ICTP SAIFR

International Centre for Theoretical Physics
South American Institute for Fundamental Research

SCHOOL ON COMPLEX NETWORKS AND APPLICATIONS TO NEUROSCIENCES

CONNECTING NETWORKS

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OVERVIEW

I.- Network Interaction

- Consequences of interaction
- Connectors

II.- Competition

- Centrality
- Strategies
- Competition parameter
- Generality of the results

III.- Cooperation

- Synchronization
- Single networks
- Interacting networks

Network Interaction

(I-III)


Hunting ExoEarths | Denisovan DNA in Full | Secrets of Semen
ScienceNews
MAGAZINE OF THE SOCIETY FOR SCIENCE & THE PUBLIC ■ SEPTEMBER 22, 2012
Linking Up
Perils and promises of connecting networks
Star Factory
Sickening Tattoos
Crime-Solving Camera

NETWORKS INTERACTING WITH OTHER NETWORKS

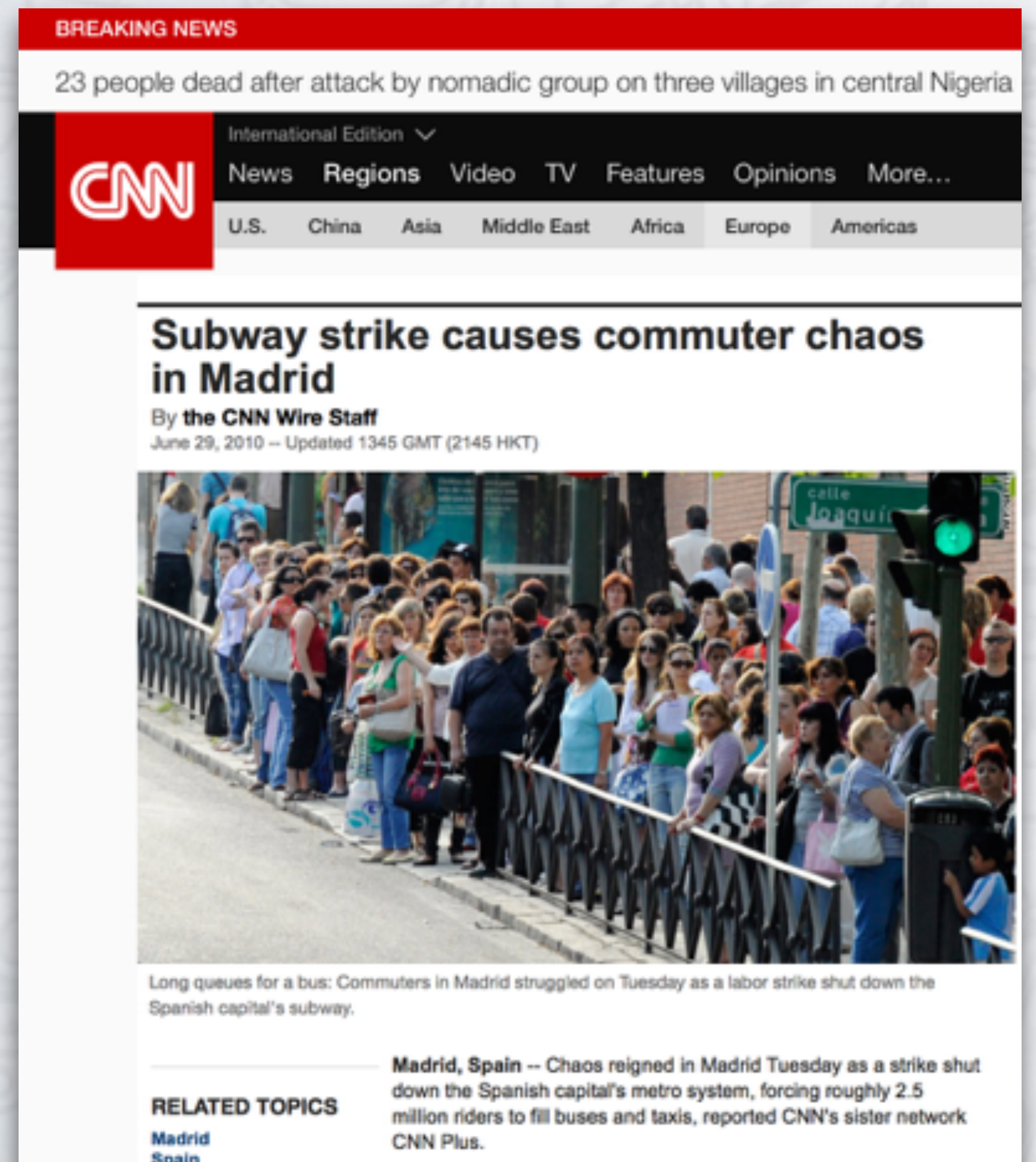
Example of interconnected networks (Madrid): (A) Suburban railway network, (B) the underground and (C) the tram lines.

The three networks **benefit** from being interconnected but also **compete** for acquiring users.

INTERACTION HAS CONSEQUENCES ON ALL NETWORKS

The case of the "savage strike"

Madrid, June 2010: A savage strike at the underground collapses all public transport networks.


BREAKING NEWS

23 people dead after attack by nomadic group on three villages in central Nigeria

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Subway strike causes commuter chaos in Madrid

By **the CNN Wire Staff**

June 29, 2010 -- Updated 1345 GMT (2145 HKT)

Long queues for a bus: Commuters in Madrid struggled on Tuesday as a labor strike shut down the Spanish capital's subway.

Madrid, Spain -- Chaos reigned in Madrid Tuesday as a strike shut down the Spanish capital's metro system, forcing roughly 2.5 million riders to fill buses and taxis, reported CNN's sister network CNN Plus.

