Introduction to Superstring Theory

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Programme

Class 1: The classical fermionic string

Class 2: The quantized fermionic string

Class 3: Partition Function

Class 4: Interactions

Outline

Class 1: The classical fermionic string

- The action and its symmetries
- Gauge fixing and constraints
- Equations of motion and boundary conditions
- Oscillator expansions

Why superstrings?

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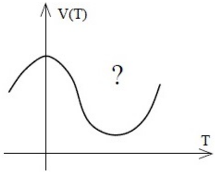

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- The critical dimension of the bosonic string is D=26
- All physical d.o.f. of bosonic string are bosonic

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- Associated with each bosonic d.o.f. $X^\mu(\sigma, \tau)$, world-sheet spinors are introduced: $\Psi^\mu(\tau, \sigma)\,, \mu = 0, \dots, D-1\,,$

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- Gliozzi, Scherk and Olive (1977): it is possible to get a model with no tachyons and equal masses and multiplicities for bosons and fermions
- Green and Schwarz (1980): this model had space-time susy

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  S_B couples D scalar fields $X^\mu(\sigma, \tau)$ to two-dim gravity $h_{\alpha\beta}$

- Ghosts are removed by physical state conditions:

$$T_{\alpha\beta} \sim \frac{\delta S_B}{\delta h^{\alpha\beta}} = 0$$

$\Longrightarrow$ the absence of negative norm states depends crucially on reparametrization invariance

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Consistency of the theory requires D=10 for the superstring

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Although this construction possesses world-sheet N=1 susy, it does not possess manifest space-time susy

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A covariant extension of the GS formalism is the pure spinor formulation.

Other possibilities?

- Extended supersymmetry?
  $N = 2$ world-sheet supersymmetry $\rightarrow$ critical dimension $D = 2$

  $N = 4$ world-sheet supersymmetry $\rightarrow$ negative critical dimension!

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  $N = 4$ world-sheet supersymmetry $\rightarrow$ negative critical dimension!

- Heterotic string: superstring modes for right-movers and bosonic string modes for left-movers. $N = 1$ susy in $D = 10$

Superstring action

We want to find the susy extension of the Polyakov action

Recall

$$S_P = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu \tag{1}$$

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$X^\mu(\tau,\sigma)$ are world-sheet scalars but space-time vectors

$\Longrightarrow$ their susy partners should be world-sheet spinors with a target space vector index

Let us consider the action

$$S = -\frac{1}{4\pi}\int d^2\sigma\sqrt{-h}\left[\frac{1}{\alpha'}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu + i\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu\right] \quad (2)$$

ψ^μ is a Majorana spinor

$$\bar{\psi} = \psi^\dagger\rho^0 = (\psi^*)^T C = \psi^T C \qquad \text{Conjugate spinor}$$

$\implies$ Majorana spinors are real

ρ^α are two dimensional Dirac matrices. A convenient basis is

$$\rho^0 = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \rho^1 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3)$$

Brief remainder on two-dimensional spinors

- The two-dimensional Dirac matrices satisfy

$$\{\rho^\alpha, \rho^\beta\} = 2h^{\alpha\beta} \tag{4}$$

They transform under coordinate transformations and are related to the constant Dirac matrices ρ^a through the zweibein:

$$\rho^\alpha = e^\alpha_a \rho^a \quad \Longrightarrow \; \{\rho^a, \rho^b\} = 2\eta^{ab} = 2\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

- We define the analogue of γ^5 in four-dimensions:

$$\bar{\rho} = \rho^0\rho^1 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Brief remainder on two-dimensional spinors

Using spinor indices

$$\bar{\chi}\Gamma\psi = \chi^A \Gamma_A{}^B \psi_B \quad \text{where} \quad \chi^A = \chi_B C^{BA}$$

where Γ is some combination of Dirac matrices.
The charge conjugation matrix ($CC^\dagger = 1$)

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Two-dimensional spinor indices take values $A = \pm$, i.e.

$$\psi_A = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \tag{5}$$

and $\psi^+ = -\psi_-, \psi^- = \psi_+$

Useful relations (exercises)

- Spin-flip property, valid for anticommuting Majorana spinors

$$\bar{\lambda}_1 \rho^{\alpha_1} \cdots \rho^{\alpha_n} \lambda_2 = (-1)^n \bar{\lambda}_2 \rho^{\alpha_n} \cdots \rho^{\alpha_1} \lambda_1 \qquad (6)$$

- Fierz identity, valid for anticommuting Majorana spinors

$$(\bar{\psi}\lambda)(\bar{\phi}\chi) = -\frac{1}{2}\{(\bar{\psi}\chi)(\bar{\phi}\lambda) + (\bar{\psi}\bar{\rho}\chi)(\bar{\phi}\bar{\rho}\bar{\lambda}) + (\bar{\psi}\rho^\alpha\chi)(\bar{\phi}\rho_\alpha\lambda)\}$$

- $\rho^\alpha \rho_\beta \rho_\alpha = 0$

- $\rho^\alpha \rho^\beta = h^{\alpha\beta} + \frac{1}{e}\epsilon^{\alpha\beta}\bar{\rho}$ with $\epsilon^{01} = 1$

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- D world-sheet Majorana fermions ψ^μ provide 2D fermionic degrees of freedom $\psi^\mu_\pm$
- $\Longrightarrow$ we have to introduce D real auxiliary scalar fields F^μ
- Together (X^μ, ψ^μ, F^μ) form an off-shell scalar multiplet of two-dimensional N=1 supersymmetry

