Class 2: The quantized fermionic string

- Canonical quantization
- Light-cone quantization
- Spectrum of the fermionic string, GSO projection

Summary

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There are two possible space-time supersymmetric GSO projections which result in the Type IIA and Type IIB superstring.

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The Poisson brackets are promoted to (anti)commutators according to

$$[\ ,\ ]_{PB} \to \frac{1}{i}[\ ,\ ]\,, \qquad \{\ ,\ \}_{PB} \to \frac{1}{i}\{\ ,\ \}$$

Canonical quantization

In this way we obtain for the equal time commutators

$$\begin{aligned} \left[X^\mu(\sigma), \dot{X}^\mu(\sigma')\right] &= 2\pi i\alpha'\eta^{\mu\nu}\delta(\sigma,\sigma') \\ \left[X^\mu(\sigma), X^\mu(\sigma')\right] &= \left[\dot{X}^\mu(\sigma), \dot{X}^\mu(\sigma')\right] = 0 \end{aligned}$$

$$\begin{aligned} \{\psi_+^\mu(\sigma), \psi_+^\nu(\sigma')\} &= \{\psi_-^\mu(\sigma), \psi_-^\nu(\sigma')\} = 2\pi\delta(\sigma-\sigma')\eta^{\mu\nu}\,, \\ \{\psi_+^\mu(\sigma), \psi_-^\nu(\sigma')\} &= 0 \end{aligned}$$

The Fourier expansion coefficients are now operators with the following (anti)commutation relations

$$\begin{aligned} [\alpha^\mu_m, \alpha^\nu_n] &= [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\delta_{m+n}\eta^{\mu\nu} \, , \\ [\tilde{\alpha}^\mu_m, \alpha^\nu_n] &= 0 \, , \\ [x^\mu, p^\nu] &= i\eta^{\mu\nu} \end{aligned}$$

$$\begin{aligned} \{b^\mu_r, b^\nu_s\} &= \{\tilde{b}^\mu_r, \tilde{b}^\nu_s\} = \delta_{r+s}\eta^{\mu\nu} \, , \\ \{b^\mu_r, \tilde{b}^\nu_s\} &= 0 \end{aligned}$$

The reality conditions become hermiticity conditions:

$$(\alpha^\mu_m)^\dagger = \alpha^\mu_{-m} \, , \qquad (\tilde{\alpha}^\mu_m)^\dagger = \tilde{\alpha}^\mu_{-m}$$

and if we rescale them as: $a^\mu_m = \frac{1}{\sqrt{m}}\alpha^\mu_m$, $(a^\mu_m)^\dagger = \frac{1}{\sqrt{m}}\alpha^\mu_{-m}$ for $m > 0$, then the a^μ_m satisfy the usual harmonic oscillator commutation relations.

Hilbert space

We define oscillators with positive (negative) mode numbers as annihilation (creation) operators.

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Recall the bosonic string: The ground state is annihilated by all positive modes and we choose it to be an eigenstate of the center of mass momentum operator α_0^μ and $\tilde{\alpha}_0^\mu$ with eigenvalue p^μ:

$$\begin{aligned} \alpha_m^\mu|0;p^\mu> &= 0\,, \qquad \tilde{\alpha}_m^\mu|0;p^\mu>= 0\,, \qquad \text{for } m>0 \\ \hat{p}^\mu|0;p^\mu> &= p^\mu|0;p^\mu> \end{aligned}$$

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To examine the states in the Hilbert space of the fermionic theory we have to distinguish between R and NS sectors.

Hilbert space

The oscillator ground state in the NS sector is defined by

$$\alpha^\mu_m|0>_{NS}= b^\mu_r|0>_{NS}= 0\,, \qquad m=1,2,\dots\,, \quad r=\frac{1}{2},\frac{3}{2},\dots$$

and the ground state in the R sector is defined as

$$\alpha^\mu_m|a>_R= b^\mu_m|a>_R= 0\,, \qquad m=1,2,\dots\,,$$

$|a>_R$ is because of the zero modes b^μ_0. Recall

$$\{b^\mu_r, b^\nu_s\} = \{\tilde{b}^\mu_r, \tilde{b}^\nu_s\} = \delta_{r+s}\eta^{\mu\nu}\,,$$

Then $\{b^\mu_0, b^\nu_0\} = \eta^{\mu\nu}$.

It is easy to check that $[b^\mu_0, M^2] = 0 \implies$
the states $|a>$ and $\prod_i b^{\mu_i}_0|a>$ are degenerate in mass.