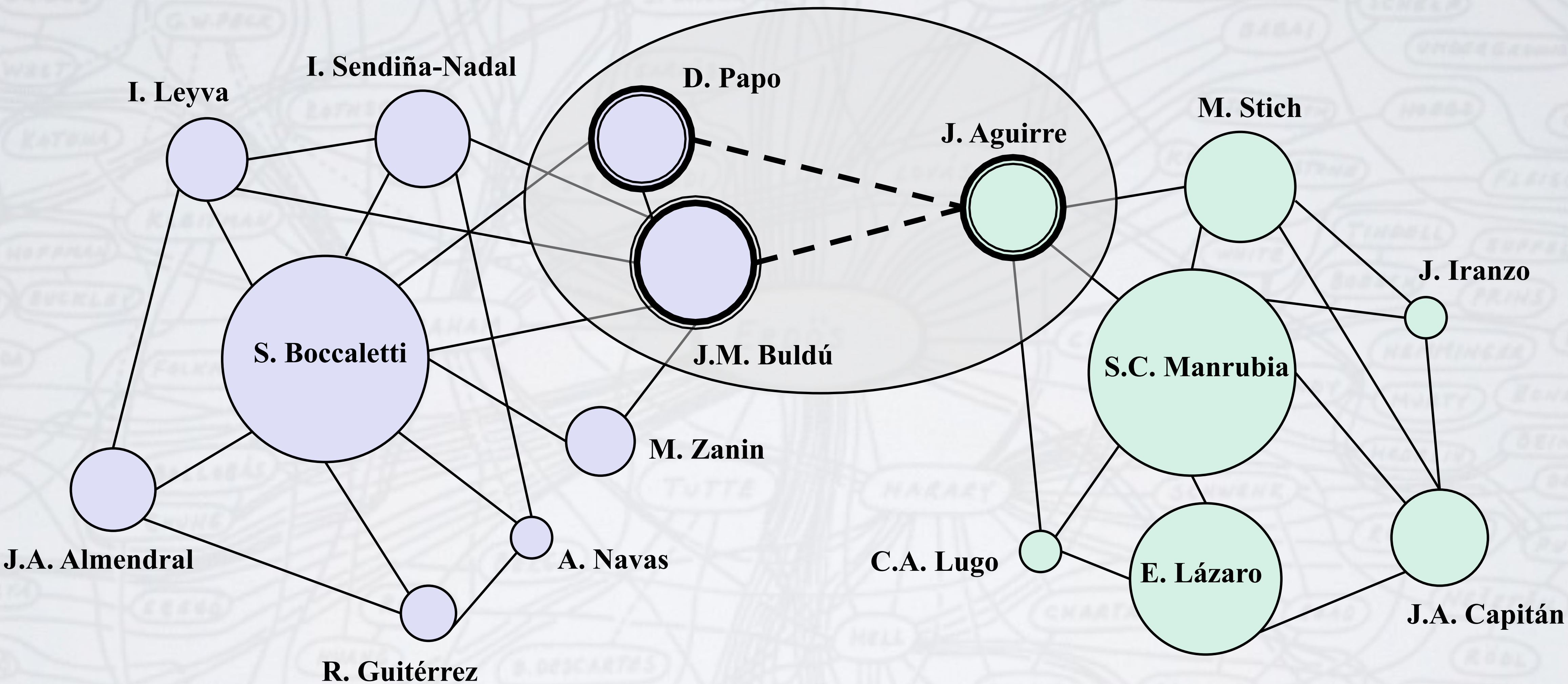
RELATED TOPICS

Madrid

Spain

INTERACTION HAS CONSEQUENCES ON ALL NETWORKS

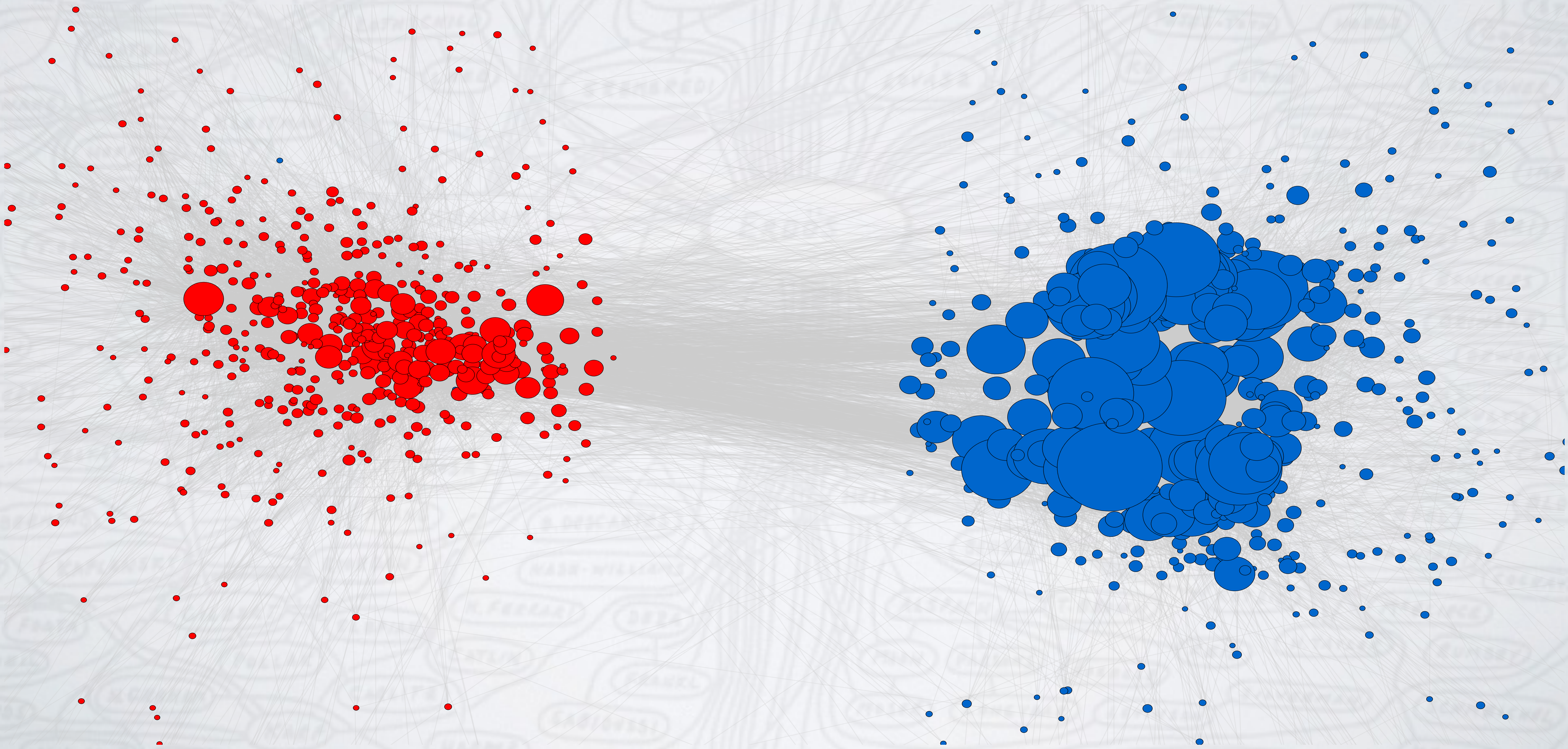
A more positive consequence: knowledge transfer



NETWORKS INTERACTING BETWEEN THEM

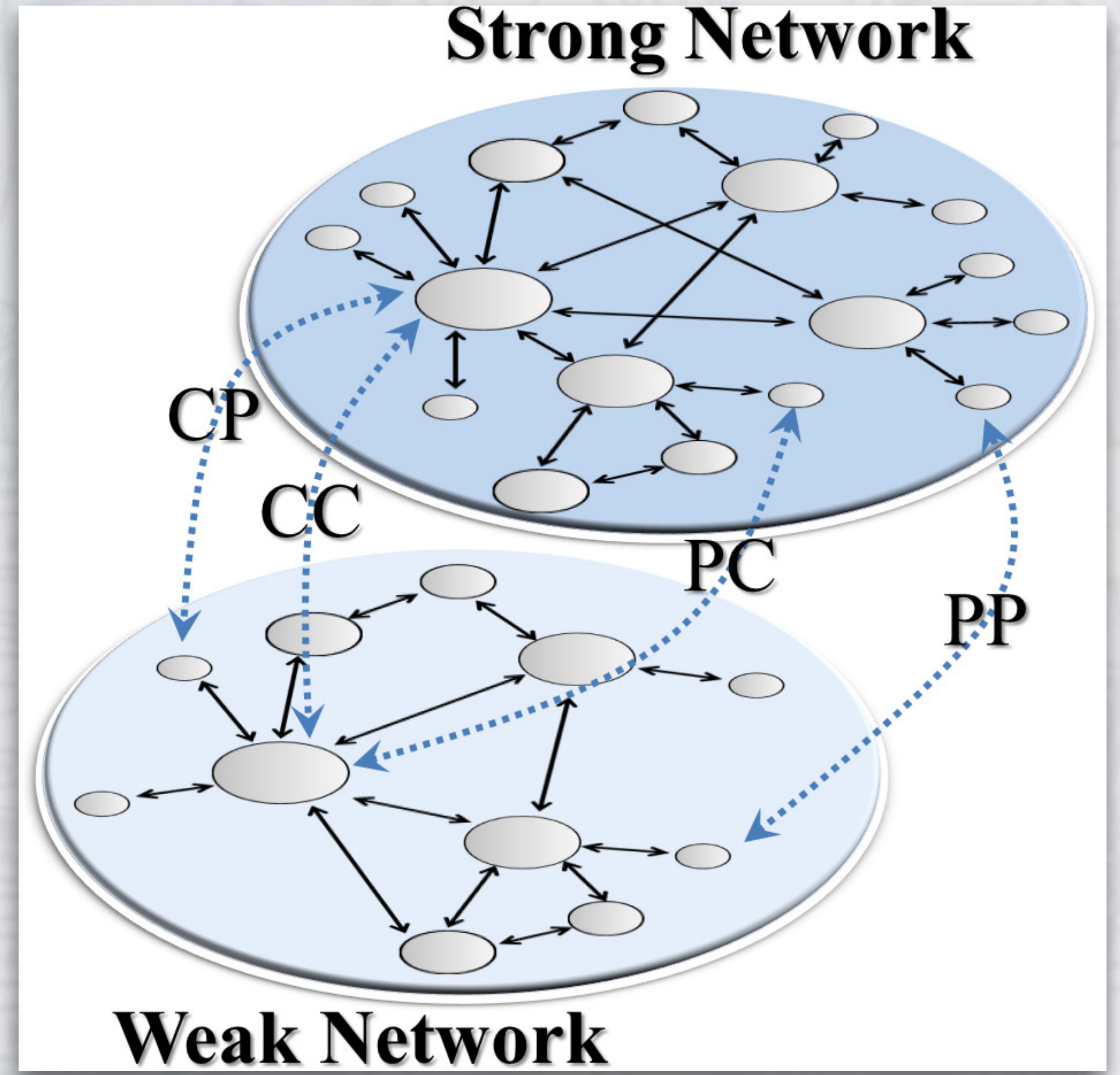
Interacting with other networks leads to changes in the structural/dynamical properties of each network, which leads to a natural question:

How to create connections between networks?



CHOOSING THE ADEQUATE CONNECTORS: GENERAL STRATEGIES

We will focus on the way **connector links** are created:



Schematic representation of the different strategies for creating connection paths between two undirected networks: CC, PP, CP and PC.

* Links are not created randomly.

* We connect nodes according to their centrality (importance).

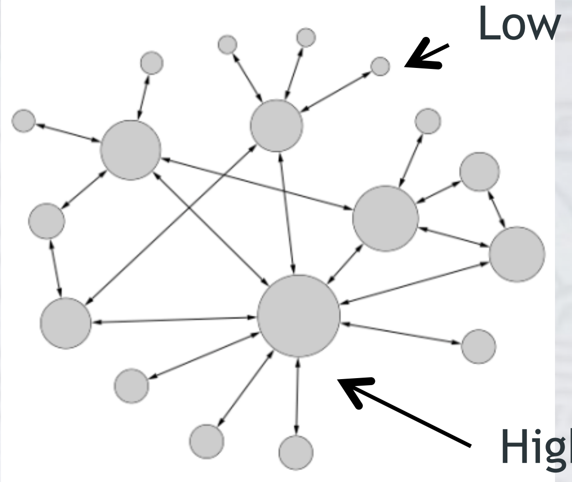
* Central nodes (C) and peripheral nodes (P).

* The strong (weak) network is the one with the highest (lowest) largest eigenvalue of the connection matrix.

CHOOSING THE ADEQUATE CONNECTORS: NODES ARE HETEROGENEOUS

- Importance of a node is quantified by the *eigenvector centrality*:

$$x_k = \gamma^{-1} \sum_j G_{kj} x_j \qquad \gamma \vec{x} = \mathbf{G} \vec{x} \qquad \vec{x}(t \to \infty) = \vec{u}_1$$



> **Interestingly, the eigenvector centrality x_k, is given by the eigenvector u_1 associated to the first eigenvalue λ_1 of G.**

- Now, consider a dynamical process on a network described as:

$$\vec{n}(t+1) = \mathbf{M} \vec{n}(t) \qquad M_{ij} \geq 0$$

  - The state vector can be expressed as:

$$\vec{n}(t) = \mathbf{M}^t \vec{n}(0) = \sum_{i=1}^{m} (\vec{n}(0) \vec{u}_i) \lambda_i^t \vec{u}_i$$

n(0): vector of initial conditions
λ_i: eigenvalue i of the matrix M
u_i: eigenvector i of the matrix M

  - Normalizing the state vector such that $|n(t)| = 1$

$$\lim_{t \to \infty} \left(\frac{\vec{n}(t)}{(\vec{n}(0) \vec{u}_1) \lambda_1^t} \right) = \vec{u}_1$$

> **The final state is described by the eigenvector u_1 associated to the largest eigenvalue λ_1.**

Competition

(II-III)

NETWORKS COMPETING BETWEEN THEM

If we want to identify the most convenient **strategies**, first we have to define a …

TARGET

↓

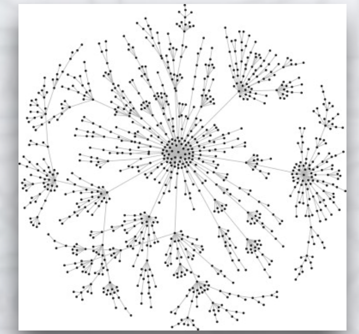
ACQUIRING CENTRALITY

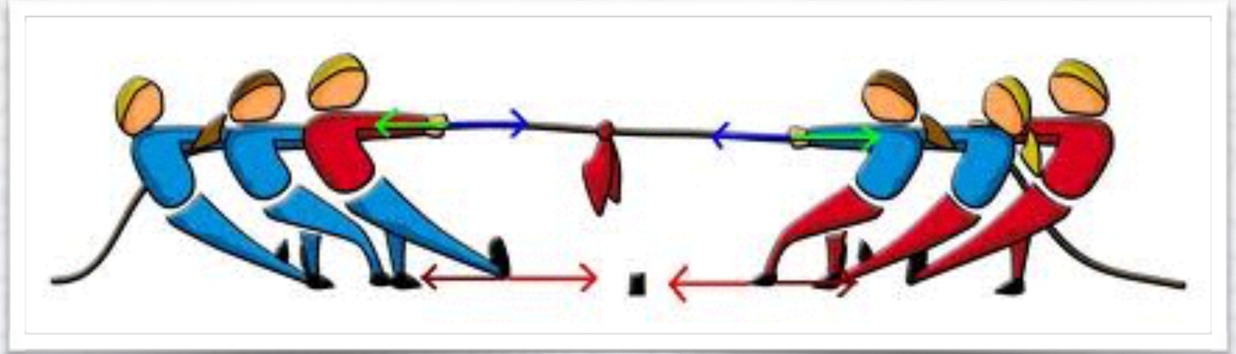
specifically, eigenvector centrality → eigenvector u_1 of the largest eigenvalue

