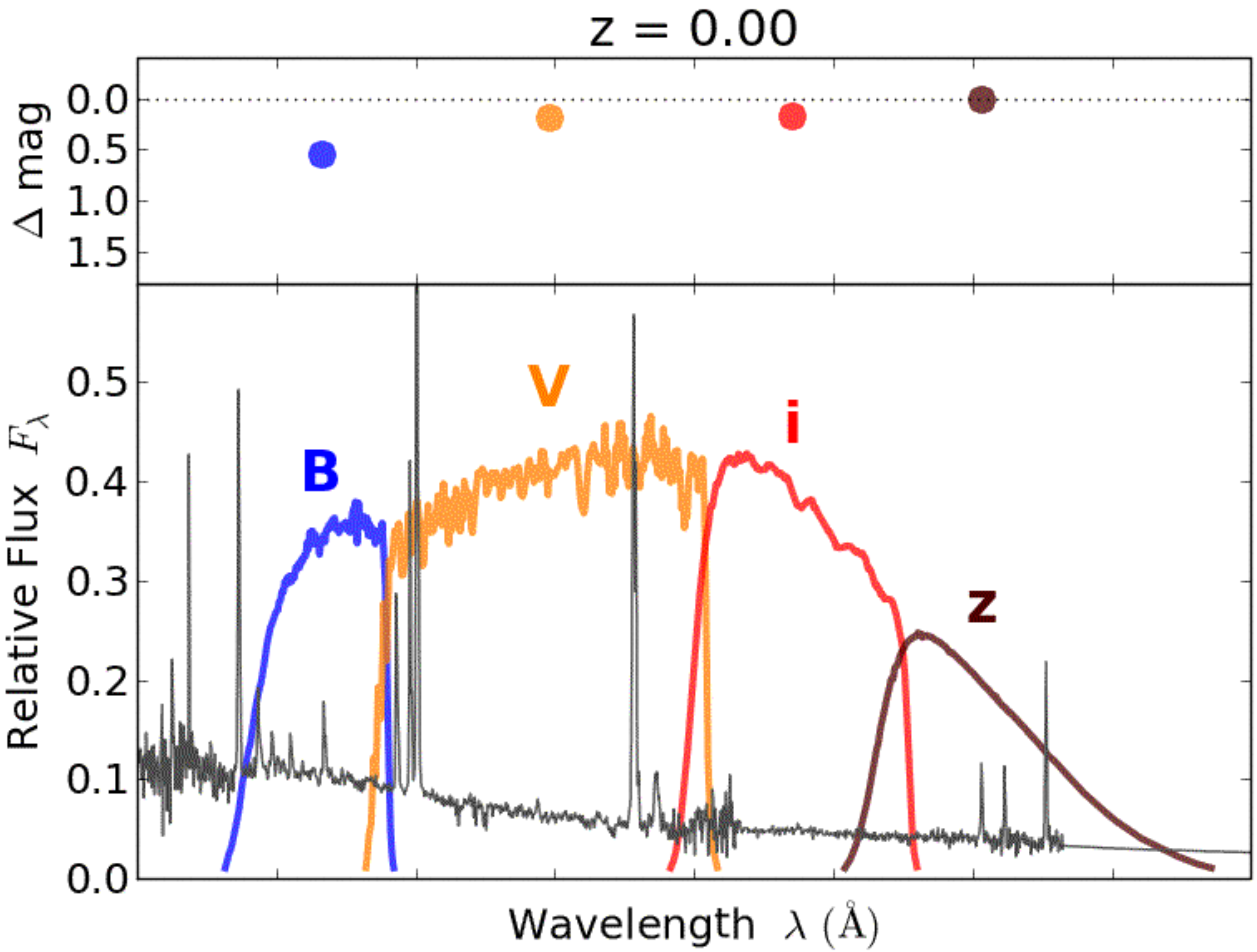
$$C_\ell^{\mathrm{EE}} = \left(\frac{3H_0^2\Omega_M}{2c^2}\right)^2 \int_0^\infty \frac{W_i(\chi)W_j(\chi)}{\chi^2} P(k=\ell/\chi, \chi)\, \mathrm{d}\chi$$

$$W_i(\chi) = \frac{a}{\chi}\int_0^\chi n_i(\chi')\, \frac{\chi'-\chi}{\chi'}\, \mathrm{d}\chi'$$