  On-shell (X^μ, ψ^μ) suffice: $S_F \propto \int d^2\sigma e\ F^\mu F_\mu$

Back to the action

$$S = -\frac{1}{4\pi}\int d^2\sigma\sqrt{-h}\left[\frac{1}{\alpha'}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu + i\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu\right]$$

The derivative is ordinary instead of covariant due to the Majorana spin-flip property: $\bar{\psi}^\mu\rho^\alpha\omega_\alpha\psi_\mu = -\bar{\psi}^\mu\rho^\alpha\omega_\alpha\psi_\mu$

It is invariant under the infinitesimal transformations:

$$\sqrt{\frac{1}{\alpha'}}\delta_\epsilon X^\mu = i\bar{\epsilon}\psi^\mu \qquad (7)$$

$$\delta_\epsilon\psi^\mu = \sqrt{\frac{2}{\alpha'}}\frac{1}{2}\rho^\alpha\partial_\alpha X^\mu\epsilon \qquad (8)$$

with ϵ a constant anticommuting infinitesimal Majorana spinor.

Supersymmetry transformations mix bosonic and fermionic fields

A basic fact about susy is

$$[\delta_1, \delta_2] X^\mu = \delta_1(\bar{\epsilon}_2 \psi^\mu) - (1 \leftrightarrow 2) = a^\alpha \partial_\alpha X^\mu$$

the commutator of two supersymmetry transformations gives a spatial translation (here on the world-sheet) with

$$a^\alpha = 2i \bar{\epsilon}_1 \rho^\alpha \epsilon_2$$

Here it is important that for Majorana spinors in two dimensions: $\bar{\epsilon}_1 \rho^\alpha \epsilon_2 = -\bar{\epsilon}_2 \rho^\alpha \epsilon_1$.

$$[\delta_1, \delta_2] \psi^\mu = a^\alpha \partial_\alpha \psi^\mu$$

Here it is necessary that ψ^μ obeys the Dirac equation: $\rho^\alpha \partial_\alpha \psi^\mu = 0$.

The gravity sector

The supergravity multiplet consists of the

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There are two reparametrizations and one local Lorentz transformation as gauge symmetries.

This leaves one bosonic degree of freedom in two-dimensions

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The complete off-shell multiplet is $(e^a_\alpha, \chi_\alpha, A)$. On-shell $(e^a_\alpha, \chi_\alpha)$

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is not locally susy.

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is not locally susy. Local susy requires the additional term:

$$S' = \frac{i}{8\pi}\int d^2\sigma e\bar{\chi}_\alpha\rho^\beta\rho^\alpha\psi^\mu\left(\sqrt{\frac{2}{\alpha'}}\partial_\beta X_\mu - \frac{i}{4}\bar{\chi}_\beta\psi_\mu\right)$$

The auxiliary field A does not appear and the auxiliary matter scalars F^μ can be eliminated via their eom. $e = |dete^a_\alpha| = \sqrt{-h}$.

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The kinetic term for the gravitino vanishes identically in two dimensions $\bar{\chi}_\alpha\Gamma^{\alpha\beta\gamma}D_\beta\chi_\gamma$ where $\Gamma^{\alpha\beta\gamma}$ is the antisymmetrized product of three Dirac matrices which vanishes in two dimensions.

The action

The complete action is:

$$
\begin{aligned}
S &= -\frac{1}{8\pi}\int d^2\sigma e\left[\frac{2}{\alpha'}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu + 2i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right. \\
&\left. \qquad -i\bar{\chi}_\alpha \rho^\beta \rho^\alpha \psi^\mu \left(\sqrt{\frac{2}{\alpha'}}\partial_\beta X_\mu - \frac{i}{4}\bar{\chi}_\beta \psi_\mu\right)\right]
\end{aligned}
$$

Symmetries

The action is invariant under the following local world-sheet symmetries

- Supersymmetry

$$
\begin{aligned}
\sqrt{\frac{2}{\alpha'}}\delta_\epsilon X^\mu &= i\bar{\epsilon}\psi^\mu\,, \\
\delta_\epsilon \psi^\mu &= \frac{1}{2}\rho^\alpha\left(\sqrt{\frac{2}{\alpha'}}\partial_\alpha X^\mu - \frac{i}{2}\bar{\chi}_\alpha\psi^\mu\right)\epsilon\,, \\
\delta_\epsilon e_\alpha{}^a &= \frac{i}{2}\bar{\epsilon}\rho^\alpha\chi_\alpha\,, \\
\delta_\epsilon \chi_\alpha &= 2D_\alpha\epsilon
\end{aligned}
$$

where $\epsilon(\tau,\sigma)$ is a Majorana spinor which parametrizes susy transformations and D_α is a covariant derivative with torsion

Symmetries

$$
\begin{aligned}
D_\alpha \epsilon &= \partial_\alpha \epsilon - \frac{1}{2}\omega_\alpha \bar{\rho}\epsilon \\
\omega_\alpha &= -\frac{1}{2}\epsilon^{ab}\omega_{\alpha ab} = \omega_\alpha(e) + \frac{i}{4}\bar{\chi}_\alpha \bar{\rho}\rho^\beta \chi_\beta \\
\omega_\alpha(e) &= -\frac{1}{e} e_{\alpha a}\epsilon^{\beta\gamma}\partial_\beta e_\gamma{}^a
\end{aligned}
$$

where $\omega_\alpha(e)$ is the spin connection without torsion

Symmetries

- Weyl transformations: $h_{\alpha\beta} \to \Omega^2(\tau,\sigma)h_{\alpha\beta}$ for $\Omega^2 = e^{2\Lambda}$

$$
\begin{aligned}
\delta_\Lambda X^\mu &= 0 \\
\delta_\Lambda \psi^\mu &= -\frac{1}{2}\Lambda\psi^\mu \\
\delta_\Lambda e_\alpha{}^a &= \Lambda e_\alpha{}^a \\
\delta_\Lambda \chi_\alpha &= \frac{1}{2}\Lambda\chi_\alpha
\end{aligned}
$$