↓

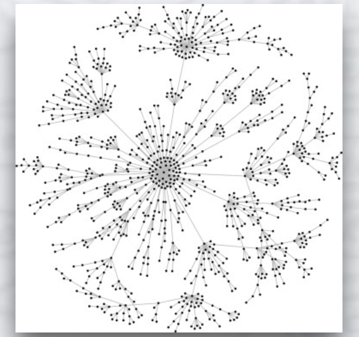
COMPETITION PROCESS

Network A



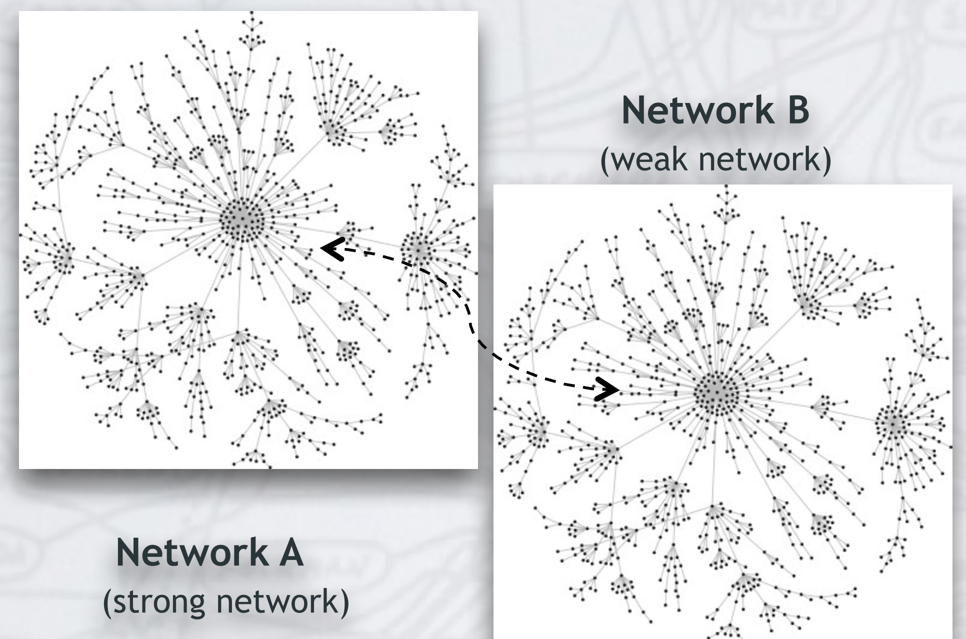


Network B



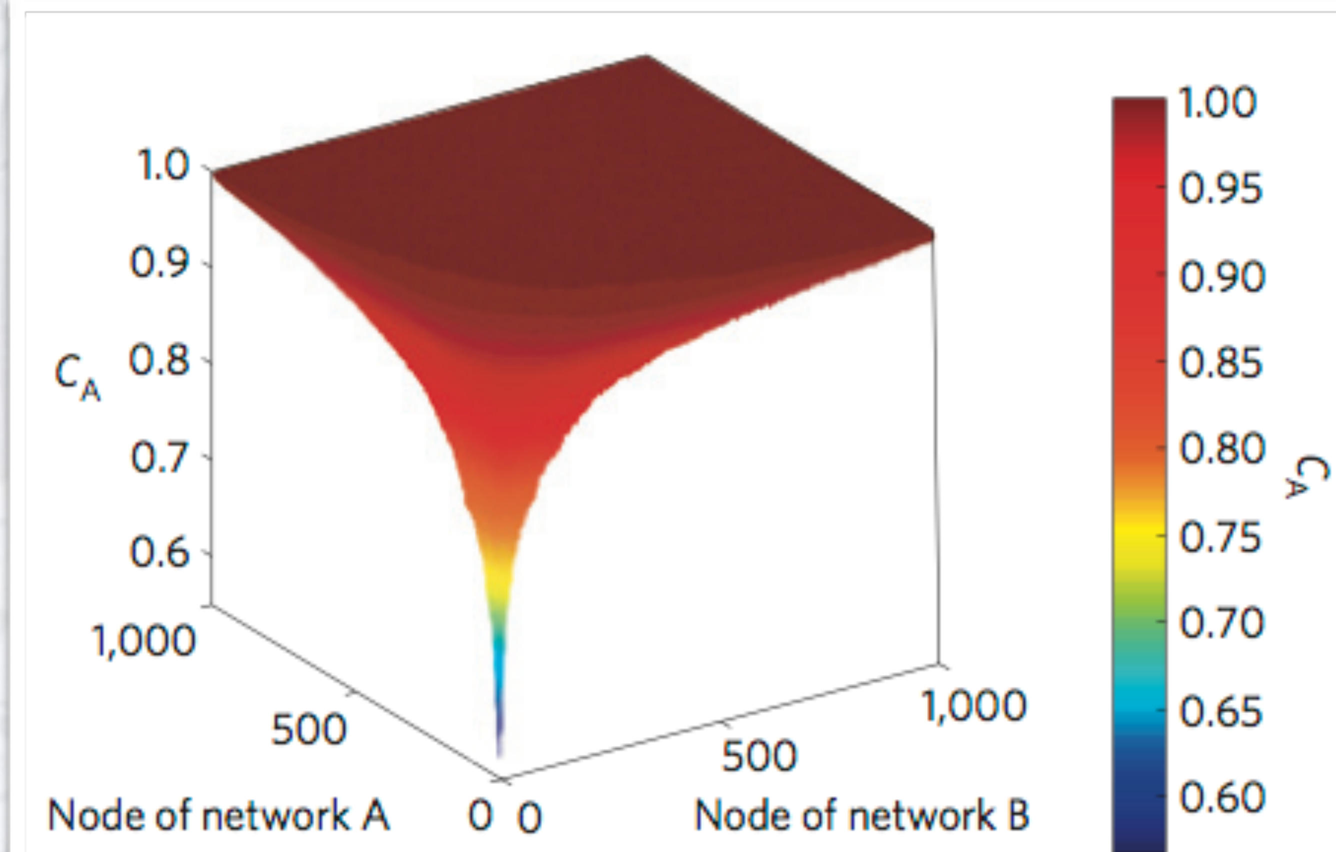
NETWORKS COMPETING FOR CENTRALITY

We connect two Barabási-Albert networks with a **unique link in all possible configurations**, according to a weighted connection matrix ***M***. Next, we calculate the eigenvector centrality acquired by each network:



* In this example, the weighted connection matrix ***M*** represents a replication & mutation process of a population of RNA sequences, where $\boldsymbol{M} = (2-m)\boldsymbol{I} + (m/3L)\boldsymbol{A}$.

* C_A is the centrality accumulated by network A and it is obtained from the eigenvector associated to the largest eigenvalue of the interconnected network, specifically from the centralities of nodes belonging to A.



Two Barabási–Albert networks A and B of size $N_A = N_B = 1{,}000$ and $L_A = L_B = 2{,}000$ links, connected by one single connector link in all possible configurations. **C_A is the centrality accumulated by network A**. The axes represent the connector nodes in networks A and B, and **nodes are numbered according to their network centrality ranking**. A is the strong network ($\lambda_{A,1} > \lambda_{B,1}$).

NETWORKS COMPETING FOR CENTRALITY

It is possible to evaluate **how the centrality of the whole network-of-networks will distribute** among each of the sub-networks:

1.- Before the connection:

$$A = \{N_A, L_A\} \quad B = \{N_B, L_B\} \xrightarrow[\textbf{A} \text{ and } \textbf{B} \text{ get connected to form } \textbf{T}]{\lambda_{A,1} > \lambda_{B,1}} T = \{N_T, L_T\} \quad L_T = L_A + L_B + L \quad N_T = N_A + N_B$$

connector links {cl}

2.- We connect the two networks:

$$P_{ij} = P_{ji} \neq 0 \quad \text{for} \quad ij \in \{cl\}$$

$$M_T = M_{AB} + \varepsilon P$$

3.- Eigenvector centralities (before and after):

$$u_{T,1} = (c_1, c_2, c_3, c_4, ..., c_{N-3}, c_{N-2}, c_{N-1}, c_N)$$
$$u_{A,1} = (c_1, c_2, c_3, c_4, ..., 0,0,0,0)$$
$$u_{B,1} = (0,0,0,0, ..., c_{N-3}, c_{N-2}, c_{N-1}, c_N)$$

4.- We quantify the centrality of each network:

$$C_A = \sum_{k \in A} (\vec{u}_1)_k \Big/ \sum_{k \in T} (\vec{u}_1)_k \qquad C_B = 1 - C_A$$

RESULTS & STRATEGIES

$$\vec{u}_{T,1} \approx \vec{u}_{A,1} + \varepsilon \frac{(\vec{u}_{A,1} P \vec{u}_{B,1})}{\lambda_{A,1} - \lambda_{B,1}} \vec{u}_{B,1}$$

A. Difference between largest eigenvalues $1/(\lambda_{A,1} - \lambda_{B,1})$

A. The higher $\lambda_{A,1}$-$\lambda_{B,1}$, the higher C_A

B. Centrality of the connector nodes $(\vec{u}_{A,1} P \vec{u}_{B,1})$

B1. The larger the number of terms in ***P*** (connector links), the lower C_A

B2. The higher the centrality of the connector nodes, the lower C_A

NETWORKS COMPETING FOR CENTRALITY

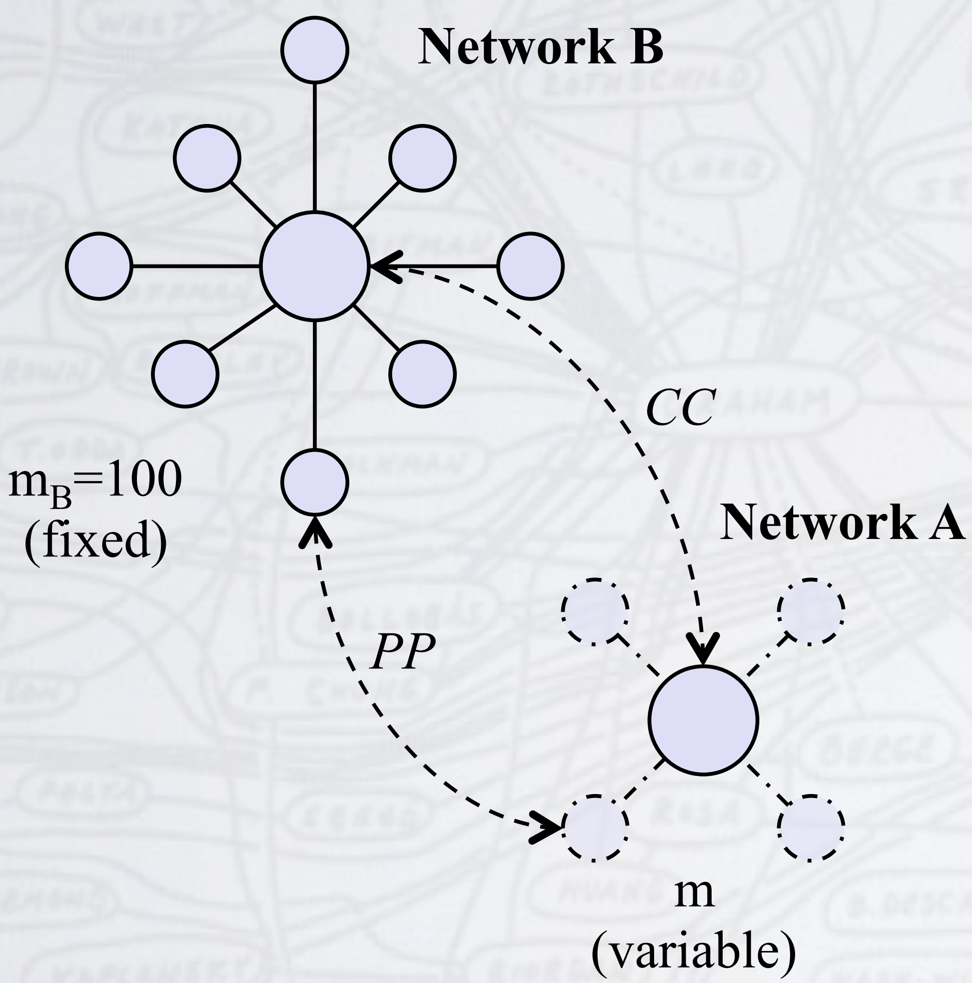
It is possible to define general strategies according to the kind of network:

