(what I think I have understood so far about) Random walks on networks

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Sept. 30^{th} 2015

Outline

1. Introduction and definitions
2. Plain random walks
3. Variations on the theme
4. Using random walks

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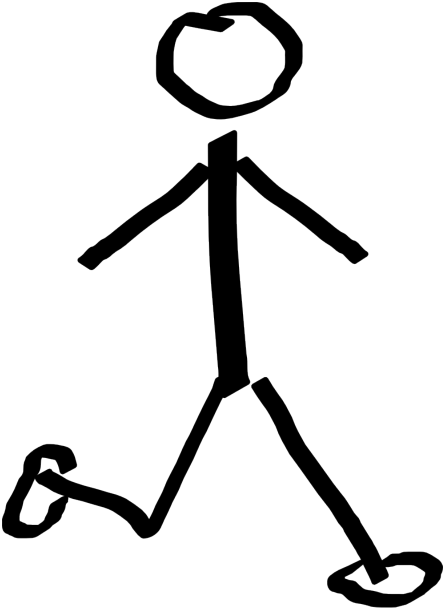

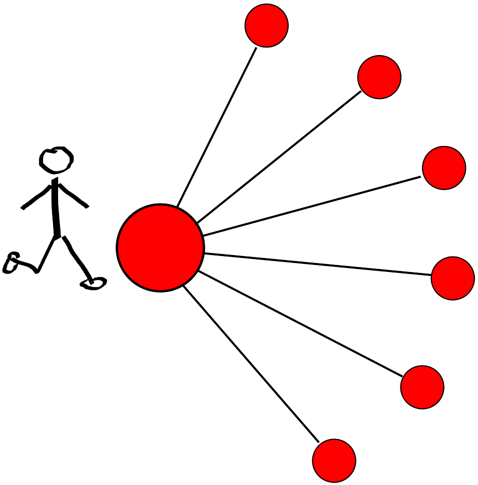

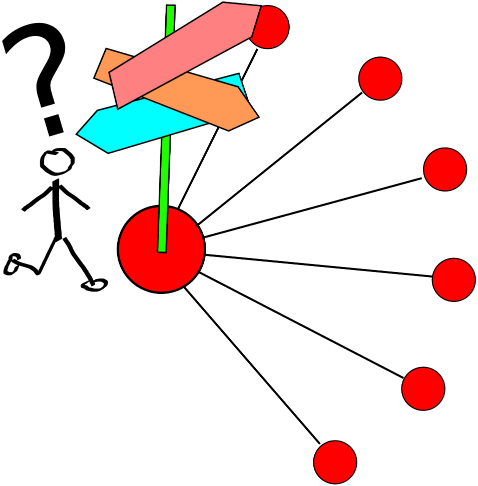

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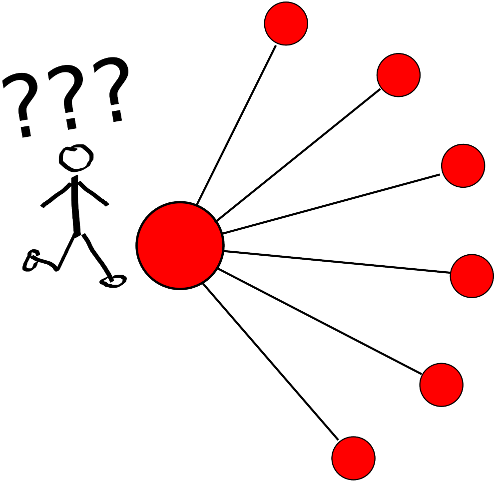
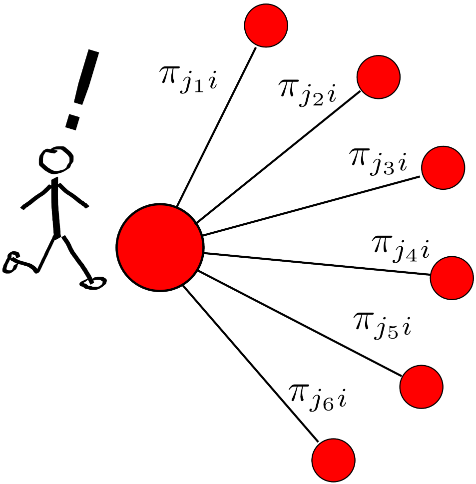
- π_{ji}: probability for a walker on node i to “jump” on node j in one time step
- $$\sum_j \pi_{ji} = 1$$




!
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$\pi_{j_2 i}$
$\pi_{j_3 i}$
$\pi_{j_4 i}$
$\pi_{j_5 i}$
$\pi_{j_6 i}$

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- Occupation probability: $p_i(t)$ is the probability of finding a walker on node i at time t

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A random walk on a graph $G(V, E)$ is a Markov chain defined by the transition matrix Π on the state space V.