- Super-Weyl transformations

$$
\begin{aligned}
\delta_\eta \chi_\alpha &= \rho_\alpha \eta \\
\delta_\eta(\text{others}) &= 0
\end{aligned}
$$

with $\eta(\tau,\sigma)$ a Majorana spinor parameter

Symmetries

- Two-dimensional Lorentz transformations

$$\begin{aligned}
\delta_l X^\mu &= 0 \\
\delta_l \psi^\mu &= -\frac{1}{2} l \bar{\rho} \psi^\mu \\
\delta_l e_\alpha{}^a &= l \epsilon^a{}_b e_\alpha{}^b \\
\delta_l \chi_\alpha &= -\frac{1}{2} l \bar{\rho} \chi_\alpha
\end{aligned}$$

- Reparametrizations

$$\begin{aligned}
\delta_\xi X^\mu &= -\xi^\beta \partial_\beta X^\mu \\
\delta_\xi \psi^\mu &= -\xi^\beta \partial_\beta \psi^\mu \\
\delta_\xi e_\alpha{}^a &= -\xi^\beta \partial_\beta e_\alpha{}^a - e_\beta{}^a \partial_\alpha \xi^\beta \\
\delta_\xi \chi_\alpha &= -\xi^\beta \partial_\beta \chi_\alpha - \chi_\beta \partial_\alpha \xi^\beta
\end{aligned}$$

Symmetries

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In addition to the local world-sheet symmetries, the action is also invariant under global space-time Poincaré transformations:

$$\begin{aligned} \delta X^\mu &= a^\mu{}_\nu X^\nu + b^\mu\,, \qquad a_{\mu\nu} = -a_{\nu\mu} \\ \delta h_{\alpha\beta} &= 0 \\ \delta \psi^\mu &= a^\mu{}_\nu \psi^\nu \\ \delta \chi_\alpha &= 0 \end{aligned}$$

Gauge fixing

We can now use local susy, reparametrizations and Lorentz transformations to gauge away two d.o.f. of the zweibein and two d.o.f. of the gravitino.

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We can now use local susy, reparametrizations and Lorentz transformations to gauge away two d.o.f. of the zweibein and two d.o.f. of the gravitino.

To do this we decompose the gravitino as

$$\begin{aligned}\chi_\alpha &= \left(h_\alpha{}^\beta - \frac{1}{2}\rho_\alpha\rho^\beta\right)\chi_\beta + \frac{1}{2}\rho_\alpha\rho^\beta\chi_\beta \\ &= \left(\frac{1}{2}\rho^\beta\rho_\alpha\chi_\beta + \frac{1}{2}\rho_\alpha\rho^\beta\chi_\beta\right) \\ &= \tilde{\chi}_\alpha + \rho_\alpha\lambda \end{aligned} \tag{9}$$

where $\tilde{\chi} = \frac{1}{2}\rho^\beta\rho_\alpha\chi_\beta$ is ρ-traceless: $\rho\cdot\tilde{\chi} = 0$ and $\lambda = \frac{1}{2}\rho^\alpha\chi_\alpha$, corresponding to a decomposition of the spin $3/2$ gravitino into helicity $\pm 3/2$ and $\pm 1/2$ components.

The same decomposition can be made for the susy transformation of the gravitino:

$$\begin{aligned}\delta_\epsilon \chi_\alpha &= 2D_\alpha \epsilon \\ &= 2(\Pi\epsilon)_\alpha + \rho_\alpha \rho^\beta D_\beta \epsilon\end{aligned}$$

where

$$(\Pi\epsilon)_\alpha = \left(h_\alpha{}^\beta - \frac{1}{2}\rho_\alpha \rho^\beta \right) D_\beta \epsilon = \frac{1}{2}\rho^\beta \rho_\alpha D_\beta \epsilon$$

maps spin 1/2 fields to ρ-traceless spin 3/2 fields.
Now we can write

$$\tilde{\chi}_\alpha = \rho^\beta \rho_\alpha D_\beta \kappa \quad (10)$$

for some spinor κ (where we used $\rho^\alpha \rho_\beta \rho_\alpha = 0$)
$\implies$ κ can be eliminated by a susy transformation $\rightarrow \chi_\alpha = \rho_\alpha \lambda$

Superconformal gauge

Reparametrizations and local Lorentz transformations allow to transform the zweibein into

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In the classical theory we can use Weyl ($\delta_\Lambda e_\alpha{}^a = \Lambda e_\alpha{}^a$) and super-Weyl ($\delta_\eta \chi_\alpha = \rho_\alpha \eta$) transformations to gauge away ϕ and λ, leaving only $e_\alpha{}^a = \delta_\alpha^a$ and $\chi_\alpha = 0$

Superconformal gauge

Reparametrizations and local Lorentz transformations allow to transform the zweibein into

$$e_\alpha{}^a = e^\phi \delta_\alpha^a \tag{11}$$

In this way we arrive at the superconformal gauge (a generalization of the conformal gauge):

$$e_\alpha{}^a = e^\phi \delta_\alpha^a \,, \qquad \chi_\alpha = \rho_\alpha \lambda \tag{12}$$

In the classical theory we can use Weyl ($\delta_\Lambda e_\alpha{}^a = \Lambda e_\alpha{}^a$) and super-Weyl ($\delta_\eta \chi_\alpha = \rho_\alpha \eta$) transformations to gauge away ϕ and λ, leaving only $e_\alpha{}^a = \delta_\alpha^a$ and $\chi_\alpha = 0$

In analogy to the bosonic case, these symmetries will be broken in the quantum theory except in the critical dimension.