- A **strong network** should connect through **peripheral** (P) nodes.
- A **weak network** should connect through **central** (C) nodes.
- The **higher the number of inter-connections** the better for the **weak network.**
- **Increasing the largest eigenvalue** of a network increases its centrality.

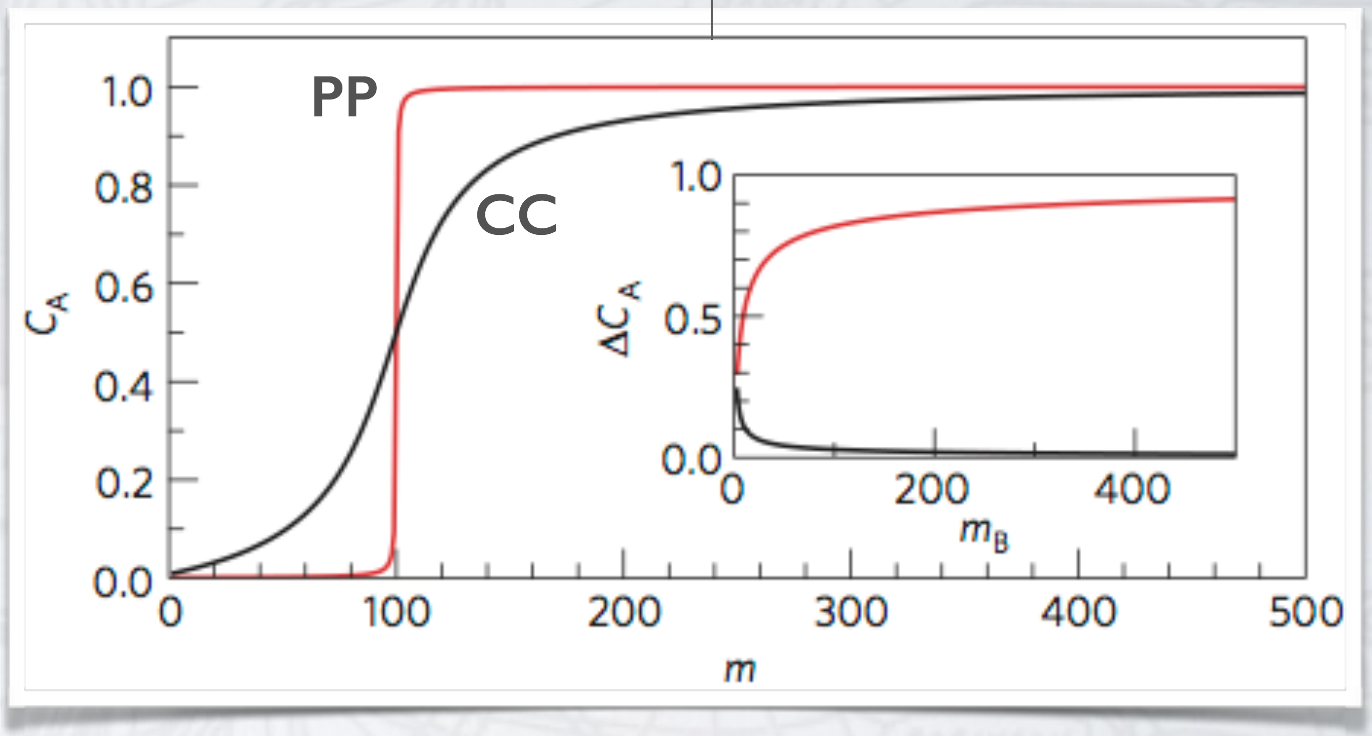
NETWORKS COMPETING FOR CENTRALITY

As we all know, **size is important** (since it is related to λ_1):

We connect two star networks of sizes m_B=100 and m (variable):



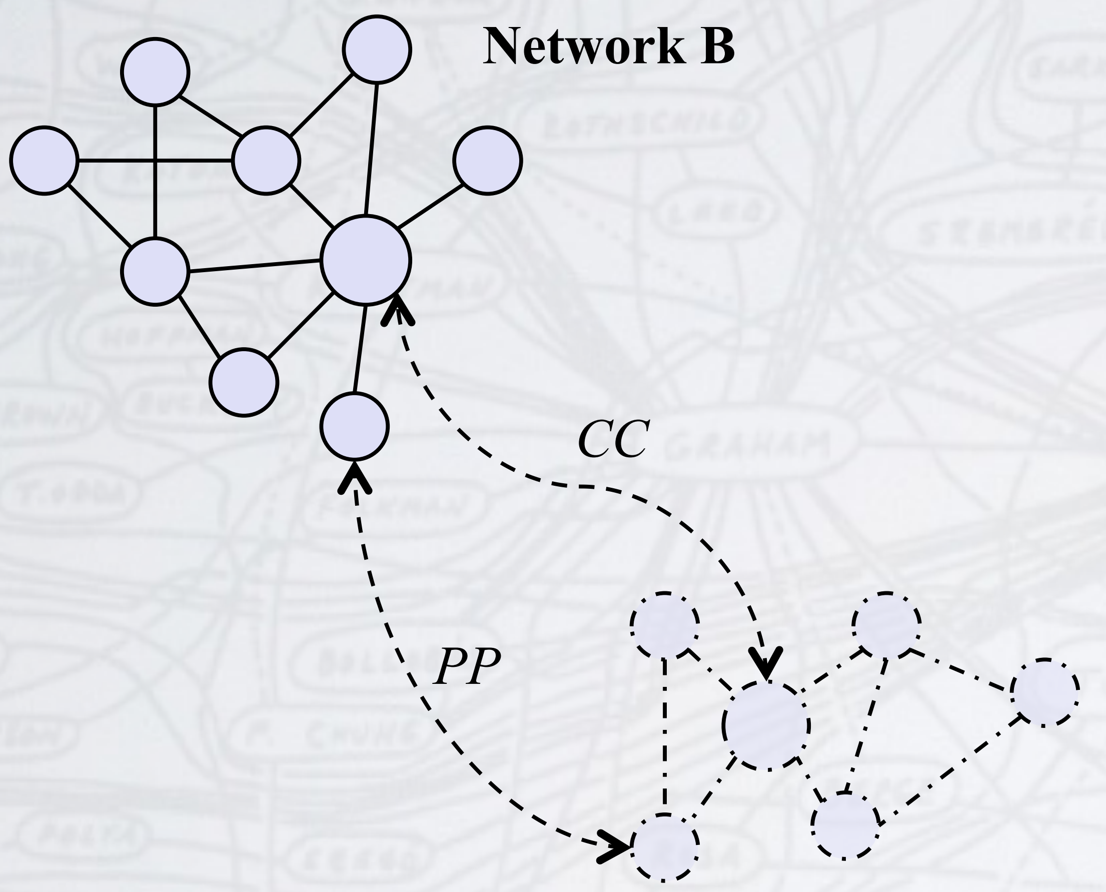
The strategy used in the interconnection determines the smoothness of the transition.



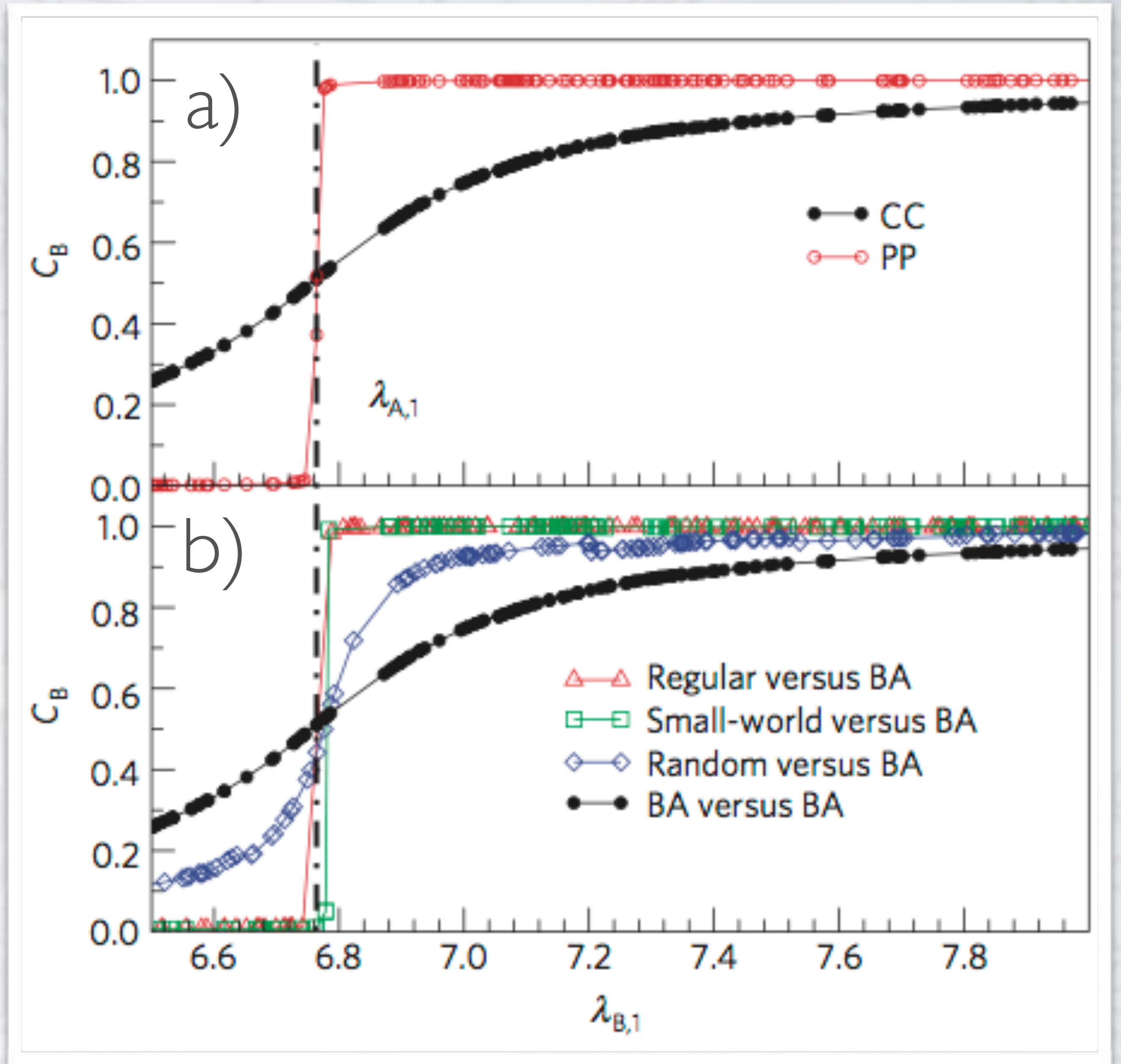
A star network ***A*** of m nodes competes, increasing its size (i.e., its λ_1), against a star network ***B*** of m_B=100 nodes. C_A depends on the size m and on the strategy used to create the connections between both networks (CC or PP). The inset shows how the increase of C_A at m=m_B depends on the network size.