Hilbert space

The mass-shell condition is determined by the zero-frequency part of the Virasoro constraints. Recall the closed bosonic string

$$L_0 = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \frac{1}{2}\alpha_0^2 = N + \frac{\alpha'}{4}p^2$$
$$\tilde{L}_0 = \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n + \frac{1}{2}\alpha_0^2 = \tilde{N} + \frac{\alpha'}{4}p^2$$

Then

$$M^2 = -p^\mu p_\mu = M_L^2 + M_R^2 \,,$$

$$\alpha' M_L^2 = 2(\tilde{N} + a) \,, \qquad \alpha' M_R^2 = 2(N + a)$$

a is the normal ordering constant accounting for the ordering ambiguity of L_0 and $M_L^2 = M_R^2$.

Normal ordering

The super-Virasoro generators are undefined without giving an operator ordering prescription. We define them by normal ordering: annihilation operators to the right of creation operators

$$
\begin{aligned}
L_n &= L_n^{(\alpha)} + L_n^{(b)} \\
L_n^{(\alpha)} &= \frac{1}{2} \sum_{m \in \mathbb{Z}} : \alpha_{-m} \cdot \alpha_{m+n} : \\
L_n^{(b)} &= \frac{1}{2} \sum_{r \in \mathbb{Z}+\phi>0} \left(r + \frac{n}{2} \right) : b_{-r} \cdot b_{n+r} :
\end{aligned}
$$

$$
G_r = \sum_{m \in \mathbb{Z}} \alpha_{-m} \cdot b_{r+m}
$$

Ambiguity in L_0 taken into account by normal ordering constant $a \to a_{NS}$ and a_R.

Level numbers

The mass operators for the fermionic string are given by the same expressions as in the bosonic case:

$$\alpha' M^2 = N + \text{constant}$$

but with level number operator: $N = N^{(\alpha)} + N^{(b)}$

$$N^{(\alpha)} = \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m \qquad N^{(b)} = \sum_{r \in \mathbb{Z}+\phi > 0} r b_{-r} \cdot b_r$$

and constant $= a_{NS}$ or a_R.
An excitation by α^{μ}_{-m} with $m > 0$ increases $\alpha' M^2$ by m units.
An excitation by b^{μ}_{-r} with $r > 0$ increases $\alpha' M^2$ by r units.

NS and R sectors

In the NS sector ($r \in \mathbb{Z} + \frac{1}{2}$) there is a unique ground state, which may then be identified with a spin zero state.

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We then represent the b_0^μ as Dirac matrices:

$$b_0^\mu|a>= \frac{1}{\sqrt{2}}(\gamma^\mu)_b^a|b>$$

γ^μ is a Dirac matrix in d dimensions, satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

NS and R sectors

We write the R ground state $|a>$, where a is an $SO(d-1,1)$ spinor index.
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We still have to implement the constraints on the states.

The R ground state

We can work out an explicit description of $|0>_R$ for d even. Define fermionic raising and lowering operators

$$\begin{aligned} b_0^{\pm} &= \frac{1}{2}(\pm b_0^0 + b_0^1)\,, \\ b_i^{\pm} &= \frac{1}{2}(b_0^{2i} \pm i b_0^{2i+1})\,, \qquad i = 1, \dots, \frac{d-2}{2} \end{aligned}$$

In this basis, the Clifford algebra reads ($i = 0, \dots, \frac{d-2}{2}$)

$$\{b_i^+, b_j^-\} = \delta_{ij}$$

Define a highest weight state $|0>_R$ satisfying $b_i^-|0>_R = 0$. Successive application of the raising operators b_i^+ generates the $2^{\frac{d}{2}}$ dimensional representation of $SO(d-1,1)$. For $d = 10$:

$$|\mathbf{s}> = |s_0, s_1, s_2, s_3, s_4>\,, \qquad \text{with } s_i = \pm\frac{1}{2}$$

The R ground state

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where

$$|0>_R = |-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}> ,$$

and b_i^+ raises s_i from $-\frac{1}{2}$ to $+\frac{1}{2}$.
This leads to a $2^{\frac{n}{2}}$ representation of the b_0^μ:

$$|\mathbf{s}> = (b_4^+)^{s_4+1/2} \cdots (b_0^+)^{s_0+1/2}|0>_R$$

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The notation **s** reflects the Lorentz properties of the spinors.