The action in superconformal gauge

In superconformal gauge the action simplifies to

$$S = -\frac{1}{4\pi}\int d^2\sigma\left[\frac{1}{\alpha'}\partial_\alpha X^\mu\partial^\alpha X_\mu + i\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu\right]$$

This is the action of a free scalar superfield in two dimensions. To arrive at this action we have rescaled $e^{\phi/2}\psi \to \psi$.

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To arrive at this action we have rescaled $e^{\phi/2}\psi \to \psi$.

World-sheet indices are now raised and lowered with the flat metric $\eta^{\alpha\beta}$ and $\rho^\alpha = \delta^\alpha_a\rho^a$.

Equations of motion

The e.o.m. derived from the action

$$S = -\frac{1}{4\pi}\int d^2\sigma \left[\frac{1}{\alpha'}\partial_\alpha X^\mu \partial^\alpha X_\mu + i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right]$$

are

$$\partial_\alpha \partial^\alpha X^\mu = 0 \tag{13}$$

$$\rho^\alpha \partial_\alpha \psi^\mu = 0\,. \tag{14}$$

As in the bosonic theory, they have to be supplemented by boundary conditions (later)

Equations of motion

The e.o.m. for the zweibein and the gravitino are:

$$T_{\alpha\beta} = 0\,, \qquad T_{F\alpha} = 0\,.$$

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$$T_{\alpha\beta} \quad = \quad \frac{2\pi}{e}\frac{\delta S}{\delta e_a^\beta} e_{a\alpha} \tag{15}$$

We can analogously define the supercurrent as the response to variations of the gravitino:

$$T_{F\alpha} = \frac{2\pi}{e}\frac{\delta S}{i\delta\bar{\chi}^\alpha} \tag{16}$$

In the superconformal gauge they are

$$\begin{aligned} T_{\alpha\beta} &= -\frac{1}{\alpha'}\left(\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\eta_{\alpha\beta}\partial^\gamma X^\mu \partial_\gamma X_\mu\right) \\ &\quad -\frac{i}{4}\left(\bar\psi^\mu \rho_\alpha \partial_\beta \psi_\mu + \bar\psi^\mu \rho_\beta \partial_\alpha \psi_\mu\right) = 0 \\ T_{F\alpha} &= -\frac{1}{4}\sqrt{\frac{2}{\alpha'}}\rho^\beta \rho_\alpha \psi^\mu \partial_\beta X_\mu = 0 \end{aligned}$$

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&\quad -\frac{i}{4}\left(\bar{\psi}^\mu \rho_\alpha \partial_\beta \psi_\mu + \bar{\psi}^\mu \rho_\beta \partial_\alpha \psi_\mu\right) = 0 \\
T_{F\alpha} &= -\frac{1}{4}\sqrt{\frac{2}{\alpha'}}\rho^\beta \rho_\alpha \psi^\mu \partial_\beta X_\mu = 0
\end{aligned}
$$

Tracelessness $T_\alpha^\alpha = 0$ follows upon using the e.o.m. and as a consequence of Weyl invariance

The analogue $\rho^\alpha T_{F\alpha} = 0$ follows from super-Weyl invariance

Conservation laws and conserved charges

The energy-momentum tensor and the supercurrent are conserved:

$$\partial^\alpha T_{\alpha\beta} = 0 \tag{17}$$

$$\partial^\alpha T_{F\alpha} = 0 \tag{18}$$

These conservation laws lead to an infinite number of conserved charges.

In light-cone coordinates on the world-sheet

$$\sigma^\pm = \tau \pm \sigma \tag{19}$$

where $ds^2 = -d\tau^2 + d\sigma^2 = -d\sigma^+ d\sigma^-$

$$\eta_{+-} = \eta_{-+} = -\frac{1}{2}\,, \qquad \eta^{+-} = \eta^{-+} = -2 \tag{20}$$

$$\eta_{++} = \eta_{--} = \eta^{++} = \eta^{--} = 0\,, \quad \partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \tag{21}$$

Analysis in light-cone coordinates on the world-sheet

The action and eom in light-cone coordinates are

$$S = -\frac{1}{2\pi}\int d^2\sigma\left[\frac{2}{\alpha'}\partial_+ X^\mu \partial_- X_\mu + i(\psi_+^\mu \partial_- \psi_{+\mu} + \psi_-^\mu \partial_+ \psi_{-\mu})\right]$$

where

$$\psi_A = \begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix}$$

and the eom

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where

$$\psi_A = \begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix}$$

and the eom

$$\partial_+\partial_- X^\mu = 0\,, \tag{22}$$

$$\partial_- \psi_+^\mu = \partial_+ \psi_-^\mu = 0\,. \tag{23}$$

Analysis in light-cone coordinates on the world-sheet

The energy-momentum tensor in light-cone coordinates is

$$\begin{aligned} T_{++} &= -\frac{1}{\alpha'}\partial_+ X \cdot \partial_+ X - \frac{i}{2}\psi_+ \cdot \partial_+ \psi_+ \,, \\ T_{--} &= -\frac{1}{\alpha'}\partial_- X \cdot \partial_- X - \frac{i}{2}\psi_- \cdot \partial_- \psi_- \,, \\ T_{+-} &= T_{-+} = 0 \end{aligned} \tag{24}$$

with $\partial_- T_{++} = \partial_+ T_{--} = 0$
And the supercurrent

$$T_{F\pm} = -\frac{1}{2}\sqrt{\frac{2}{\alpha'}}\psi_\pm \cdot \partial_\pm X \tag{25}$$

with

$$\partial_- T_{F+} = \partial_+ T_{F-} = 0 \tag{26}$$

Solutions

From the e.o.m.