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- τ-step evolution:

$$P(t+\tau) = \Pi P(t+\tau-1) = \Pi^2 P(t+\tau-2) = \ldots = \Pi^\tau P(t)$$

Question: If we know that the walker was at node i at time 0, where can we find it after t time steps??

Stationary occupation probability

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- $$P(t+1) = \Pi P(t) \tag{1}$$
- Does the limit:
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  exist?
- Can we find a P^* such that $P^* = \Pi P^*$, i.e., a fixed point for the dynamics of Eq. (1)?

Perron-Frobenius theorem...

Given a non-negative irreducible (aperiodic) matrix $M = \{m_{ij}\}$

- The largest eigenvalue $\lambda_{\max}$ of M (in modulus) is real and positive
- $\lambda_{\max}$ is simple
- The eigenvector associated to $\lambda_{\max}$ is the only positive eigenvector of M

...plus the power method...

- If M is a non-negative irreducible aperiodic matrix, then the sequence of vectors

$$x(t+1) = Mx(t)$$

converges to a vector $\tilde{x}$ which is parallel to the eigenvector associated to the largest eigenvalue of M

...give the answer

- $$P(t+1) = \Pi P(t) \tag{2}$$

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- If Π is irreducible and aperiodic (G is connected and contains one odd cycle)
- $\Longrightarrow$
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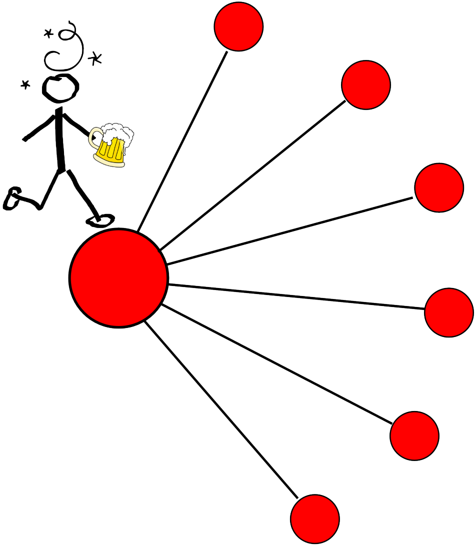
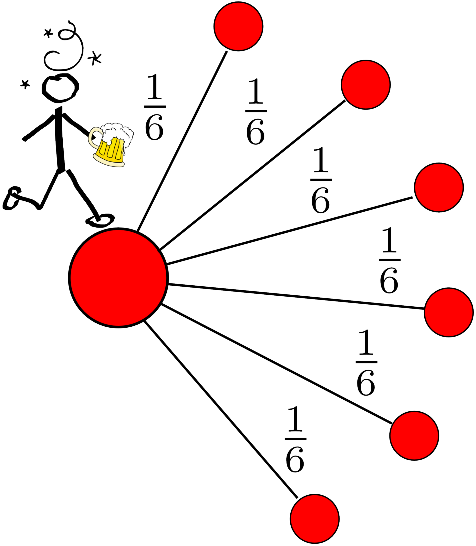
  exists and is equal to the first eigenvector of Π
- which is positive and is called the stationary occupation probability distribution associated to Π

Outline

Simplest example: "The Drunken"

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The drunken equation (Plain random walk)

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- Stationary probability distribution:

$$p_i^* = \frac{k_i}{2K}$$

P^* for plain random walks

- Probability of going from i to j in t steps:

$$W_{i\to j}(t) = \sum_{j_1,j_2,\ldots,j_{t-1}} \pi_{j_1,j} \times \pi_{j_2,j_1} \times \ldots \times \pi_{j,j_{t-1}}$$

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By comparing $W_{i\to j}$ and $W_{j\to i}$ (undirected graphs):

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and by imposing the normalisation condition $\sum_j p_j^* = 1$ we get:

$$\sum_j p_j^* k_i = \sum_j p_i^* k_j \quad \Rightarrow \quad p_i^* = \frac{k_i}{2K}$$

Let's time our walker!

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- Coverage time: average time needed for a walker to visit all the nodes of the graph

Interesting facts

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- Coverage time: I am sure you don't want to know!