The Lorentz generators

$$\Sigma^{\mu\nu} = -\frac{i}{4}\left[\gamma^{\mu}, \gamma^{\nu}\right]$$

satisfy the $SO(d-1,1)$ algebra

$$i\left[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}\right] = \eta^{\nu\rho}\Sigma^{\mu\sigma} - \eta^{\nu\sigma}\Sigma^{\mu\rho} - \eta^{\mu\rho}\Sigma^{\nu\rho} + \eta^{\mu\sigma}\Sigma^{\nu\rho}$$

The generators $\Sigma^{2i,2i+1}$ commute and can be simultaneously diagonalized. In terms of raising and lowering operators

$$S_i = i^{\delta_{i,0}}\Sigma^{2i,2i+1} = b_i^{+} b_i^{-} - \frac{1}{2}$$

so $|\mathbf{s} >$ is a simultaneous eigenstate of the S_i with eigenvalues s_i. The half-integer values show that this is a spinor representation. The spinors form the $2^{d/2}$ dimensional representation of the Lorentz algebra $SO(d-1,1)$

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$$\Gamma = i^{-(d-2)/2}\gamma^0\gamma^1\cdots\gamma^{d-1} = 2^{d/2}S_0S_1\cdots S_{d/2}$$

which has the properties

$$(\Gamma)^2 = 1\,, \quad \{\Gamma, \gamma^\mu\} = 0\,, \quad [\Gamma, \Sigma^{\mu\nu}] = 0$$

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Then Γ is diagonal, taking the value $+1$ when the s_a include an even number of $-\frac{1}{2}s$ and -1 for an odd number of $-\frac{1}{2}$.

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In $d = 4$, the Dirac representation is the familiar four-dimensional one, which separates into two two-dimensional Weyl repr.

$$\mathbf{4}_{Dirac} = \mathbf{2} + \mathbf{2}'$$

In $d = 10$ the representations are

$$\mathbf{32}_{Dirac} = \mathbf{16} + \mathbf{16}'$$

Hilbert space

Recall in the bosonic string, $\eta^{00} = -1 \implies [\alpha^0_m, \alpha^0_{-m}] = -m$, and $\alpha^0_{-m}|0>$, with $m > 0$, have negative norm

$$< 0|\alpha^0_m \alpha^0_{-m}|0> < 0$$

They are called ghosts
The physical state conditions: Virasoro constraints, allowed to decouple the ghosts.
The constraints $L_n|\phi>= 0$, cannot be implemented $\forall n$ since

$$<\phi|\,[L_n, L_{-n}]\,|\phi>=<\phi|2nL_0|\phi> + \frac{c}{12}n(n^2-1) <\phi|\phi>$$

At most:

$$\begin{aligned} L_n|\mathrm{phys}> &= 0\,, \quad n > 0 \\ (L_0 + a)|\mathrm{phys}> &= 0 \end{aligned}$$

For the closed string there are similar constraints involving $\tilde{L}_n$ and the level matching condition $(L_0 - \tilde{L}_0)|\mathrm{phys}>= 0$

For the fermionic string we need the super Virasoro algebra.

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{d}{8}m(m^2 - 2\phi)\delta_{m+n} \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r} \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{d}{2}\left(r^2 - \frac{\phi}{2}\right)\delta_{r+s}
\end{aligned}$$

with $\phi = 0$ for R and $\phi = 1/2$ for NS.

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A straightforward way to derive it is to evaluate the (anti)commutators between two states which are annihiated by all annihilation operators or to use techniques of SCFT

Hilbert space

Due to the anomalies in the super Virasoro algebra, it is again impossible to impose $L_m|\mathrm{phys}>= G_r|\mathrm{phys}>= 0, \forall m, r \implies$

$$
\begin{aligned}
L_n|\mathrm{phys}> &= 0\,, \quad n > 0 \\
G_r|\mathrm{phys}> &= 0\,, \quad r > 0 \\
(L_0 + a)|\mathrm{phys}> &= 0\,,
\end{aligned}
$$

in the NS sector, and

$$
\begin{aligned}
L_n|\mathrm{phys}> &= 0\,, \quad n > 0 \\
G_r|\mathrm{phys}> &= 0\,, \quad r \geq 0 \\
L_0|\mathrm{phys}> &= 0\,,
\end{aligned}
$$

in the R sector

Hilbert space

Note we have not included a normal ordering constant in the R sector. There are several reasons for this.

From the super Virasoro algebra we find $G_0^2 = L_0$, i.e. if we had $(L_0 - \mu^2)|\mathrm{phys}>= 0$ we also need $(G_0 - \mu)|\mathrm{phys}>= 0$. However $G_0 = \sum_m \alpha_{-m} \cdot b_m$ has no normal ordering ambiguity.