$$\partial_+\partial_- X^\mu = 0 \implies \quad X^\mu(\tau,\sigma) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-)$$
$$\partial_-\psi_+^\mu = \partial_+\psi_-^\mu = 0 \implies \quad \psi_+^\mu = \psi_+^\mu(\sigma^+)\,,\ \psi_-^\mu = \psi_-^\mu(\sigma^-)$$

the fields can be split into left- and right-movers

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the fields can be split into left- and right-movers

and from the conservation laws

$$\partial_- T_{++} = \partial_+ T_{--} = 0\,, \qquad \partial_- T_{F+} = \partial_+ T_{F-} = 0 \qquad \implies$$

T_{++} and T_{F+} are functions of σ^+ only whereas
T_{--} and T_{F-} are functions of σ^- only.

Boundary conditions

Varying X^μ in the action such that $\delta X^\mu(\tau_0) = 0 = \delta X^\mu(\tau_1)$ gives:

$$\begin{aligned}\delta S_P &= \frac{1}{2\pi\alpha'} \int_{\tau_0}^{\tau_1} d\tau \int_0^l d\sigma \sqrt{-h} \delta X_\mu \nabla^2 X^\mu \\ &\quad - \frac{1}{2\pi\alpha'} \int_{\tau_0}^{\tau_1} d\tau \sqrt{-h}\, \delta X_\mu \partial_\sigma X^\mu \Big|_{\sigma=0}^{\sigma=l}\end{aligned}$$

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The boundary term vanishes if

$$\partial_\sigma X^\mu(\tau, 0) = \partial_\sigma X^\mu(\tau, l) = 0$$

These are Neumann boundary conditions on X^μ: the ends of the open string move freely in space-time

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These are Neumann boundary conditions on X^μ: the ends of the open string move freely in space-time

Surface term also vanishes if fields are periodic $\rightarrow$ closed string

$$X^\mu(\tau, l) = X^\mu(\tau, 0)\,, \qquad \partial_\sigma X^\mu(\tau, l) = \partial_\sigma X^\mu(\tau, 0)$$

Boundary conditions

To derive the e.o.m. for the fermions we impose
$\delta\psi^\mu(\tau_0) = \delta\psi^\mu(\tau_1) = 0$
Further we have to impose bdry cond such that the boundary term

$$\delta S = \frac{1}{2\pi} \int_{\tau_0}^{\tau_1} d\tau (\psi_+ \cdot \delta\psi_+ - \psi_- \cdot \delta\psi_-)|_{\sigma=0}^{\sigma=l} \tag{27}$$

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$\delta\psi^\mu(\tau_0) = \delta\psi^\mu(\tau_1) = 0$
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$$\delta S = \frac{1}{2\pi}\int_{\tau_0}^{\tau_1} d\tau(\psi_+\cdot\delta\psi_+ - \psi_-\cdot\delta\psi_-)|_{\sigma=0}^{\sigma=l} \qquad (27)$$

vanishes. For the closed string this requires

$$(\psi_+\cdot\delta\psi_+ - \psi_-\cdot\delta\psi_-)(\sigma) = (\psi_+\cdot\delta\psi_+ - \psi_-\cdot\delta\psi_-)(\sigma + l)$$

which is solved by

$$\psi_+^\mu(\sigma) = \pm\psi_+^\mu(\sigma + l) \qquad (28)$$
$$\psi_-^\mu(\sigma) = \pm\psi_-^\mu(\sigma + l) \qquad (29)$$

and the same conditions on $\delta\psi_\pm$. Antiperiodicity of ψ is possible as they are fermions on the world-sheet.

Ramond and Neveu-Schwarz boundary conditions

Periodic bdy cond are called Ramond boundary conditions

$$\psi_+^\mu(\sigma) = +\psi_+^\mu(\sigma + l)$$
$$\psi_-^\mu(\sigma) = +\psi_-^\mu(\sigma + l)$$

Anti-periodic bdy cond are called Neveu-Schwarz boundary conditions

$$\psi_+^\mu(\sigma) = -\psi_+^\mu(\sigma + l)$$
$$\psi_-^\mu(\sigma) = -\psi_-^\mu(\sigma + l)$$

Space-time Poincaré invariance requires that we impose the same boundary conditions in all directions μ.
This also guarantees that $T_{F\pm}$ have definite periodicity.

Boundary conditions

Fermions on the world-sheet satisfy

$$\psi^\mu(\sigma + l) = e^{2\pi i\phi}\psi^\mu(\sigma) \begin{cases} \phi = 0 \text{ for the R sector} \\ \phi = \frac{1}{2} \text{ for the NS sector} \end{cases} \tag{30}$$

More general phases are not allowed for real ψ.
The conditions for the two spinor components ψ_+ and ψ_- can be chosen independently, leading to four possibilities

$$(\mathrm{R}, \mathrm{R}) \quad (\mathrm{NS}, \mathrm{NS}) \quad (\mathrm{NS}, \mathrm{R}) \quad (\mathrm{R}, \mathrm{NS})$$

We shall see that string states in the (R,R) and (NS,NS) sectors are space-time bosons while those in the (R, NS) and (NS, R) sectors are space-time fermions.