NETWORKS COMPETING FOR CENTRALITY

If a network cannot grow, it may reorganize (**act locally, think globally!**):



Network A reorganizes its internal structure to increase $\lambda_{A,1}$



a) Two connected Barabási-Albert (BA) (N_A=N_B=200 nodes, L_A=L_B=400 links), where network B reorganizes and overcomes network A ($\lambda_{A,1}$=6.76). b) Different initial structures for network B (CC strategy).

EVALUATING THE COMPETITION IN REAL NETWORKS

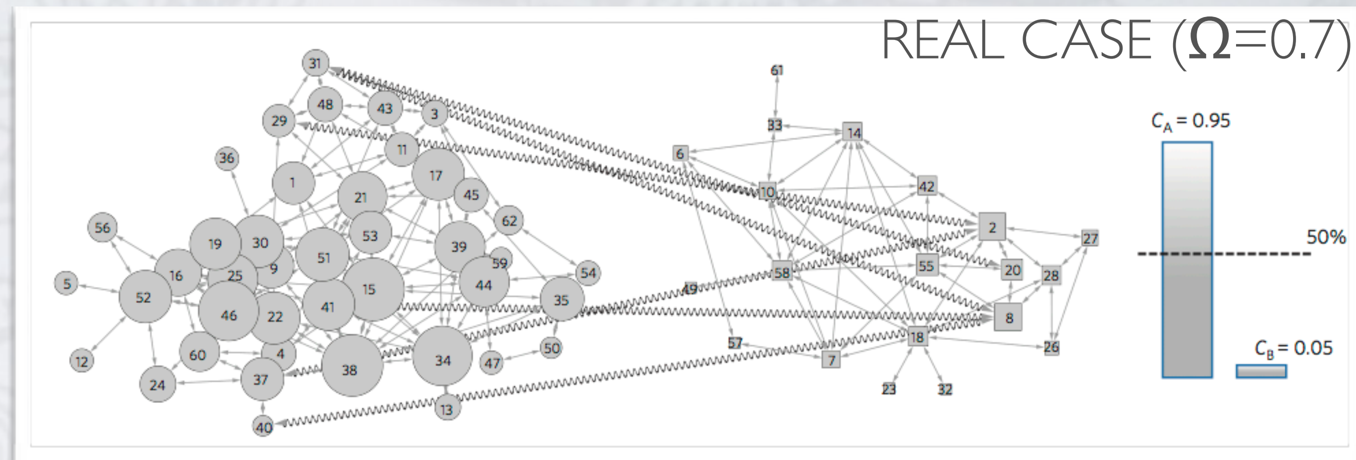
It is possible to define a **competition parameter** that indicates which network benefited from the structure of connections in real cases:

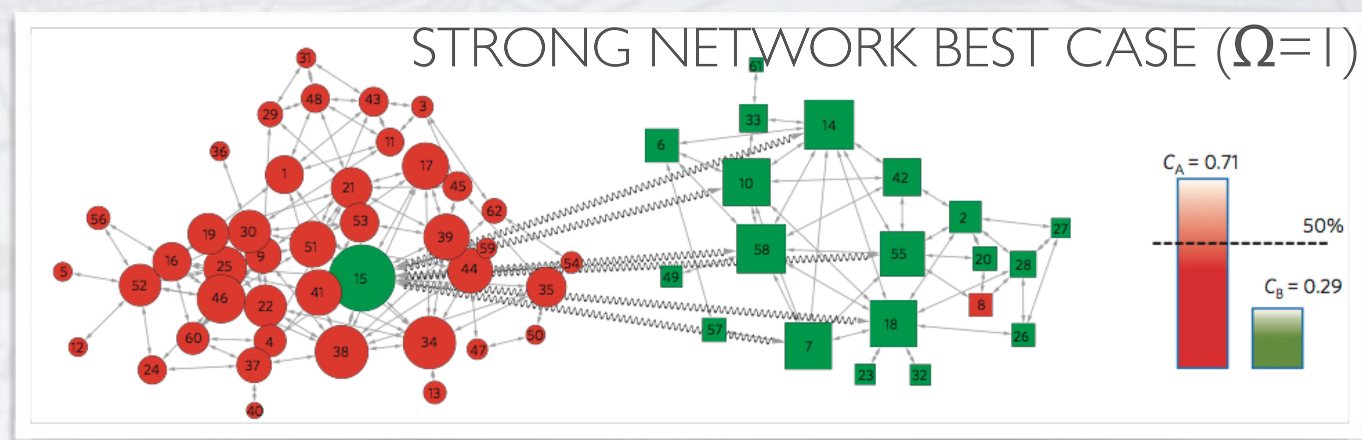
$$\Omega = \frac{2\left(C_A - C_A^{\min}\right)}{C_A^{\max} - C_A^{\min}} - 1$$

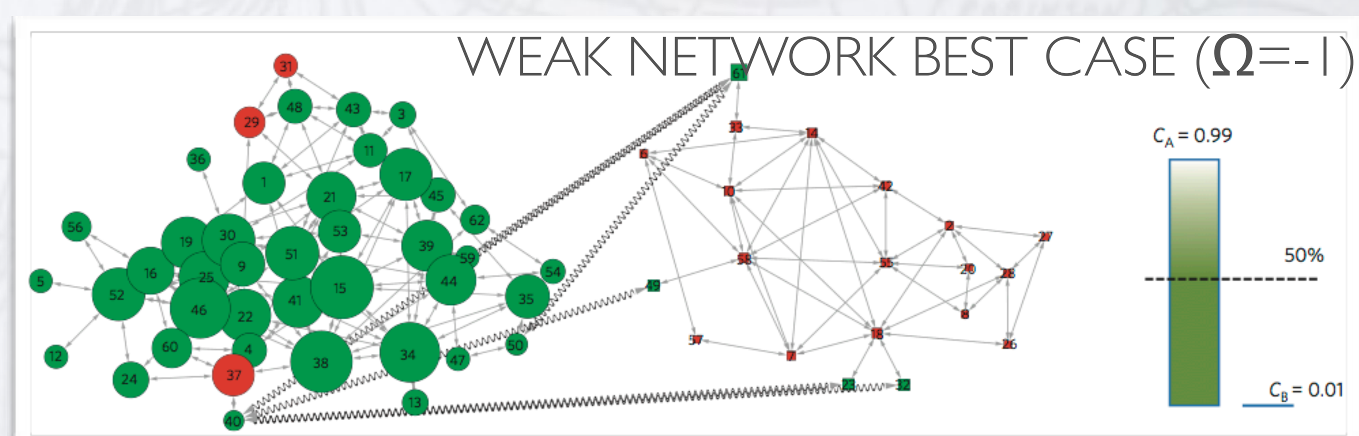
STRONG NETWORK BEST CASE (Ω=1)

WEAK NETWORK BEST CASE (Ω=-1)

BALANCE OF STRATEGIES (Ω=0)



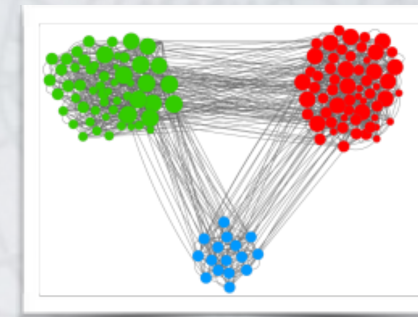




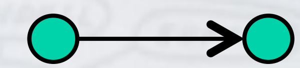
Dolphin network of Doubtful Sound (Ω=0.7)

EVALUATING THE COMPETITION IN REAL NETWORKS

The same methodology can be extended to other cases and applications:



- It is adaptable to **M interacting networks**.
- It also can be applied to **directed networks**.



- Any process related with the **first eigenvector of the transition matrix** and, in general, network processes described by n(t+1)=Mn(t):
  - Importance of nodes in a network (e.g., pagerank)
  - Disease spreading (SI or SIR models)
  - Rumor propagation (MT or DK models)
  - Population dynamics (e.g., RNA evolutionary processes)

Maki-Thompson model
(rumor propagation)

y(t): Probability of hearing a rumor
α: spreading rate
β: blocking rate
u_1: first eigenvector (centrality)

$$\vec{y}(t) \sim e^{[(\alpha+\beta)\lambda_1-\beta]t}\vec{u}_1$$

Cooperation

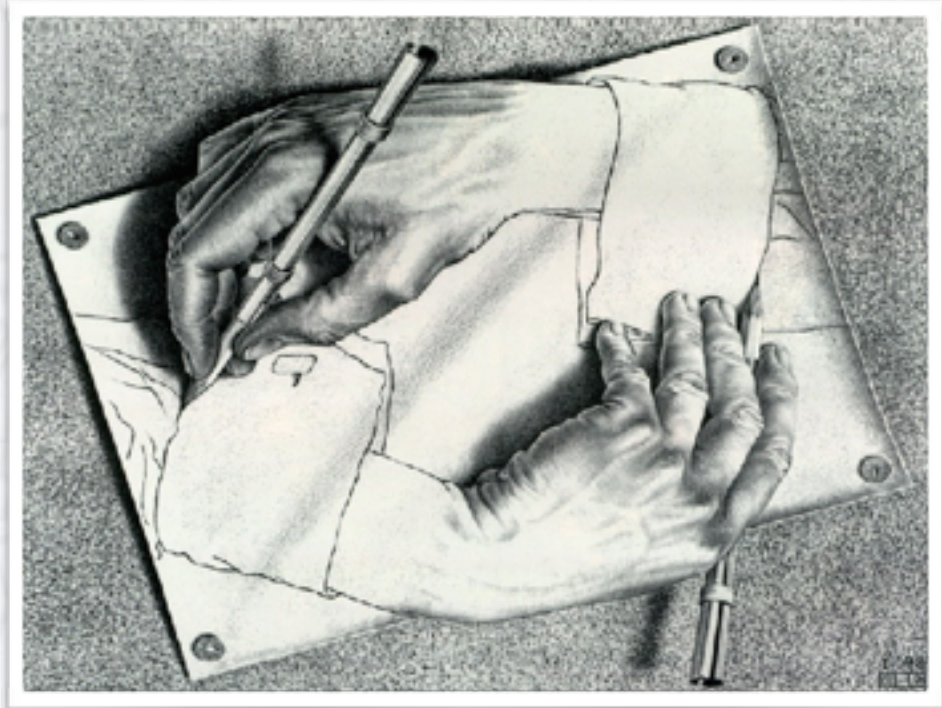
(III-III)