Also, G_0 is anti-commuting whereas the normal ordering constant is a commuting c-number.

When we discuss the spectrum we'll find that $\mu = 0$ is correct.

Hilbert space

For the closed string we have of course a second set of conditions for left-movers and also

$$(L_0 - \tilde{L}_0)|\mathrm{phys}>= 0\,,$$

again expressing that no point on a closed string is distinguished.

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again expressing that no point on a closed string is distinguished.

Consider the ground state in the R sector. Physical state $\implies$ $G_0|a>= 0$. With

$$G_0 = \sum_{m\in\mathbb{Z}} \alpha_{-m}\cdot b_m \quad \mathrm{and} \quad \alpha_m^\mu|a>_R = b_m^\mu|a>_R = 0\,, \qquad m = 1, 2, \ldots$$

we find

$$G_0|a>= \alpha_{0\mu}b_0^\mu|a>\propto p_\mu(\gamma^\mu)^a_b|b>= 0$$

Canonical quantization

$$G_0|a>=\alpha_0 \cdot b_0|a>\propto p_\mu(\gamma^\mu)^a_b|b>=0$$

Introducing the polarization spinor u_s and defining the state $|u>=u_s|s> \implies |u>$ is a physical state if $p^\mu\gamma_\mu u=0$, i.e. if u satisfies the massless Dirac equation.

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One can prove a no-ghost theorem stating that the ghosts decouple in the critical dimension d for a particular value of the normal ordering constant.

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One can prove a no-ghost theorem stating that the ghosts decouple in the critical dimension d for a particular value of the normal ordering constant.

For the fermionic string, the superconformal symmetry is enough to allow for ghost decoupling for $d=10$ and $a=-1/2$

Light-Cone quantization

We will not prove the no-ghost theorem but instead discuss the non-covariant light-cone quantization where the constraints are solved explicitly.

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In the bosonic theory the light-cone gauge is obtained by

$$X^+ = \frac{2\pi\alpha'}{l} p^+ \tau$$

which fixes the gauge completely.

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This gauge is again possible in the fermionic theory and also completely eliminates the reparametrization invariance.

Light-cone quantization

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In going to super-conformal gauge we have fixed it partially, leaving only transformations satisfying $\partial_+ \epsilon^- = \partial_- \epsilon^+ = 0$.

This freedom can now be used to transform ψ^+ away:

$$X^+ = \frac{2\pi\alpha'}{l} p^+ \tau \,, \qquad \psi^+ = \frac{1}{\sqrt{2}}(\psi^0 + \psi^1) = 0 \,,$$

or, equivalently $b_r^+ = 0, \forall r$. We can now solve the constraints.

Solving the constraints

The bosonic constraints $T_{\pm\pm} = 0$ lead to

$$\partial_\pm X^- = \frac{1}{2p^+}\frac{l}{2\pi}\left(\frac{1}{\alpha'}\partial_\pm X^i \partial_\pm X^i + i\psi^i_\pm \partial_\pm \psi^i_\pm\right)$$

and the fermionic constraints $T_{F\pm} = 0 \implies$

$$\psi^-_\pm = \frac{2}{\alpha' p^+}\frac{l}{2\pi}\psi^i_\pm \partial_\pm X^i\,,$$

which leaves only the transverse coordinates X^i and ψ^i as independent dof.

Solving the constraints

The corresponding oscillator expressions are

$$\alpha_m^- = \frac{1}{\sqrt{2\alpha'}p^+}\left\{\sum_n : \alpha_n^i \alpha_{m-n}^j \delta_{ij} + \sum_r \left(\frac{m}{2} - r\right) : b_r^i b_{m-r}^i : + a\delta_m\right\}$$

and

$$b_r^- = \sqrt{\frac{2}{\alpha'}}\frac{1}{p^+}\sum_q \alpha_{r-q}^i b_q^i$$

For the closed string we have to supplement these expressions with the right-movers.
For the open string there is a $1/2$ factor in the r.h.s.

Solving the constraints

For the closed string the level matching condition $L_0 - \tilde{L}_0 = 0$ leads to

$$N_{tr}^{(\alpha)} + N_{tr}^{(b)} = \tilde{N}_{tr}^{(\alpha)} + \tilde{N}_{tr}^{(b)}$$

The mass operators are now

$$\alpha' M_L^2 = 2(\tilde{N}_{tr}^{(\alpha)} + \tilde{N}_{tr}^{(b)} + \tilde{a}) , \qquad \alpha' M_R^2 = 2(N_{tr}^{(\alpha)} + N_{tr}^{(b)} + a) ,$$

$$M_L^2 = M_R^2$$

as a consequence of level matching and we assumed $a = \tilde{a}$.
We have to determine the normal ordering constants.

