

For this problem you need the following formulae. The specific angular momentum (i.e., per unit rest mass) of a particle in a circular geodesic at radius r around a black hole of mass M and spin parameter $a = jM$ is

$$u_\phi = \pm \frac{\sqrt{Mr} \left(r^2 \mp 2a\sqrt{Mr} + a^2 \right)}{r \left(r^2 - 3Mr \pm 2a\sqrt{Mr} \right)^{1/2}} . \quad (1)$$

Here the upper sign is for prograde orbits and the lower sign is for retrograde orbits. The specific energy of a particle in that same circular orbit is

$$-u_t = \frac{r^2 - 2Mr \pm a\sqrt{Mr}}{r(r^2 - 3Mr \pm 2a\sqrt{Mr})^{1/2}} \quad (2)$$

where again the upper sign is for prograde orbits and the lower sign is for retrograde orbits. The radius of the innermost stable circular orbit is

$$r_{\text{ISCO}} = M \left\{ 3 + Z_2 \mp [(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2} \right\} , \quad (3)$$

where once again the upper sign is for prograde and the lower is for retrograde. Here we use

$$Z_1 = 1 + \left(1 - j^2 \right)^{1/3} \left[(1 + j)^{1/3} + (1 - j)^{1/3} \right] \quad (4)$$

and

$$Z_2 = \left(3j^2 + Z_1^2 \right)^{1/2} . \quad (5)$$