Boundary conditions for the open string

For the open string the variation

$$\delta S = \frac{1}{2\pi}\int_{\tau_0}^{\tau_1} d\tau(\psi_+ \cdot \delta\psi_+ - \psi_- \cdot \delta\psi_-)|_{\sigma=0}^{\sigma=l}$$

has to be canceled on each boundary, i.e. at $\sigma = 0$ and $\sigma = l$, separately. This leads to

$$\psi_+^\mu(0) = \pm\psi_-^\mu(0)\,, \qquad \psi_+^\mu(l) = \pm\psi_-^\mu(l)$$

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$$\psi_+^\mu(0) = \pm\psi_-^\mu(0)\,, \qquad \psi_+^\mu(l) = \pm\psi_-^\mu(l)$$

To preserve space-time Poincaré invariance we have to impose the same conditions on all μ.

Boundary conditions for the open string

Without loss of generality we specify

$$\psi_+^\mu(0) = \psi_-^\mu(0)\,, \qquad \psi_+^\mu(l) = \eta\psi_-^\mu(l) \tag{31}$$

where $\eta = \pm 1$. Only the relative sign in the boundary conditions at $\sigma = 0$ and $\sigma = l$ is relevant and by a redefinition $\psi_- \to \pm\psi_-$, which leaves the action invariant, we can always move the sign to the $\sigma = l$ boundary.

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We then have to distinguish beetween two sectors:
$\eta = +1$ is the Ramond sector
$\eta = -1$ is the Neveu Schwarz sector
States in the R sector will turn out to be space-time fermions.
States in the NS sector will turn out to be space-time bosons.

Superconformal algebra

To find the algebra satisfied by $T_{\alpha\beta}$ and $T_{F\alpha}$ we need the equal τ Poisson brackets.
In conformal gauge

$$\begin{aligned}\left[X^\mu(\sigma), \dot{X}^\mu(\sigma')\right]_{PB} &= 2\pi\alpha'\eta^{\mu\nu}\delta(\sigma,\sigma') \\ \left[X^\mu(\sigma), X^\mu(\sigma')\right]_{PB} &= \left[\dot{X}^\mu(\sigma), \dot{X}^\mu(\sigma')\right]_{PB} = 0\end{aligned}$$

$$\begin{aligned}\{\psi_+^\mu(\sigma), \psi_+^\nu(\sigma')\} &= \{\psi_-^\mu(\sigma), \psi_-^\nu(\sigma')\} = -2\pi i\delta(\sigma-\sigma')\eta^{\mu\nu}\,, \\ \{\psi_+^\mu(\sigma), \psi_-^\nu(\sigma')\} &= 0\end{aligned}$$

Using these brackets one finds

Superconformal algebra

$$
\begin{aligned}
\left[T_{\pm\pm}(\sigma), T_{\pm\pm}(\sigma')\right] &= \pm\left(2T_{\pm\pm}(\sigma')\partial' + \partial' T_{\pm\pm}(\sigma')\right) 2\pi\delta(\sigma-\sigma') \\
\left[T_{\pm\pm}(\sigma), T_{F\pm}(\sigma')\right] &= \pm\left(\frac{3}{2}T_{F\pm}(\sigma')\partial' + \partial' T_{F\pm}(\sigma')\right) 2\pi\delta(\sigma-\sigma') \\
\left\{T_{F\pm}(\sigma), T_{F\pm}(\sigma')\right\} &= \pm\frac{i}{2}T_{\pm\pm}(\sigma')2\pi\delta(\sigma-\sigma')
\end{aligned}
$$

We can also verify the supersymmetry transformations

$$
\begin{aligned}
\left[T_{F\pm}(\sigma), \sqrt{\frac{2}{\alpha'}}X^{\mu}(\sigma')\right] &= \frac{1}{2}\psi^{\mu}_{\pm}(\sigma)2\pi\delta(\sigma-\sigma') \\
\left\{T_{F\pm}(\sigma), \psi^{\mu}_{\pm}(\sigma')\right\} &= \frac{i}{2}\sqrt{\frac{2}{\alpha'}}\partial_{\pm}X^{\mu}(\sigma')2\pi\delta(\sigma-\sigma')
\end{aligned}
$$

Oscillator expansions

We now solve the classical equations of motion in conformal gauge taking into account the boundary conditions.

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We have to distinguish between closed and open strings

The treatment for the bosonic coordinates is identical to the bosonic string. Let us briefly recall it.

Oscillator expansions: Closed bosonic string

The general solution of the two-dimensional wave equation $\partial_+\partial_- X^\mu = 0$, compatible with the periodicity condition

$$X^\mu(\sigma,\tau) = X^\mu(\sigma + l, \tau)$$

is

$$X^\mu(\sigma,\tau) = X^\mu_R(\tau-\sigma) + X^\mu_L(\tau+\sigma)$$

where

$$\begin{aligned} X^\mu_R(\tau-\sigma) &= \frac{1}{2}x^\mu + \frac{\pi\alpha'}{l}p^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-\frac{2\pi}{l}in(\tau-\sigma)} \\ X^\mu_L(\tau+\sigma) &= \frac{1}{2}x^\mu + \frac{\pi\alpha'}{l}p^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-\frac{2\pi}{l}in(\tau+\sigma)} \end{aligned}$$

Oscillator expansions: Closed bosonic string

If we define

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$$

we can write

$$\begin{aligned}
\partial_- X^\mu &= \dot{X}_R^\mu = \frac{2\pi}{l}\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{+\infty} \alpha_n^\mu e^{-\frac{2\pi}{l} in(\tau-\sigma)} \\
\partial_+ X^\mu &= \dot{X}_L^\mu = \frac{2\pi}{l}\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{+\infty} \tilde{\alpha}_n^\mu e^{-\frac{2\pi}{l} in(\tau+\sigma)}
\end{aligned}$$