FROM COMPETITION TO COOPERATION

Instead of competing, networks may be interested in collaborating…

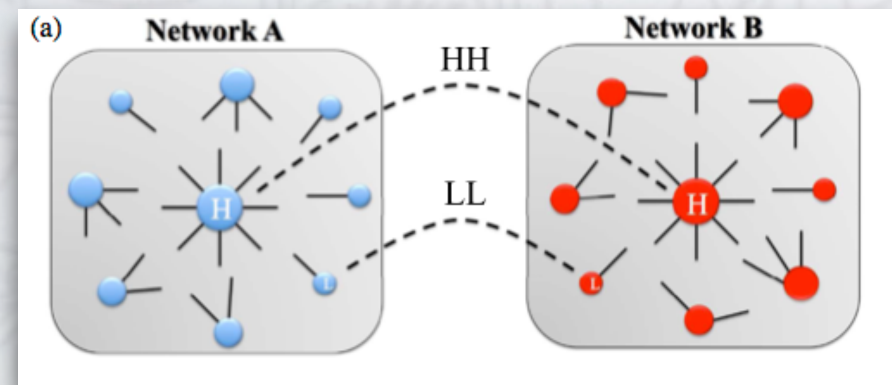
COMPETITION

COOPERATION

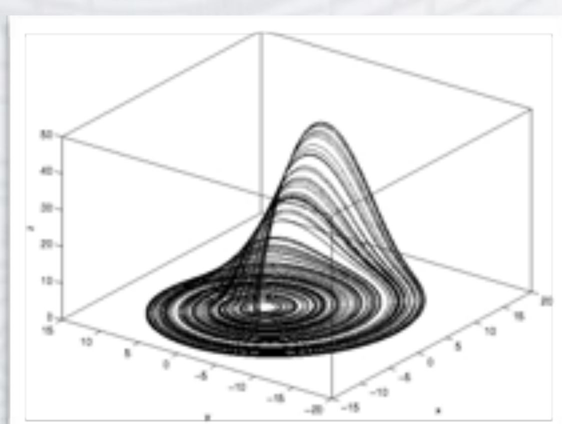


NETWORKS COOPERATING TO ACHIEVE COMPLETE SYNCHRONIZATION

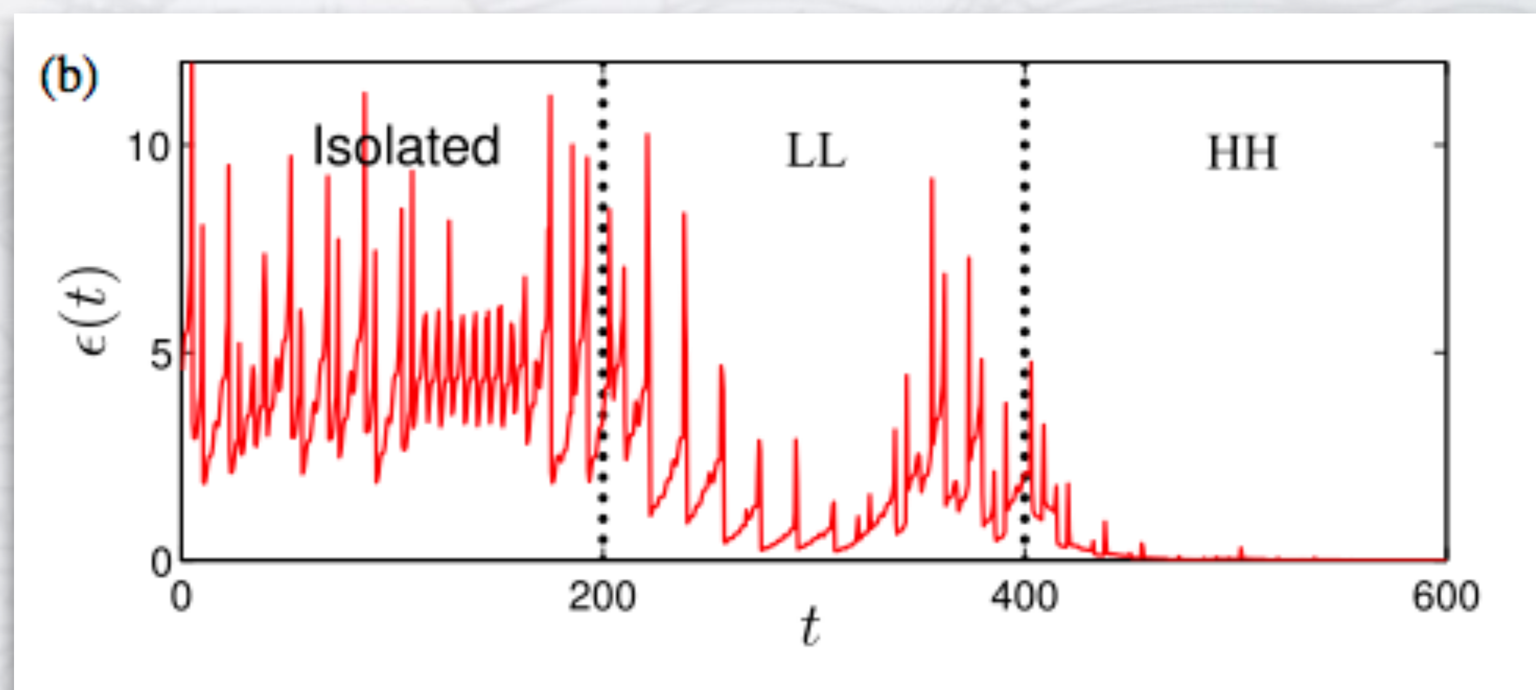
Suppose we are two networks, what is our best connection strategy to achieve complete synchronization?



Schematic representation of the different strategies for creating connection paths between two undirected networks. **High-degree nodes (H) and Low-degree nodes (L)**.



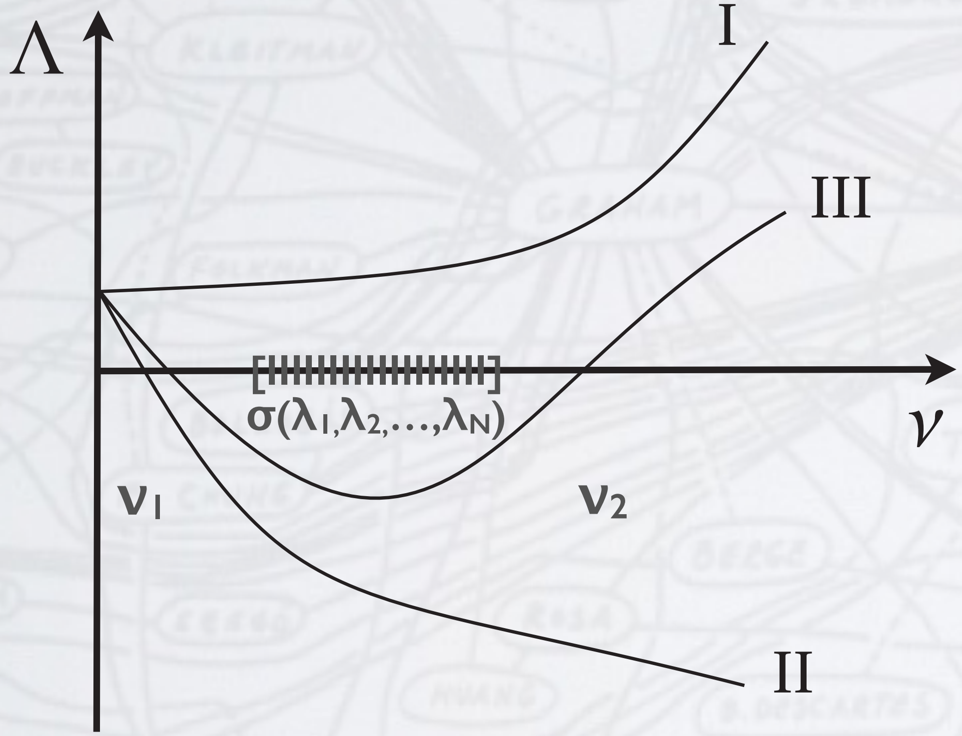
Synchronization error **ε**(t) of two interconnected Barabási-Albert networks of N=200 Rössler oscillators at three different stages: isolated, interconnected following a LL strategy, and replacing the LL connection with a HH one.



COMMON OBJECTIVE: COMPLETE SYNCHRONIZATION
IN DIFFUSIVELY COUPLED IDENTICAL SYSTEMS

The Master Stability Function* (MSF) is a tool to evaluate the **stability of the synchronized state of diffusively coupled dynamical systems**:

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) + \sigma \sum_{i=1}^{N} a_{ij} w_{ij} [\mathbf{H}(\mathbf{x}_j) - \mathbf{H}(\mathbf{x}_i)] = \mathbf{F}(\mathbf{x}_i) - \sigma \sum_{j=1}^{N} G_{ij} \mathbf{H}(\mathbf{x}_j)$$



ν is related with σλi where σ is the coupling strength and λi are the eigenvalues of the Laplacian matrix (G=S-W) and $\lambda_1 < \lambda_2 < \ldots < \lambda_N$.