Photometric
redshifts

Background
evolution

Photometric Redshifts



http://www.stsci.edu/~dcoe

Photometric Redshift Methods

- Templates
- Machine Learning

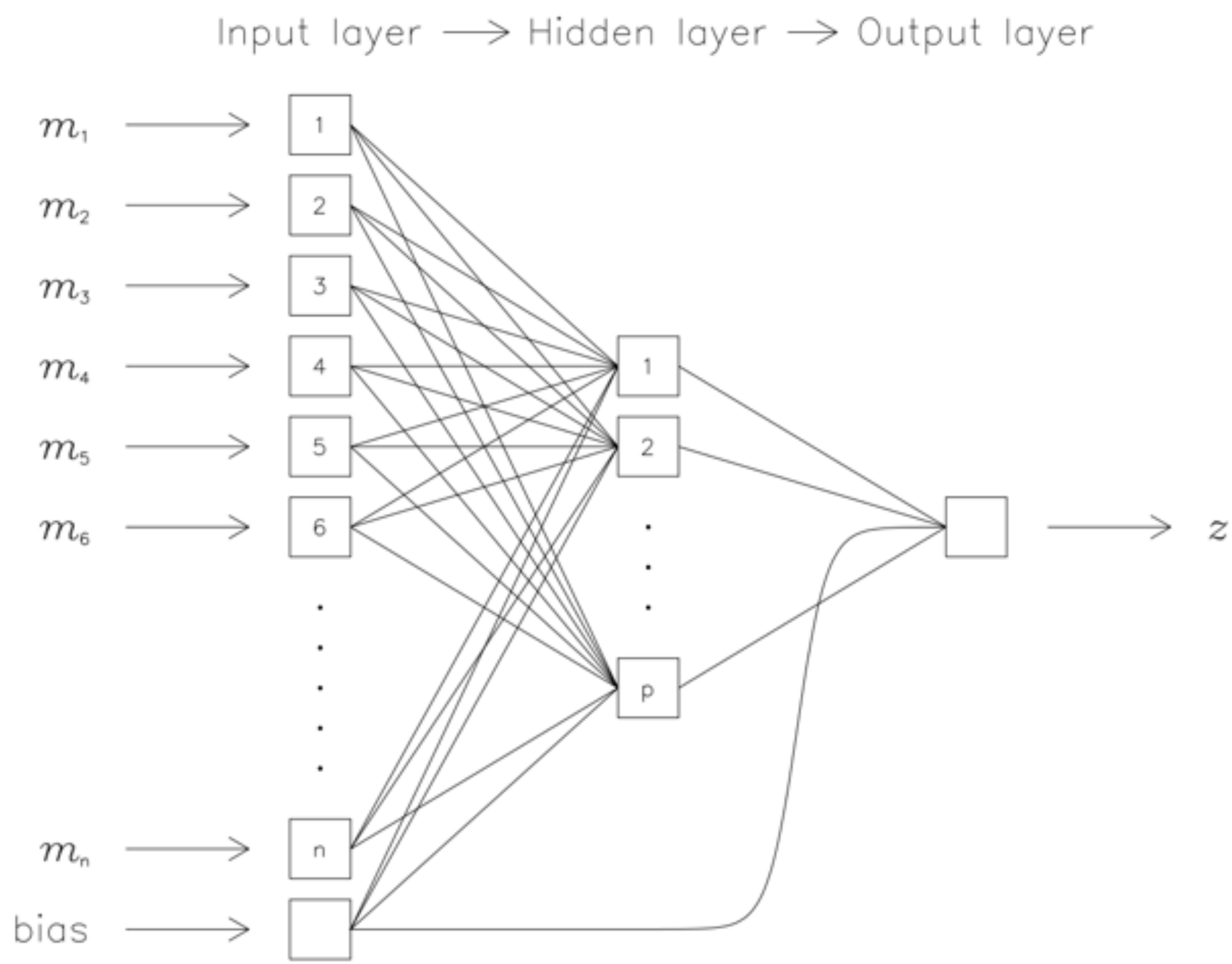