Oscillator expansions: Closed bosonic string

From the Poisson brackets for the X^μ, we derive the brackets for the $\alpha_n^\mu, \tilde{\alpha}_n^\mu, x^\mu, p^\mu$

$$\begin{aligned} [\alpha_m^\mu, \alpha_n^\nu]_{PB} &= [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu]_{PB} = -im\delta_{m+n}\eta^{\mu\nu}\,, \\ [\tilde{\alpha}_m^\mu, \alpha_n^\nu]_{PB} &= 0\,, \\ [x^\mu, p^\nu]_{PB} &= \eta^{\mu\nu} \end{aligned}$$

Oscillator expansions: Open bosonic string

For the open string we have to require $X'^{\mu} = 0$ at $\sigma = 0$ and $\sigma = l$. The general solution of the wave equation subject to these bdry cond is

$$X^{\mu}(\tau, \sigma) \;=\; x^{\mu} + \frac{2\pi\alpha'}{l} p^{\mu}\tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^{\mu} e^{-i\frac{\pi}{l}n\tau} \cos\left(\frac{n\pi\sigma}{l}\right)$$

from which we get, with $\alpha_0^{\mu} = \sqrt{2\alpha'} p^{\mu}$,

$$\partial_{\pm} X^{\mu} = \frac{1}{2}(\dot{X}^{\mu} \pm X'^{\mu}) = \frac{\pi}{l}\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{+\infty} \alpha_n^{\mu} e^{-\frac{\pi i n}{l}(\tau \pm \sigma)}$$

Then

$$[\alpha_m^{\mu}, \alpha_n^{\nu}]_{PB} = -im\delta_{m+n}\eta^{\mu\nu}\,, \qquad [x^{\mu}, p^{\nu}]_{PB} = \eta^{\mu\nu}$$

Oscillator expansions of fermionic fields: closed string

The fermionic fields require some care. We have to distinguish between two choices of boundary cond for each chirality.

The general solutions of the two-dimensional Dirac equation with periodic (R) and antiperiodic (NS) bdy cond are

$$\psi_+^\mu(\sigma,\tau) = \sqrt{\frac{2\pi}{l}} \sum_{r\in\mathbb{Z}+\phi} \tilde{b}_r^\mu e^{-2\pi i r(\tau+\sigma)/l}$$

$$\text{where} \quad \begin{cases} \phi = 0 & \text{(R)} \\ \phi = \frac{1}{2} & \text{(NS)} \end{cases}$$

$$\psi_-^\mu(\sigma,\tau) = \sqrt{\frac{2\pi}{l}} \sum_{r\in\mathbb{Z}+\phi} b_r^\mu e^{-2\pi i r(\tau-\sigma)/l}$$

The reality of the Majorana spinors translates into

$(b_r^\mu)^* = b_{-r}^\mu, (\tilde{b}_r^\mu)^* = \tilde{b}_{-r}^\mu$

Oscillator expansions of fermionic fields: closed string

In terms of the fermionic oscillator modes, the anticommutators

$$
\begin{aligned}
\{\psi_+^\mu(\sigma), \psi_+^\nu(\sigma')\} &= \{\psi_-^\mu(\sigma), \psi_-^\nu(\sigma')\} = -2\pi i \delta(\sigma - \sigma')\eta^{\mu\nu} \,, \\
\{\psi_+^\mu(\sigma), \psi_-^\nu(\sigma')\} &= 0
\end{aligned}
$$

translate to

$$
\begin{aligned}
\{b_r^\mu, b_s^\nu\} &= -i\delta_{r+s}\eta^{\mu\nu} \,, \\
\{\tilde{b}_r^\mu, \tilde{b}_s^\nu\} &= -i\delta_{r+s}\eta^{\mu\nu} \,, \\
\{b_r^\mu, \tilde{b}_s^\nu\} &= 0
\end{aligned}
$$

Next we decompose the generators of conformal and superconformal transformations into modes

Oscillator expansions of $T_{\pm\pm}$ and $T_{F\pm}$

The conservation equations

$$\partial_- T_{++} = 0\,, \qquad \partial_+ T_{--} = 0$$

$\implies$ the existence of an infinite number of conserved charges: for any function $f(\sigma^+)$ we have $\partial_-(f(\sigma^+)T_{++}) = 0$ and the corresponding charges are

$$L_f = \frac{1}{\pi\alpha'}\int_0^l d\sigma^+ f(\sigma^+)T_{++}(\sigma^+)$$

and similarly for T_{--}.
We can choose for $f(\sigma^\pm)$ a complete set satisfying the periodicity condition appropriate for the closed string:

$$f_m(\sigma^\pm) = exp\left(\frac{2\pi i}{l} m\sigma^\pm\right) \quad \text{for all integers } m$$

Energy-momentum tensor and supercurrent

We then define the super-Virasoro generators as the corresponding charges at $\tau = 0$

$$
\begin{aligned}
L_n &= -\frac{l}{4\pi^2}\int_0^l d\sigma\, e^{-\frac{2\pi i}{l}n\sigma}\, T_{--}\,, \\
\tilde{L}_n &= -\frac{l}{4\pi^2}\int_0^l d\sigma\, e^{\frac{2\pi i}{l}n\sigma}\, T_{++}\,,
\end{aligned}
$$

$$
\begin{aligned}
G_r &= -\frac{1}{\pi}\sqrt{\frac{l}{2\pi}}\int_0^l d\sigma\, e^{-2\pi i r\sigma/l}\, T_{F-}(\sigma) \\
\tilde{G}_r &= -\frac{1}{\pi}\sqrt{\frac{l}{2\pi}}\int_0^l d\sigma\, e^{2\pi i r\sigma/l}\, T_{F+}(\sigma)
\end{aligned}
$$