Normal ordering constants

In the NS sector of the closed string we have

$$
\begin{aligned}
a_{NS} = \tilde{a}_{NS} &= \frac{1}{2}(d-2)\left(\sum_{n=0}^{\infty} n - \sum_{r=1/2}^{\infty} r\right) = \frac{1}{2}(d-2)(-\frac{1}{12} - \frac{1}{24}) \\
&= -\frac{1}{16}(d-2)
\end{aligned}
$$

In the R sector of the closed string, the sum in the fermionic sector is over the integers and then it cancels the contribution from the bosonic sector $\implies a_R = \tilde{a}_R = 0$.

We used ζ functions regularization:
$\sum_{n=1}^{\infty}(n+a) = \zeta(-1, a) = -\frac{1}{12}(6a^2 - 6a + 1)$

The spectrum

We first discuss the open string.

$$\alpha' M^2 = N_{tr}^{(\alpha)} + N_{tr}^{(b)} + a \, ,$$

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NS sector:

- The ground state is the oscillator vacuum with $\alpha' M^2 = a$.
- The first excited state is $b^i_{-1/2}|0>$ with $\alpha' M^2 = \frac{1}{2} + a$. This is a vector of $SO(d-2)$ which must be massless.
  Lorentz invariance requires that physical states fall into reps of little group of SO(d-1,1) which is SO(d-1) for massive particles and SO(d-2) for massless particles. $\implies$ $a = -\frac{1}{2}$.
  Using $a = -\frac{1}{16}(d-2)$ $\implies$ $d = 10$

The spectrum

- At the next excitation level we have $\alpha^i_{-1}|0>$ and $b^i_{-1/2}b^j_{-1/2}|0>$ with $\alpha' M^2 = \frac{1}{2}$ comprising $8 + 28$ bosonic states.

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- At the next excitation level we have $\alpha^i_{-1}|0>$ and $b^i_{-1/2}b^j_{-1/2}|0>$ with $\alpha' M^2 = \frac{1}{2}$ comprising $8 + 28$ bosonic states.
- It can be shown that these and all the other massive light-cone states, which are tensors of $SO(8)$, combine uniquely to tensors of $SO(9)$, the little group for massive states in ten dimensions.

The spectrum

R sector:

We already know that the R ground state is a spinor of $SO(9,1)$.

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In $d = 10$ it is even possible to impose both simultaneously leaving 8 independent on-shell components, the components of a Majorana-Weyl spinor of SO(8) $\implies$ as required by SUSY

The spectrum

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Using the description

$$\begin{aligned} b_0^\pm &= \frac{1}{2}(\pm b_0^0 + b_0^1)\,, \\ b_i^\pm &= \frac{1}{2}(b_0^{2i} \pm i b_0^{2i+1})\,, \qquad i = 1, \ldots, \frac{d-2}{2} \end{aligned}$$

In LCG one only has the raising and lowering zero modes $b_i^\pm$ with $i = 1, \ldots, 4 \implies$ the degenerate R ground state can be described by the 16 states $|s_1, s_2, s_3, s_4 >$ with $s_i = \pm\frac{1}{2}$.

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We can choose the ground state to have either one of two possible chiralities: $|a>$ and $|\dot{a}>$.

The spectrum

The ground state $|a>$ contains all states with $\sum_i s_i \in 2\mathbb{Z}$ and $|\dot{a}>$ all states with $\sum_i s_i = 2\mathbb{Z} + 1$.

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Tachyon free spectrum follows from requirement of modular invariance and space-time supersymmetry from the vanishing of the one-loop partition function

$\alpha' M^2$	States and $SO(8)$ repr.	Little group	$(-1)^F$	Repr. little group
$-\frac{1}{2}$	$\|0>$ (1)	$SO(9)$	-1	(1)
0	$b^i_{-1/2}\|0>$ $(8)_v$	$SO(8)$	$+1$	$(8)_v$
$+\frac{1}{2}$	$\alpha^i_{-1}\|0>,\ b^i_{-1/2}b^j_{-1/2}\|0>$ $(8)_v,\ (28)$	$SO(9)$	-1	(36)
$+1$	$b^i_{-1/2}b^j_{-1/2}b^k_{-1/2}\|0>$ $(56)_v$ $\alpha^i_{-1}b^j_{-1/2}\|0>$ $(1)+(28)+(35)$ $b^i_{-3/2}\|0>$ $(8)_v$	$SO(9)$	$+1$	$(84)+(44)$