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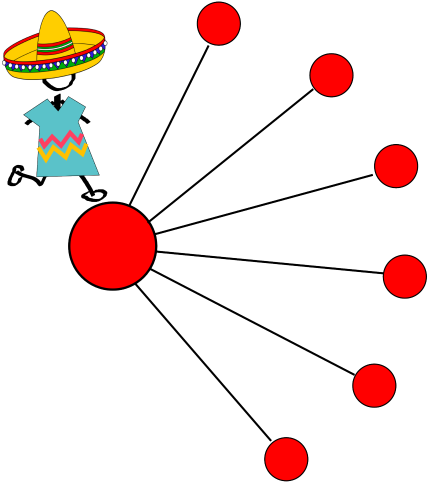
First Problem

G is not primitive $\Longrightarrow$ no stationary occupation probability (limit cycles)

The Lazy

Lazy random walk



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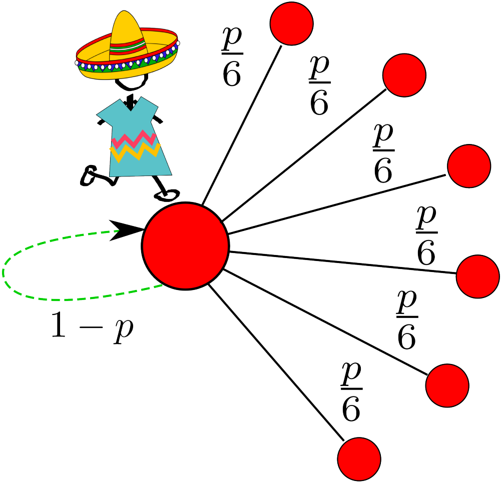

Moves with probability p

Remains still with probability

$1-p$

The “Lazy” equation (lazy RW)

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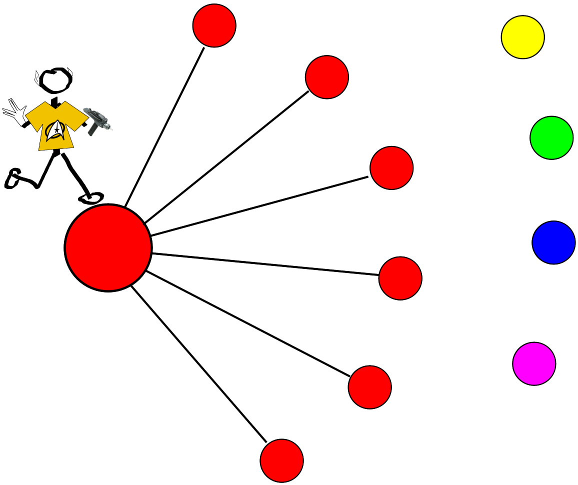
- $$P(t+1) = ((1-p)I + p\Pi)P(t)$$
- If G is connected then $M = (1-p)I + p\Pi$ is primitive (degenerate odd-length cycles)

Second problem:

G is directed $\Longrightarrow$ walkers will remain trapped on nodes having $k_{out} = 0$ and nodes with $k_{in} = 0$ will never be visited

The "Smart"



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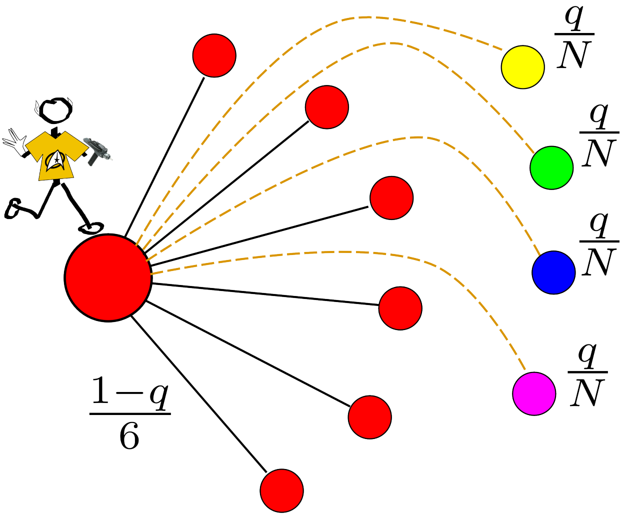