Class I system: Not synchronizable

Class II system: $\sigma \lambda_2 > \nu_1$

↓

(the higher, the better)

Class III system: $\sigma \lambda_2 > \nu_1$

$\sigma \lambda_N < \nu_2$

↓

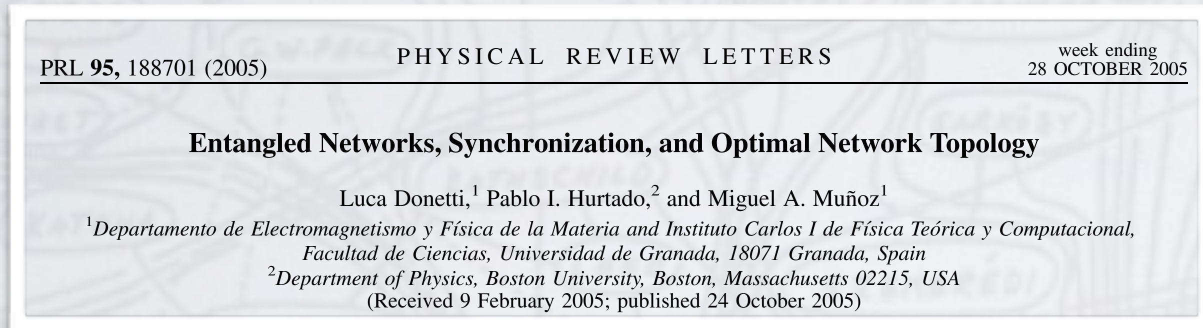
$r = \lambda_N / \lambda_2$

(the lower, the better)

*** Pecora & Carroll, PRL 1998**

STRATEGY FOR A NETWORK IN ISOLATION

If we are considering a **single network**, the best strategy is to connect peripheral nodes and to disconnect central nodes:


PRL 95, 188701 (2005) PHYSICAL REVIEW LETTERS week ending 28 OCTOBER 2005

Entangled Networks, Synchronization, and Optimal Network Topology

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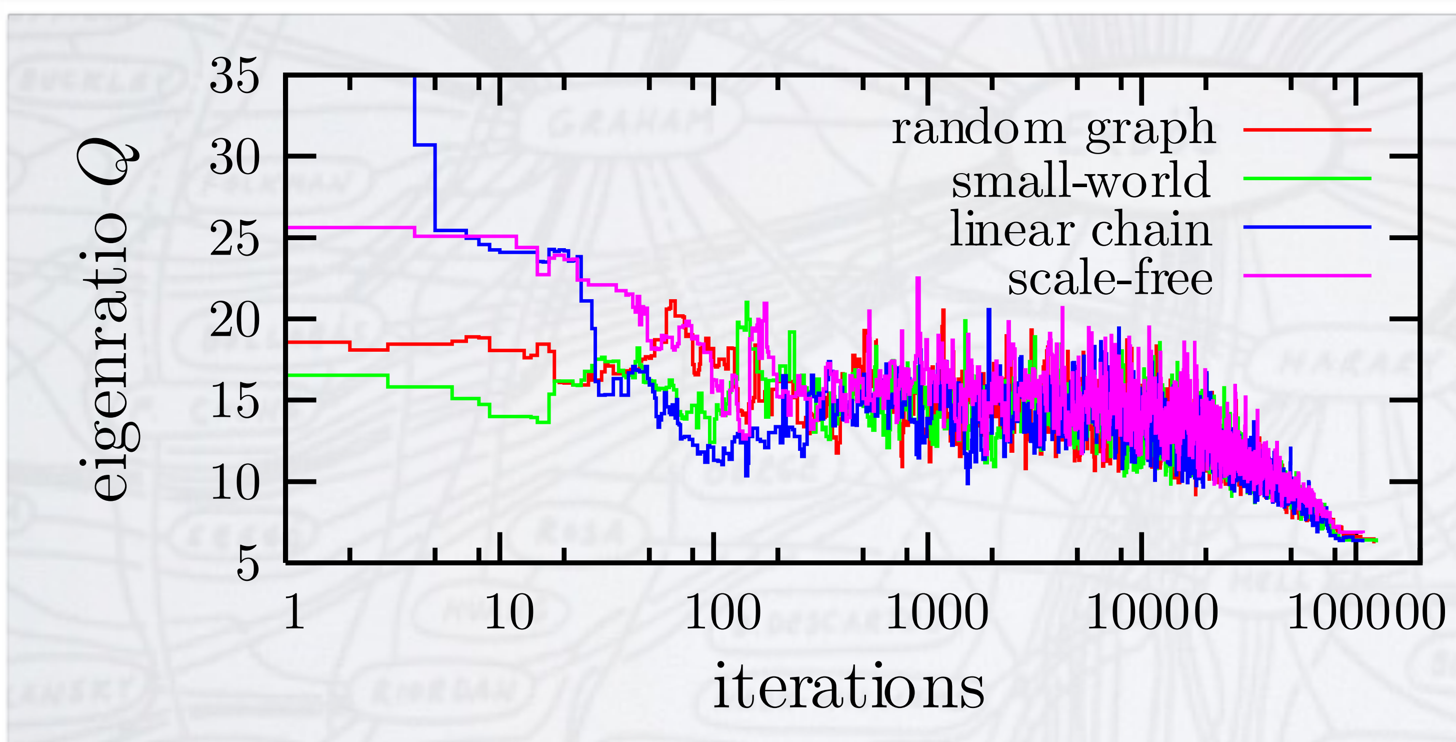


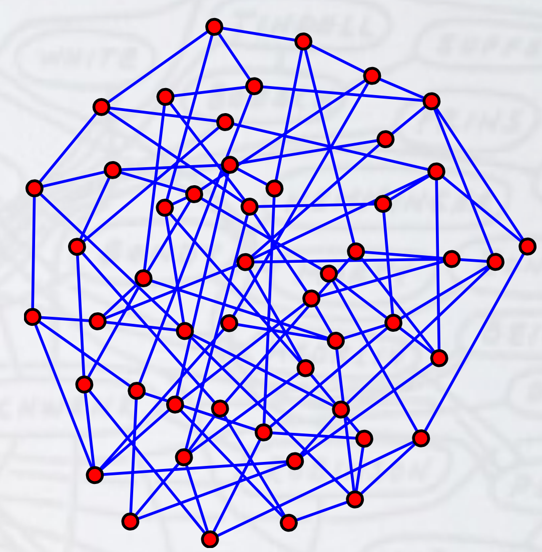
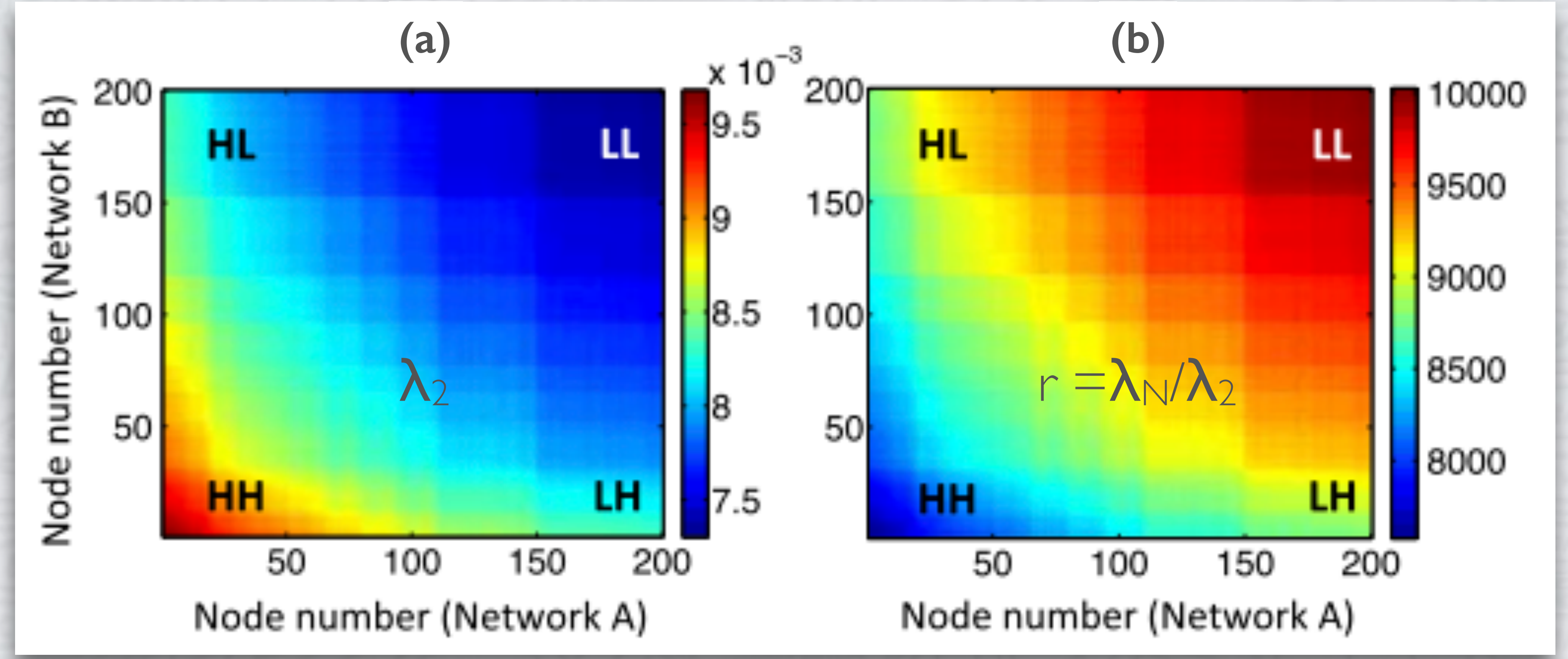


FIG. 1 (color online). Eigenvalue ratio Q as a function of the number of algorithmic iterations. Starting from different initial conditions, with $N = 50$, and $\langle k \rangle = 4$, the algorithm converges to networks, as the depicted one (b), with very similar values of Q.

Networks reorganizing to enhance synchronization (i.e., to minimize the eigenratio Q): no matter the initial structure, they become more homogeneous.

NETWORKS COOPERATING TO ACHIEVE COMPLETE SYNCHRONIZATION

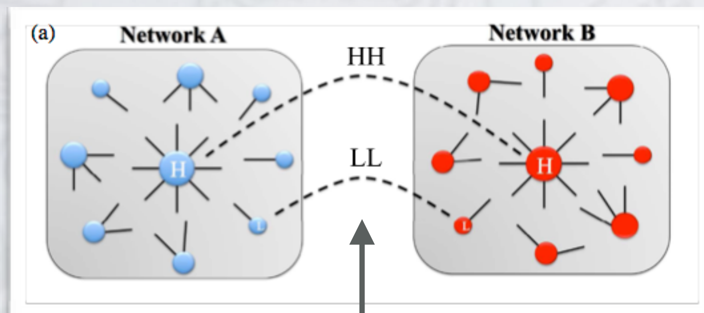
Should we connect or disconnect the hubs?



(a) λ_2 of the network-of-networks obtained from connecting two Barabási-Albert networks (N=200) with one interlink, in all possible configurations. The node numbers are ordered according to the node degree and, when coinciding, the eigenvector centrality. (b) Eigenratio $r = \lambda_N / \lambda_2$ for the same case as (a).