Table: R sector

$\alpha' m^2$	States and $SO(8)$ repr.	Little group	$(-1)^F$	Repr. littl group
0	$\vert a>$ $(8)_s$ $\vert \dot{a}>$ $(8)_c$	$SO(8)$	$+1$ -1	$(8)_s$ $(8)_c$
$+1$	$\alpha^i_{-1}\vert a>, \quad b^i_{-1}\vert \dot{a}>$ $(8)_c+(56)_c \quad (8)_s+(56)_s$ $\alpha^i_{-1}\vert \dot{a}>, \quad b^i_{-1}\vert a>$ $(8)_s+(56)_s \quad (8)_c+(56)_c$	$SO(9)$	$+1$ -1	(128) (128)

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Turning the argument around, we motivate the GSO projection by requiring a space-time supersymmetric spectrum.

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We also want to get rid of the tachyon

Define a quantum number which is the eigenvalue of the operator $(-1)^F$, where F is the world-sheet fermion number.

Assigning the NS vacuum $(-1)^F|0>=-|0>$, we can write in the NS sector $F=\sum_{r>0} b^i_{-r} b^i_r - 1$.

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A general state in the NS sector, $\alpha^{i_1}_{-n_1} \cdots \alpha^{i_N}_{-n_N} b^{j_1}_{-r_1} \cdots b^{j_M}_{-r_M} |0>$ has $(-1)^F = (-1)^{M+1}$ and all states with M even are projected out.

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In particular the tachyon (with fermionic oscillation number $M = 0$) disappears.

In the R sector, the equivalent of $(-1)^F$ is a generalized chirality operator

$$(-1)^F = 16 b_0^2 \cdot b_0^9 (-1)^{\sum_{n>0} b_{-n}^i b_n^i}$$

where $\gamma = 16 b_0^2 \cdots b_0^9$ is the chirality operator in the eight transverse dimensions and $\sum_{n>0} b_{-n}^i b_n^i$ the world-sheet fermion number operator.

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$$\{(-1)^F, \psi^\mu\} = 0$$

and the eigenvalues of the R ground states are ± 1 depending on the chirality.

We define

$$(-1)^F|a>=16\prod_{i=2}^{9} b_0^i|a>=+1|a> \quad \text{and} \quad (-1)^F|\dot{a}>=-1|\dot{a}>$$

Then a state in the R sector $\alpha^{i_1}_{-n_1}\cdots\alpha^{i_N}_{-n_N} b^{j_1}_{-m_1}\cdots b^{j_M}_{-m_M}|a>$ has $(-1)^F=(-1)^M(-1)^{\sum_i \delta_{m_i,0}}$.

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And the analogous state built on the $|\dot{a}>$ ground state has $(-1)^F = -(-1)^M(-1)^{\sum_i \delta_{m_i,0}}$.

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And the analogous state built on the $|\dot{a}>$ ground state has $(-1)^F=-(-1)^M(-1)^{\sum_i \delta_{m_i,0}}$.

The GSO projection then amounts to demanding that all the states have either $(-1)^F=1$ or $(-1)^F=-1 \implies$ susy spectrum.

Of course the consistency of the truncation requires that in the interacting theory no projected-out states are produced.

This follows from demanding locality of the operator product algebra of the vertex operators for all allowed states

GSO projection: closed string

To obtain the closed string spectrum, take the tensor product of two open string spectra, obeying the constraint
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We can choose between two possible chiralities for the left and right R ground state.

$(L_0 - \tilde{L}_0)|\mathrm{phys}>= 0$ in each sector $\implies$ $M_R^2 = M_L^2$ and then the closed string states are products of open string states at the same mass level.