Photo-z Errors

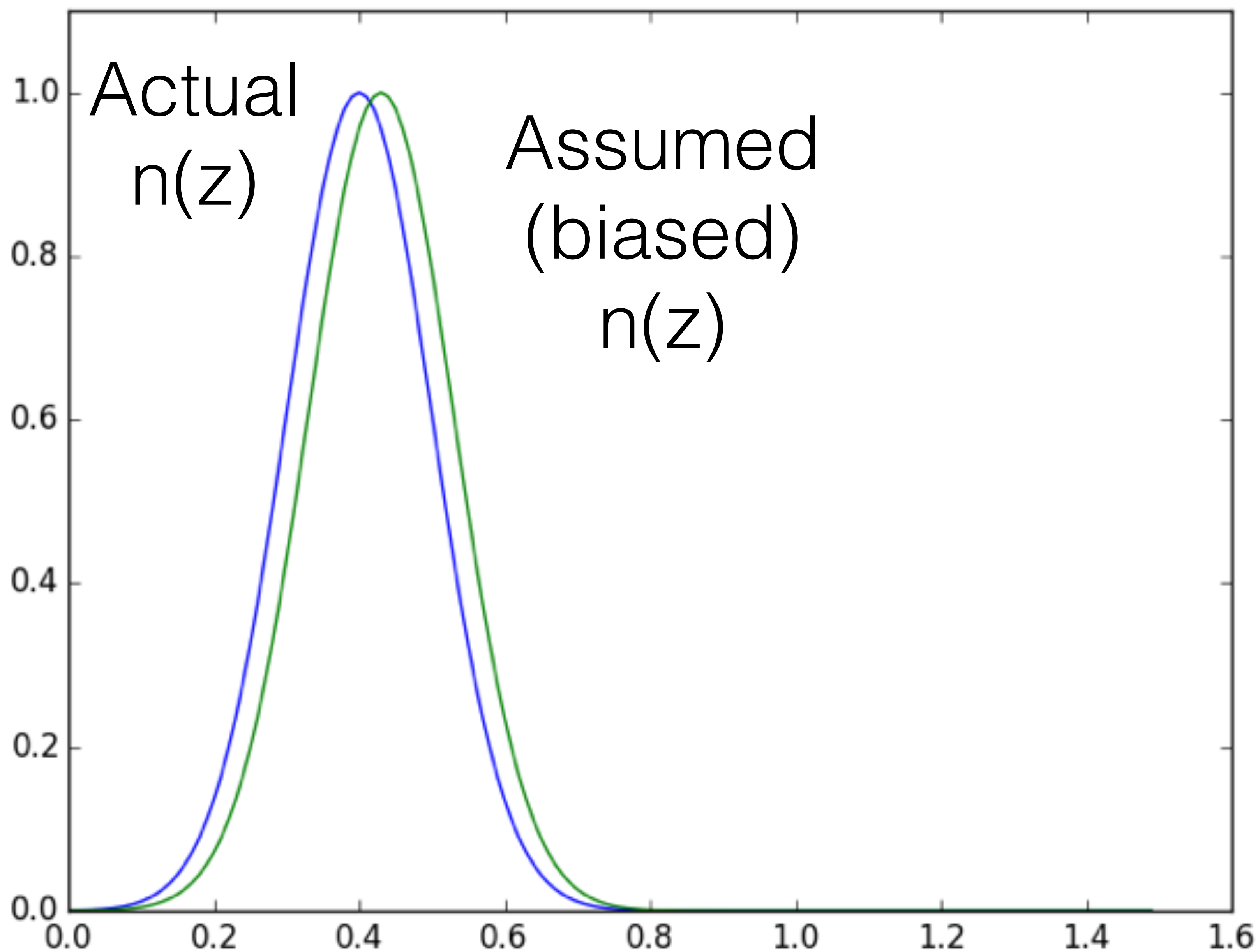
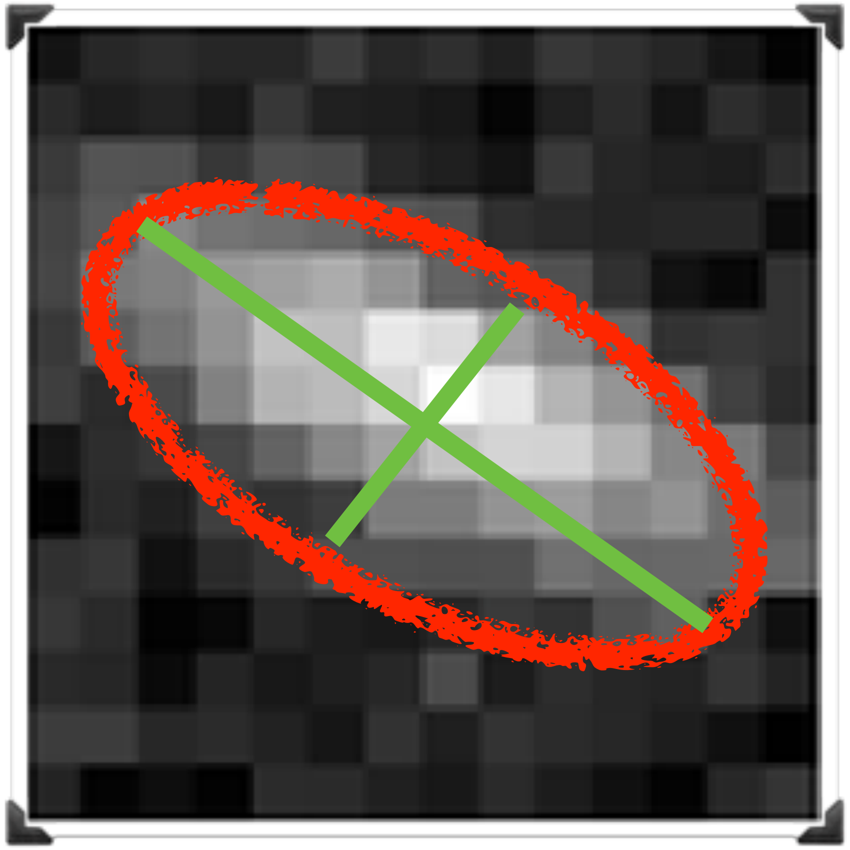