In terms of oscillators $L_m = L_m^{(\alpha)} + L_m^{(b)}$

$$\begin{aligned} L_n^{(\alpha)} &= \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{-m}\cdot\alpha_{m+n} \\ L_n^{(b)} &= \frac{1}{2}\sum_{r}\left(r+\frac{n}{2}\right) b_{-r}\cdot b_{n+r} \\ G_r &= \sum_{m} \alpha_{-m}\cdot b_{r+m} \end{aligned}$$

Note $\sum b_{-r}\cdot b_{n+r} = 0$ as it corresponds to $\partial_-(\psi_-\psi_-)$. It was included to make the expression look more symmetric.

In terms of oscillators $L_m = L_m^{(\alpha)} + L_m^{(b)}$

$$L_n^{(\alpha)} = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{-m}\cdot\alpha_{m+n}$$

$$L_n^{(b)} = \frac{1}{2}\sum_{r}\left(r+\frac{n}{2}\right) b_{-r}\cdot b_{n+r}$$

$$G_r = \sum_{m} \alpha_{-m}\cdot b_{r+m}$$

Note $\sum b_{-r}\cdot b_{n+r} = 0$ as it corresponds to $\partial_-(\psi_-\psi_-)$. It was included to make the expression look more symmetric.

From the definition we see $T_{F\pm}$ has the same periodicity as the $\psi_\pm^\mu$

$$T_{F\pm} = -\frac{1}{2}\sqrt{\frac{2}{\alpha'}}\psi_\pm\cdot\partial_\pm X$$

periodic in the R-sector and antiperiodic in the NS-sector $\Longrightarrow$ the mode numbers are integer and half-integer respectively.

In terms of oscillators $L_m = L_m^{(\alpha)} + L_m^{(b)}$

$$L_n^{(\alpha)} = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{-m}\cdot\alpha_{m+n}$$

$$L_n^{(b)} = \frac{1}{2}\sum_{r}\left(r+\frac{n}{2}\right) b_{-r}\cdot b_{n+r}$$

$$G_r = \sum_{m} \alpha_{-m}\cdot b_{r+m}$$

Note $\sum b_{-r}\cdot b_{n+r} = 0$ as it corresponds to $\partial_-(\psi_-\psi_-)$. It was included to make the expression look more symmetric.
From the definition we see $T_{F\pm}$ has the same periodicity as the $\psi_\pm^\mu$

$$T_{F\pm} = -\frac{1}{2}\sqrt{\frac{2}{\alpha'}}\psi_\pm\cdot\partial_\pm X$$

periodic in the R-sector and antiperiodic in the NS-sector $\implies$ the mode numbers are integer and half-integer respectively.

Classical super-Virasoro algebra

The generators L_m and G_r satisfy the following reality conditions

$$L_m^* = L_{-m}\,, \qquad G_r^* = G_{-r}$$

Using the basic brackets, one can now verify

$$\begin{aligned} [L_m, L_n] &= -i(m-n)L_{m+n} \\ [L_m, G_r] &= -i(\frac{1}{2}m - r)G_{m+r} \\ \{G_r, G_s\} &= -2iL_{r+s} \end{aligned}$$

It can also be derived from the Poisson brackets for $T_{\pm\pm}$ and $T_{F\pm}$ and the definitions of $L_m, G_r, \tilde{L}_m, \tilde{G}_r$. For the closed string there are two copies of this algebra, one for the left- and one for the right-movers.

Oscillator expansions of fermionic fields: open string

For the open string we also expand the fermionic fields in modes and implement the bdy cond

$$\psi^\mu_+(0) = \psi^\mu_-(0)\,, \qquad \psi^\mu_+(l) = \eta\psi^\mu_-(l)$$

The bdy cond relate the left- and right-moving modes and there is only one set of oscillators.

$$\psi^\mu_\pm(\sigma,\tau) \;=\; \sqrt{\frac{\pi}{l}}\sum_r b^\mu_r e^{-\pi i r(\tau\pm\sigma)/l} \qquad \text{where } r \in \begin{cases} \mathbb{Z} & \text{(R)} \\ \mathbb{Z}+\frac{1}{2} & \text{(NS)} \end{cases}$$

Energy-momentum tensor and supercurrent

We now derive the mode expansions of the Virasoro generators for the open string.

Energy-momentum tensor and supercurrent

We now derive the mode expansions of the Virasoro generators for the open string. The bdy cond mix left- and right-movers and consequently T_{++} and T_{--}

$$\begin{aligned} L_n &= -\frac{l}{2\pi^2}\int_{-l}^{l} d\sigma\, e^{\frac{i\pi}{l}n\sigma}\, T_{++}(\sigma) \\ G_r &= -\frac{1}{\pi}\sqrt{\frac{l}{\pi}}\int_{-l}^{l} d\sigma\, e^{i\pi r\sigma/l}\, T_{F+}(\sigma) \end{aligned}$$

$$\begin{aligned} \Longrightarrow\ L_n &= \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{-m}\cdot\alpha_{m+n} + \frac{1}{2}\sum_{r}\left(r+\frac{n}{2}\right) b_{-r}\cdot b_{m+r} \\ G_r &= \sum_{m} \alpha_{-m}\cdot b_{r+m} \quad \text{with} \begin{cases} r\in\mathbb{Z} & \text{for R} \\ r\in\mathbb{Z}+\frac{1}{2} & \text{for NS} \end{cases} \end{aligned}$$

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