Closed fermionic string spectrum

Table: (NS, NS)-sector

$\alpha' M^2$	States and $SO(8)$ representation	Little group	$(-1)^F$	$(-1)^{\tilde{F}}$
-2	$\|0>_L \times \|0>_R$ (1) (1)	$SO(9)$	-1	-1
0	$\tilde{b}^i_{-1/2}\|0>_L \times b^j_{-1/2}\|0>_R$ $(8)_v$ (1)	$SO(8)$	$+1$	$+1$

Representation contents with respect to the little group: (1) and $(1) + (28) + (35)_v$

(R,R)-sector

$\alpha' M^2$	States and $SO(8)$ repr.	Repr little group	$(-1)^F$	$(-1)^{\tilde{F}}$
0	$\|a>_L \times \|b>_R$ $(8)_s \quad (8)_s$	$(1) + (28) + (35)_s$	$+1$	$+1$
	$\|\dot{a}>_L \times \|\dot{b}>_R$ $(8)_c \quad (8)_c$	$(1) + (28) + (35)_c$	-1	-1
	$\|\dot{a}>_L \times \|b>_R$ $(8)_c \quad (8)_s$	$(8)_v + (56)_v$	-1	$+1$
	$\|a>_L \times \|\dot{b}>_R$ $(8)_s \quad (8)_c$	$(8)_v + (56)_v$	$+1$	-1

Table: (R,NS)-sector

$\alpha' M^2$	States and $SO(8)$ repr	Repr little group	$(-1)^F$	$(-1)^{\tilde{F}}$
0	$\lvert a >_L \times b^i_{-1/2} \lvert 0 >_R$ $(8)_s \qquad (8)_v$	$(8)_c + (56)_c$	$+1$	$+1$
	$\lvert \dot{a} >_L \times b^i_{-1/2} \lvert 0 >_R$ $(8)_c \qquad (8)_v$	$(8)_s + (56)_s$	-1	$+1$

Table: (NS, R)-sector

$\alpha' M^2$	States and $SO(8)$ repr	Repr. little group	$(-1)^F$	$(-1)^{\tilde{F}}$
0	$\tilde{b}^i_{-1/2}\|0>_L \times \|a>_R$ $(8)_v \quad (8)_s$	$(8)_c + (56)_c$	$+1$	$+1$
	$\tilde{b}^i_{-1/2}\|0>_L \times \|\dot{a}>_R$ $(8)_v \quad (8)_c$	$(8)_s + (56)_s$	$+1$	-1

Massive states and GSO projection

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This leads to two inequivalent possibilities: $(-1)^F = (-1)^{\tilde{F}}$ or $(-1)^F = -(-1)^{\tilde{F}}$.

Type IIB

The theory with $(-1)^F = (-1)^{\tilde{F}}$ has no tachyon and the following massless states

$$
(\mathrm{IIB}) \quad \begin{array}{ll} \mathrm{Bos}: & [(1)+(28)+(35)_v]_{(NS,NS)} + [(1)+(28)+(35)_s]_{(R,R)} \\ \mathrm{Fermi} & [(8)_c+(56)_c]_{(NS,R)} + [(8)_c+(56)_c]_{(R,NS)} \end{array}
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128 bosonic and 128 fermionic states, indicating a supersymmetric spectrum.

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128 bosonic and 128 fermionic states, indicating a supersymmetric spectrum.

The massless spectrum is that of type IIB supergravity in ten dimensions.

Bosonic spectrum of type IIB SUGRA

$$[(1) + (28) + (35)_v]_{(NS,NS)} + [(1) + (28) + (35)_s]_{(R,R)}$$

The $(35)_v$ represents the on-shell dof of a graviton.

The two (28)'s represent two antisymmetric tensor fields

The $(35)_s$ represents a rank four self-dual antisymmetric tensor

In addition there are two real scalars

Fermionic spectrum of type IIB SUGRA

$$[(8)_c + (56)_c]_{(NS,R)} + [(8)_c + (56)_c]_{(R,NS)}$$

The $(56)_c$'s are two on-shell gravitinos with spin $3/2 \implies N = 2$ supersymmetry

The $(8)_c$'s are two spin $1/2$ fermions, called dilatinos.

Since both gravitinos are of the same handedness, this is a chiral theory

Together a gravitino and a dilatino form a reducible vector spinor ψ_μ whose traceless part $\gamma^\mu \psi_\mu = 0$ is the gravitino and whose trace part is the dilatino.

Type IIA

The theory with $(-1)^F = -(-1)^{\tilde{F}}$ has no tachyon and the following massless states

$$
(\text{IIA}) \quad \begin{array}{l} \text{Bos}: \quad [(1) + (28) + (35)_v]_{(NS,NS)} + [(8_v) + (56)_v]_{(R,R)} \\ \text{Fermi} \quad [(8)_c + (56)_c]_{(NS,R)} + [(8)_s + (56)_s]_{(R,NS)} \end{array}
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Bosonic spectrum of type IIA SUGRA

$$[(1) + (28) + (35)_v]_{(NS,NS)} + [(8_v) + (56)_v]_{(R,R)}$$

The $(35)_v$ represents the on-shell dof of a graviton.