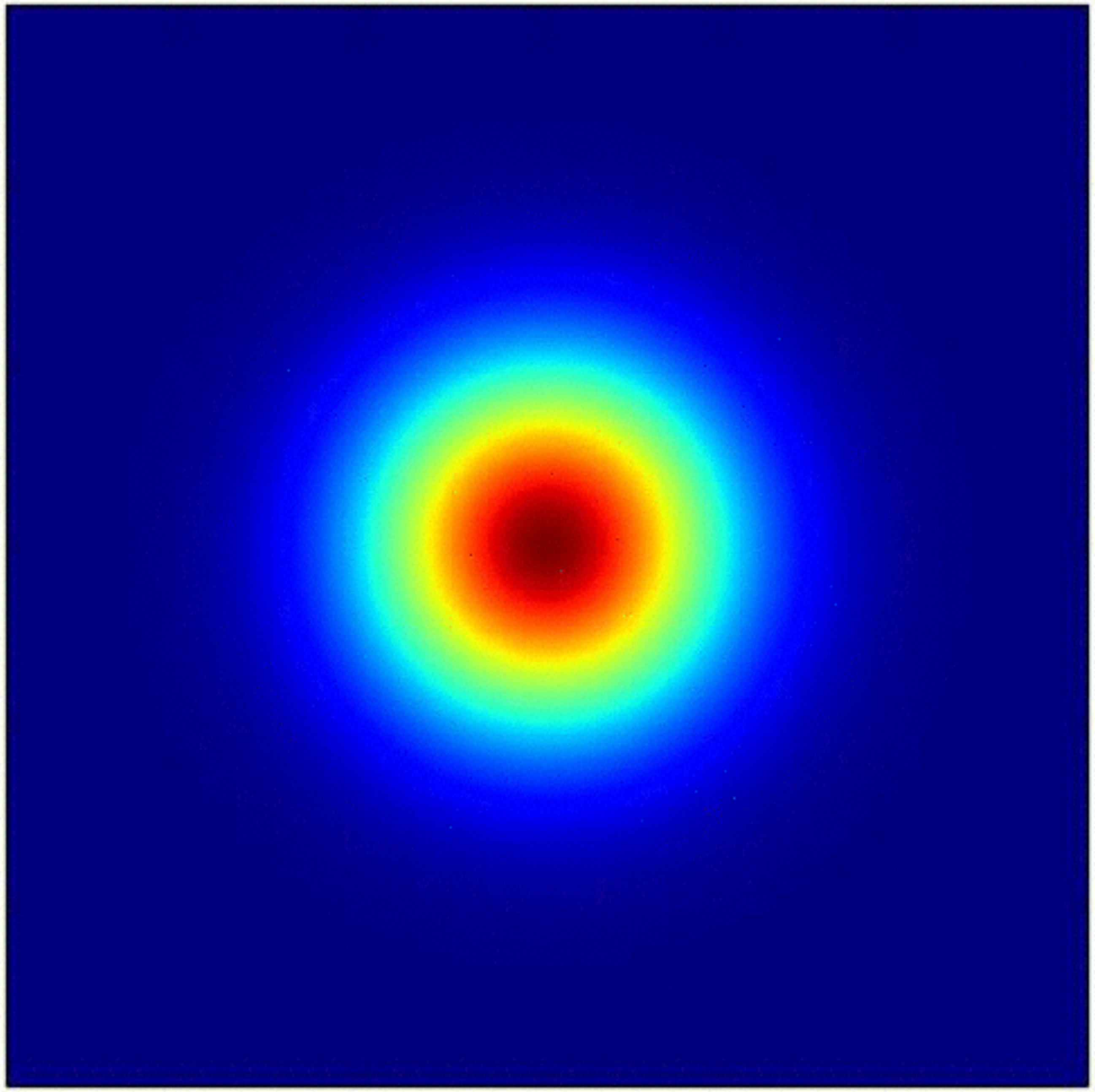
Photo-z Errors

- Let $n_i(z)$ -> $n_i(z+b_i)$
- Our likelihood now includes an additional set of parameters b_i
- Marginalize over b_i