NETWORKS COOPERATING BETWEEN THEM

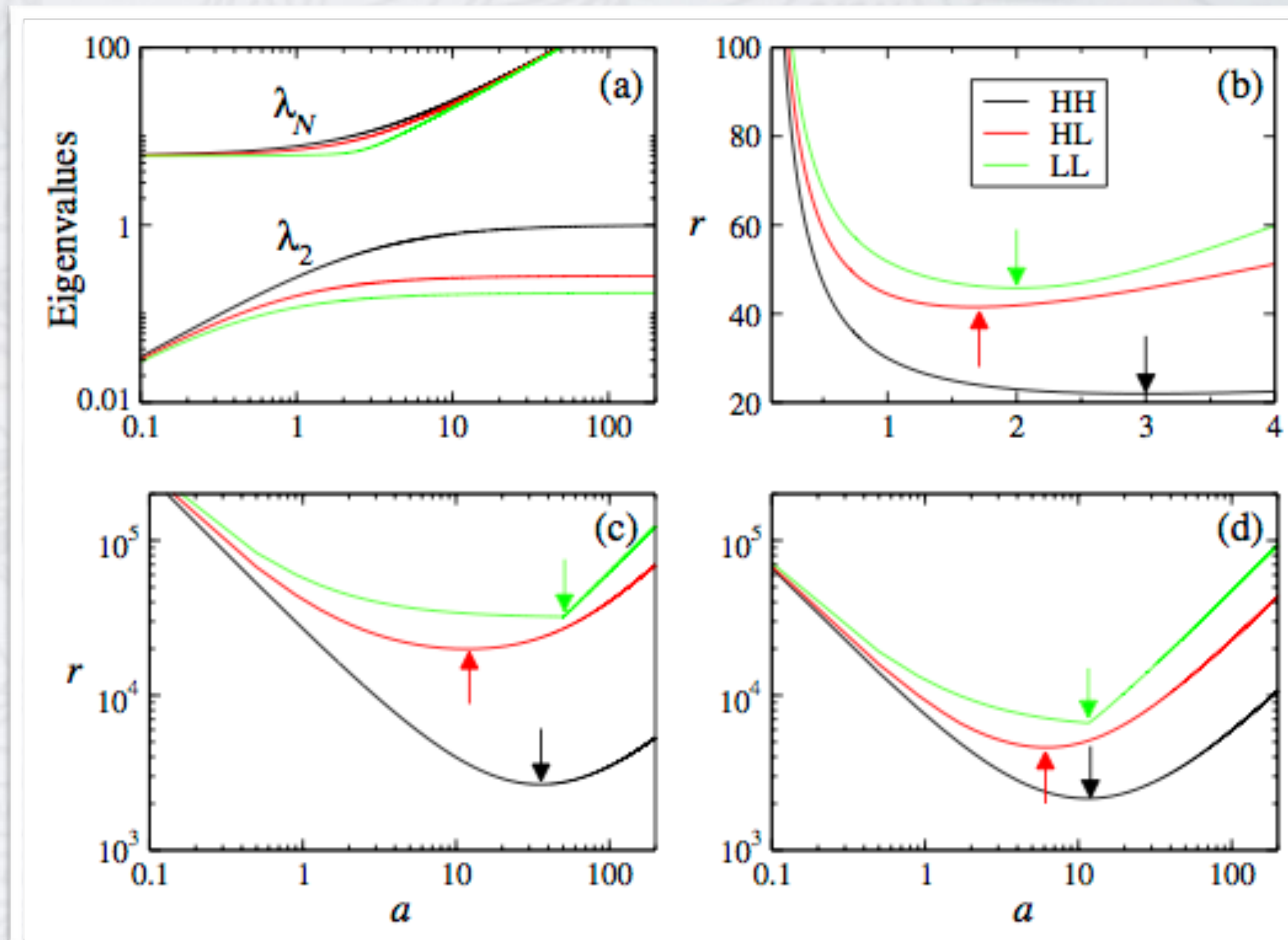
The strategy influences the ability to synchronize but also depends on the transition from a network-of-networks to a "single network"



a

(coupling strength)

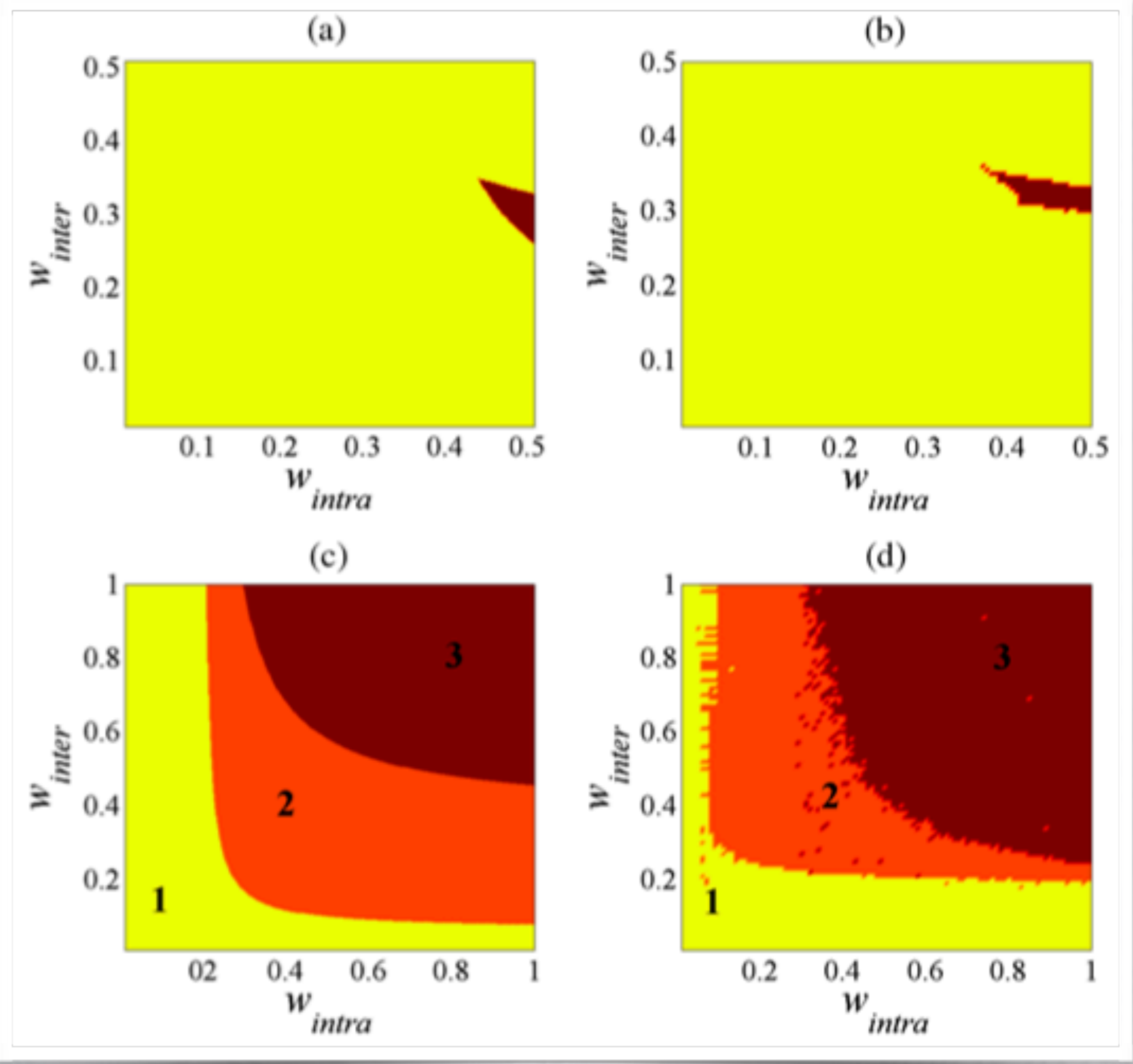
FIG. 2 (color online). Synchronizability for two networks connected by a single interlink of weight a. (a) λ_2 and λ_N for two star networks of 6 nodes each. (b),(c),(d) Eigenratio r for (b) two star networks ($N = 6$), (c) two scale-free networks ($N = 500$), and (d) two Erdős-Rényi random networks ($N = 500$). Three connecting strategies are shown: HH (black), HL (red), and LL (green). The minima of the curves (arrows) correspond to maximum synchronizability [34]. Plots (a)–(b) were obtained analytically and (c)–(d) numerically.



High-degree nodes (H) and Low-degree nodes (L).

COOPERATION IN REAL SYSTEMS: ELECTRONIC CIRCUITS

Can we translate these conclusions to real systems?



Theoretical predictions hold on in experiments where a certain parameter mismatch exists.

FIG. 4 (color online). Experimental verification of the phenomenology presented here. (a) and (b) show the regions of complete synchronization of two star networks of type III Rössler systems coupled by a HH strategy. Neither HL (LH) nor LL strategies lead to synchronization, as predicted by the theory and confirmed by the experiments (not shown here). (c) and (d) depict class II Rössler systems. Regions correspond to (1) no synchronization, (2) complete synchronization with the HH strategy, and (3) complete synchronization with the HH and the LL strategies. Results are theoretical [(a) and (c)] and experimental [(b) and (d)]. The zeroes of the MSF are $\nu_1 = 0.107$ and $\nu_2 = 2.863$ for class III and $\nu_c = 0.0651$ for class II.

TAKE HOME MESSAGE

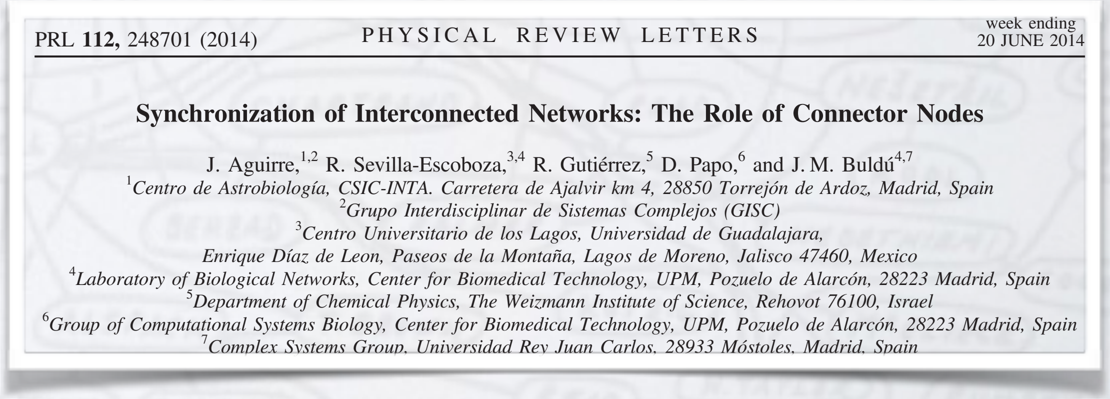
Just one and simple message…

… **be aware of the way networks are connected between them!**


nature physics
LETTERS
PUBLISHED ONLINE: 24 FEBRUARY 2013 | DOI: 10.1038/NPHYS2556

Successful strategies for competing networks

J. Aguirre[1]*, D. Papo[2] and J. M. Buldú[2,3]*


PRL 112, 248701 (2014) PHYSICAL REVIEW LETTERS week ending 20 JUNE 2014

Synchronization of Interconnected Networks: The Role of Connector Nodes

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