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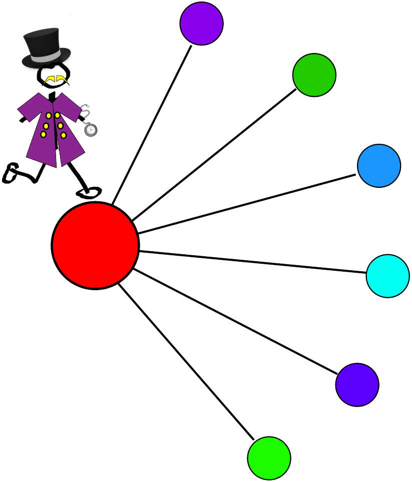
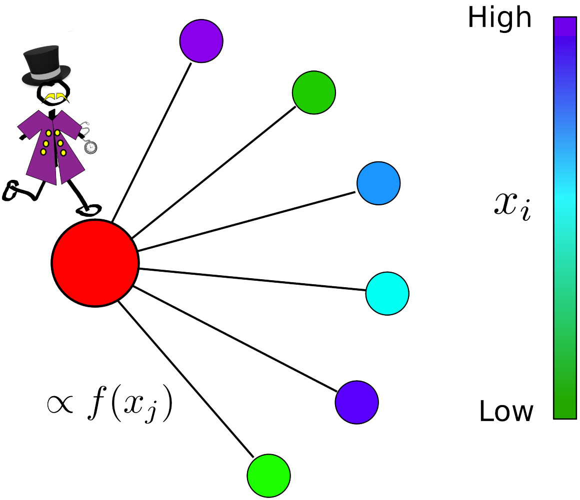
Third problem:

In plain walks $p_i^* \propto k_i$ (WWW)

The "Snob"




$\propto f(x_j)$
High
x_i
Low

The "Snob" equation (Biased RW)

- Transition probability:

$$\pi_{ji} = \frac{a_{ij} f_j}{\sum_\ell a_{i\ell} f_\ell}$$

where $f_i = f(x_i)$

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- Stationary probability distribution:

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where:

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$$c_i = \sum_\ell a_{i\ell} f_\ell$$

- Entropy rate:

$$h = \frac{\sum_i f_i \sum_j a_{ij} f_j \log(f_j) - \sum_i f_i c_i \log(c_i)}{\sum_i c_i f_i}$$

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- $\Longrightarrow$ p^* is determined by degree-degree correlations!

Interesting facts

- If G has no degree correlations:

$$p_i^* = \frac{k_i^{\alpha+1}}{N\langle k^{\alpha+1}\rangle}$$

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  Return time:

$$r_i = N$$

  The mean first-passage time T is minimal

Outline

1. Introduction and definitions
2. Plain random walks
3. Variations on the theme
4. Using random walks

The Explorer

Random walks and modularity

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- If we consider $t = \Delta\tau = 1$ (discrete-time random walk), then we have:

$$R(1) = \sum_{i,j} \frac{a_{ij}}{k_i}\frac{k_i}{2K}\delta_{c_i,c_j} - p_i^* p_j^* \delta_{c_i,c_j} = \frac{1}{2K}\sum_{i,j}\left(a_{ij} - \frac{k_i k_j}{2K}\right)\delta_{c_i,c_j}$$

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- So the modularity of a partition is just the one-step stability of a plain random walk on the graph with that partition

The Engineer

Node centrality

Problem: how we measure the relative importance of nodes?

PageRank

- The "Smart" transition matrix (RW with teleportation):

$$\Phi(q) = \frac{q}{N}\mathbf{1}\,\mathbf{1} + (1-q)\Pi$$

- $\Phi(q)$ satisfies all the hypotheses of Perron-Frobenius
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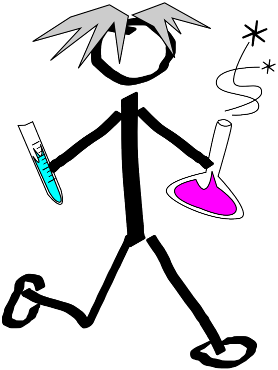
- This eigenvector is used to compute the PageRank score of a node

The Alchemist



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- Look at the typical fluctuations of H_i in an interval of length ε and call them $F(\varepsilon)$
- Look at how $F(\varepsilon)$ scales with the size ε (Detrended fluctuation analysis)