Shear Errors



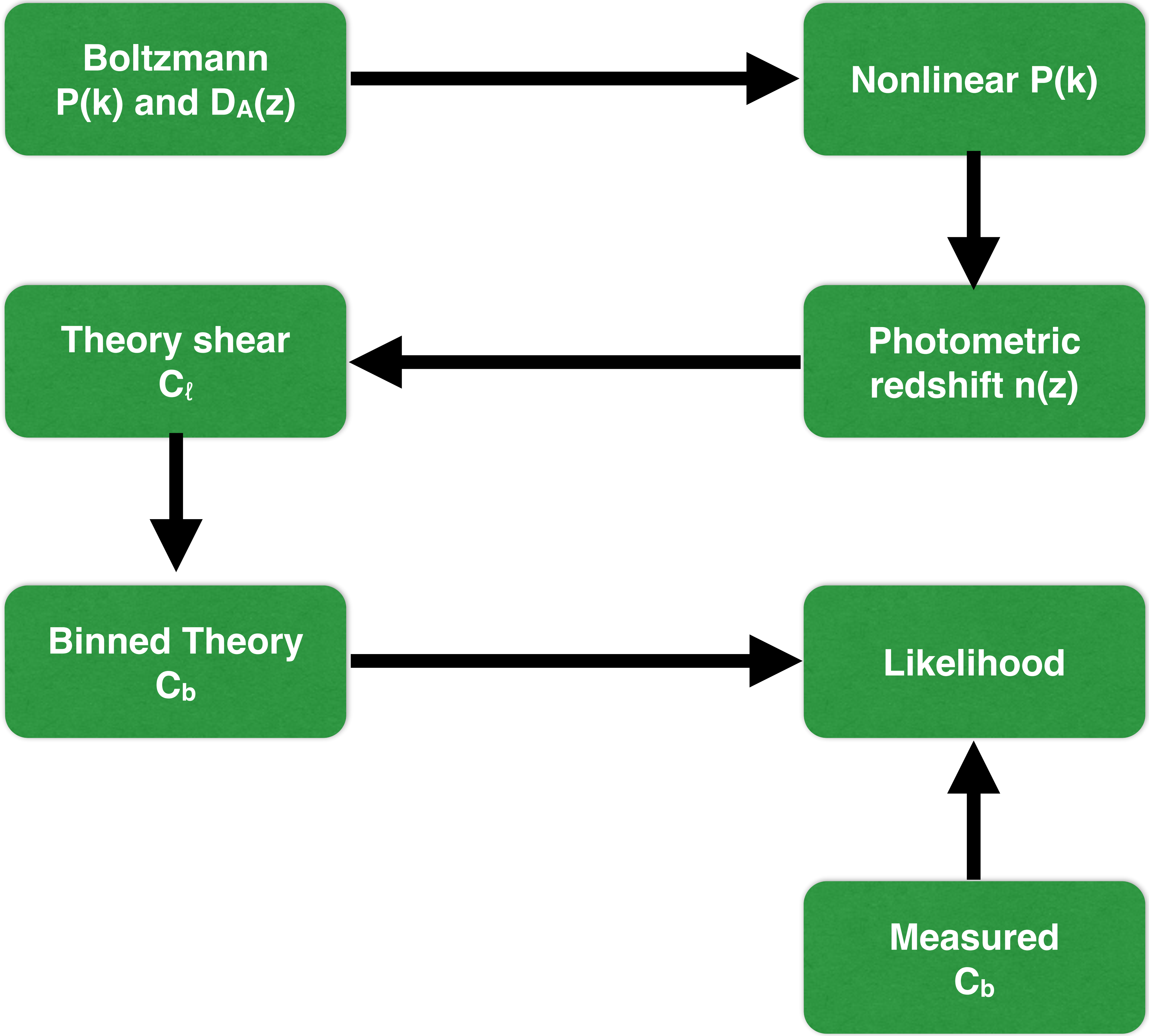
$$e = \frac{a-b}{a+b}$$

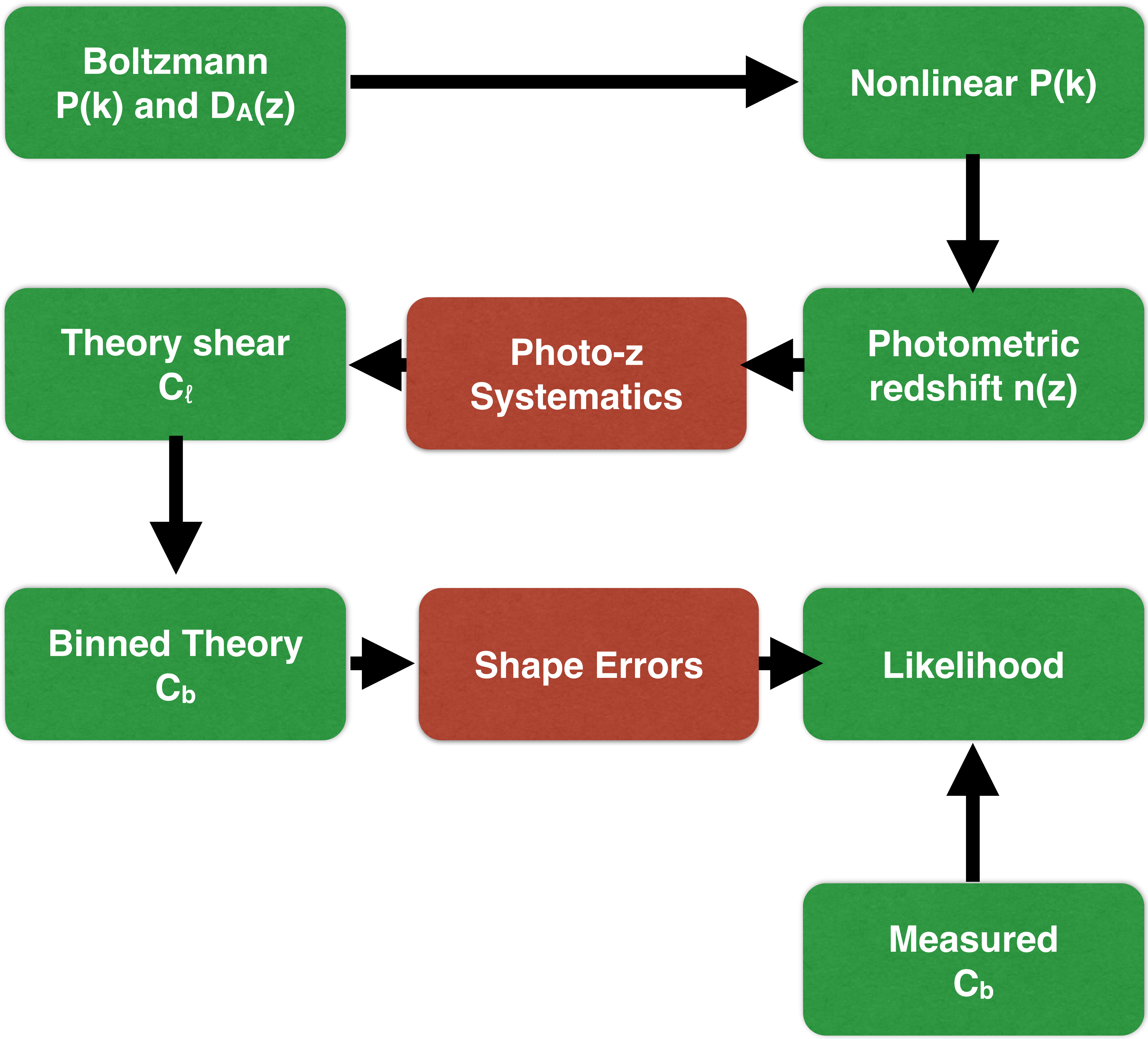


Shear Errors

$$e \to (1+m)e$$
$$\implies C_\ell \to (1+2m)C_\ell$$

- Introduce nuisance parameters with multiplicative errors


Boltzmann
P(k) and $D_A(z)$
Nonlinear P(k)
Theory shear
C_ℓ
Photometric
redshift n(z)
Binned Theory
C_b
Likelihood
Measured
C_b


Boltzmann
P(k) and $D_A(z)$
Nonlinear P(k)
Theory shear
C_ℓ
Photo-z
Systematics
Photometric
redshift n(z)
Binned Theory
C_b
Shape Errors
Likelihood
Measured
C_b