The $(56)_v$ represents an antisymmetric rank three tensor

The (28) represents an antisymmetric rank two tensor field

One vector $(8)_v$ and one real scalar, the dilaton

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$$[(8)_c + (56)_c]_{(NS,R)} + [(8)_s + (56)_s]_{(R,NS)}$$

The (56)'s are two on-shell gravitinos with spin $3/2 \implies N = 2$ supersymmetry

The (8)'s are two spin $1/2$ fermions, called dilatinos, one of each handedness

Since both gravitinos are of opposite handedness, this is a non-chiral theory

This massless spectrum can be obtained by dimensional reduction of eleven-dimensional supergravity.

Supercharges

Both supergravity theories, type IIA and IIB have N=2 susy. $\implies$ there are two fermionic generators $Q^I, I = 1, 2$ which are Majorana-Weyl spinors of $SO(1, 9)$.

Together they have 32 real components. Often this is expressed by saying that the type II supergravities have 32 supercharges.

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Compactifying on a torus to four dimensions $\rightarrow$ N=8 susy with eight fermionic generators $Q^I, I = 1, \ldots, 8$ which are Majorana spinors of $SO(1, 3)$. Again the number of supercharges is 32.

Denoting the amount of susy by the number of supercharges is independent of space-time dimension and is invariant under compactification on a torus.

Other compactifications

Compactification on manifolds with curvature breaks some or all supersymmetries and reduces the number of supercharges

Compactification on Calabi-Yau manifolds preserve one quarter, i.e. eight supercharges, that is N=2 supersymmetry in $d = 4$.

The type I supergravity that we consider next has 16 supercharges

Type I superstring

Let us look at the unoriented closed string.

Its states are a subset of those of the left-right symmetric type IIB theory, i.e. those which are symmetric under world-sheet parity Ω which interchanges left- and right-movers:

$$\begin{aligned} \Omega X^\mu(\sigma,\tau)\Omega^{-1} &= X^\mu(l-\sigma,\tau)\,, \\ \Omega\psi^\mu_\pm(\sigma,\tau)\Omega^{-1} &= \psi^\mu_\mp(l-\sigma,\tau)\,. \end{aligned}$$

In terms of oscilator modes:

$$\begin{aligned} \Omega\alpha^\mu_n\Omega^{-1} &= \tilde{\alpha}^\mu_n\,, & \Omega\tilde{\alpha}^\mu_n\Omega^{-1} = \alpha^\mu_n\,, \\ \Omega b^\mu_r\Omega^{-1} &= e^{-2\pi i r}\tilde{b}^\mu_r\,, & \Omega\tilde{b}^\mu_r\Omega^{-1} = e^{-2\pi i r}b^\mu_r\,, \end{aligned}$$

with $r \in \mathbb{Z}$ for the R-sector and $r \in \mathbb{Z}+\frac{1}{2}$ for the NS-sector

Type I superstring

Ω interchanges the (NS, R) sector with the (R, NS) sector

We also have to define how Ω acts on closed string ground states.

Since the R ground states are space-time fermions with odd Grassmann parity, one defines

$$\begin{aligned}
(\mathrm{NS},\mathrm{NS}): &\quad \Omega(|0>_L \times |0>_R) = |0>_L \times |0>_R \\
(\mathrm{R},\mathrm{R}): &\quad \Omega(|a>_L \times |b>_R) = -|b>_L \times |a>_R \\
(\mathrm{NS},\mathrm{R}): &\quad \Omega(|0>_L \times |a>_R) = |a>_L \times |0>_R \\
(\mathrm{R},\mathrm{NS}): &\quad \Omega(|a>_L \times |0>_R) = |0>_L \times |a>_R
\end{aligned}$$

Spectrum of Type I superstring

Among the massless (NS,NS) sector states the (1) + (35) survive, and among the (R,R) states the (28).

Among the fermions a diagonal combination of the $(8)_c + (56)_c$ survives.

The massless closed string spectrum of the non-orientabe theory is

$$\begin{aligned} \text{Bosons}: &\quad [(1) + (35)_v]_{(NS,NS)} + [(28)]_{(R,R)} \\ \text{Fermions}: &\quad [(8)_c + (56)_c]_{(NS,R)+(R,NS)} \end{aligned}$$

These are the states of N=1 SUGRA in ten dimensions which is the massless closed string sector of the type I superstring theory.

Consistency requires the addition of so-called twisted sectors which are open strings giving rise to massless gauge bosons

Path integral quantization