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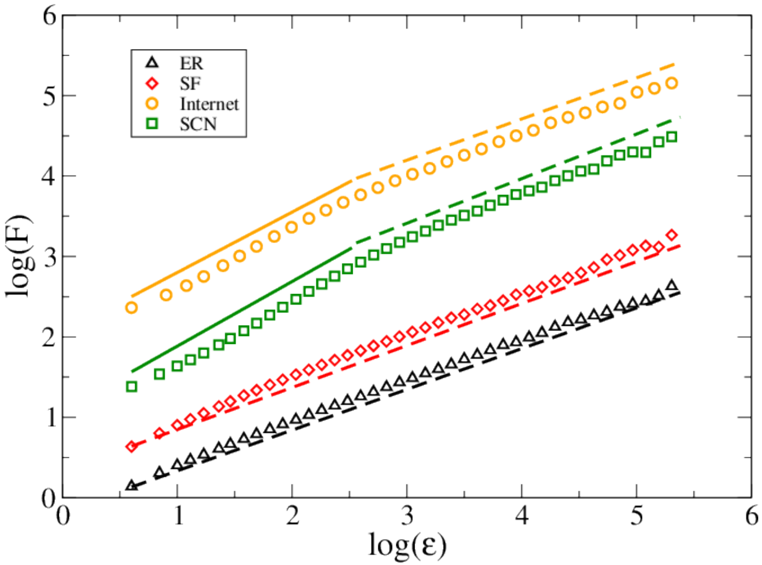
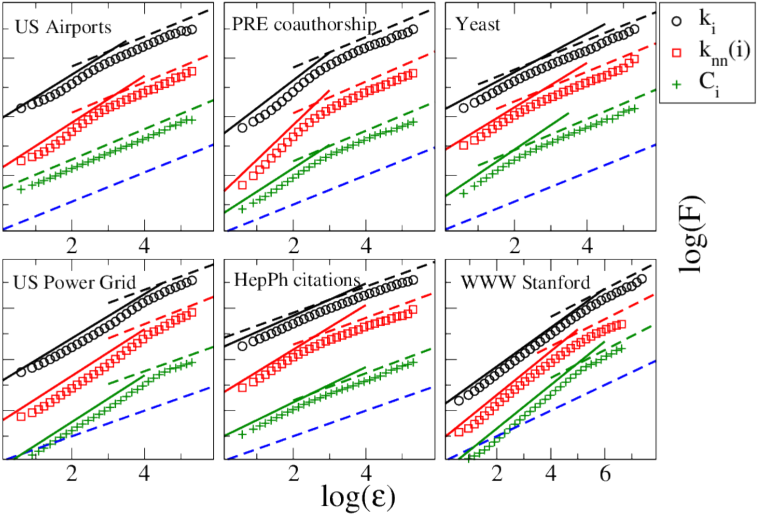
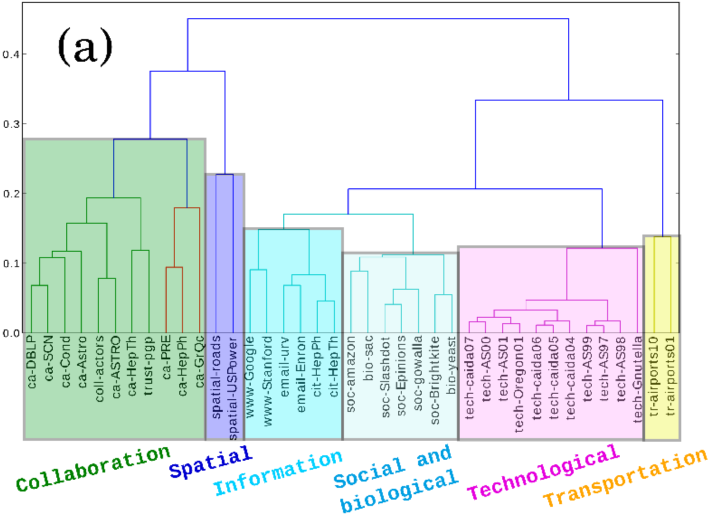



US Airports
PRE coauthorship
Yeast
k_i
$k_{nn}(i)$
C_i
US Power Grid
HepPh citations
WWW Stanford
2
4
6
log(F)
log(ε)

Taxonomy


(a)
0.4
0.3
0.2
0.1
0.0
ca-DBLP
ca-SCN
ca-Cond
ca-Astro
coll-actors
ca-ASTRO
ca-HepTh
trust-pgp
ca-PRE
ca-HepPh
ca-GrQc
spatial-roads
spatial-USPower
www-Google
www-Stanford
email-urv
email-Enron
cit-HepPh
cit-HepTh
soc-amazon
bio-sac
soc-Slashdot
soc-Epinions
soc-gowalla
soc-Brightkite
bio-yeast
tech-caida07
tech-AS00
tech-AS01
tech-Oregon01
tech-caida06
tech-caida05
tech-caida04
tech-AS99
tech-AS97
tech-AS98
tech-Gnutella
tr-airports10
tr-airports01
Collaboration
Spatial
Information
Social and biological
Technological
Transportation

Take-home message

Random walks at 9:00 am might be boring...

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...but are a nice process to understand the structure and dynamics of complex networks ;)

THANK YOU!

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