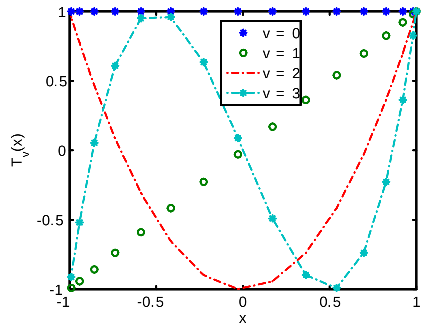

FIG. 1: Plots of the Chebyshev polynomials $T_v(x)$. The symbols denote the values at the 14 mesh-points ξ_k, which are the zero's of T_{14}.They are not eqi-spaced.

I. LECTURE 5, CHEBYSHEV POLYNOMIALS

Chebyshev Polynomials $T_v(x)$, $v = 0, 1, 2...$ provide a very useful set of basis functions for expansion purposes. The variable x is contained in the interval $-1 \rightarrow +1$, and is related to an angle θ by $x = \cos\theta$. The last equation shows that the $x's$ are projections on the $x-$axis of the tip of a radius vector of unit length that describes a semi-circle as θ goes from 0 to π. In terms of the $x-$variable the T_n's are given by

$$\begin{aligned} T_0 &= 1 \\ T_1 &= x \\ T_2 &= 2x^2 - 1 \\ T_{n+1} &= 2xT_n - T_{n-1} \end{aligned} \tag{1}$$

In terms of the θ variable they are given by

$$T_n = \cos(n\ \theta); \quad 0 \leq \theta \leq \pi. \tag{2}$$

It is clear from Eq.(2) that $-1 \leq T_n(x) \leq 1$, and that the larger the index n, the more zeros these polynomials have, as is shown in Fig 1 The T_n's are orthogonal to each other with the weight function $\rho(x) = (1 - x^2)^{-1/2}$. The integral $\mathcal{I}$

$$\mathcal{I}_{n,m} = \int_{-1}^{+1} T_n(x)\ T_m(x)\ (1 - x^2)^{-1/2}\ dx = \int_0^\pi\ \cos(n\theta)\ \cos(m\theta)\ d\theta \tag{3}$$

has the value 0 if $n \neq m$, and the values $\pi/2$ if $n = m \neq 0$ and π if $n = m = 0$.

They also obey a discrete orthogonality

$$\frac{\pi}{N} \sum_{k=1}^{N} T_n(\xi_k)\ T_m(\xi_k) = \frac{\pi}{2} \delta_{n.m}(1 + \delta_{0n}) \quad n < N,\ m < N \tag{4}$$

where the ξ_k are the zero's of T_N, given by

$$\xi_k = \cos[\frac{\pi}{N}(k - 1/2)], \quad k = 1, 2, ...N. \tag{5}$$

This set of support (or mesh) -points are the ones used by Deloff [2] and also in our own work, because they do not attain the values ± 1, thus avoiding possible singularities in the functions being calculated at these points. By contrast, the support points used by Trefethen [1] are

$$x_k = \cos[\frac{\pi}{N}k], \quad k = 0, 1, 2, ...N. \tag{6}$$

They do include the points ± 1, and are especially useful to construct differentiation matrices for functions that are not periodic, because these functions can be made periodic by extending the angles of the support points (previously defined in $0 \leq \theta \leq \pi$) all the way around the circle from $\pi \leq \theta \leq 2\pi$. Since at the mesh-points the angles are equi-spaced, Fourier series and FTT's can be calculated with this variable, another useful property.

The Chebyshev polynomials can be calculated as a function of x by either using Eq. (2), with $\theta = \arccos(x)$, or by means of the recursion relations

$$T_n(x) = 2x\ T_{n-1}(x) - T_{n-2}(x), \quad n \geq 2 \tag{7}$$

As can be seen from Fig. 1, at the end points their values are

$$T_n(1) = 1, \text{ and } T_n(-1) = (-)^n \text{ for all } n \tag{8}$$

Many additional properties are given in text books [4].

Assignment 6 a):

1.Start from an equispaced discrete set of angles θ contained between 0 and π. Use the symbols $'*'$ or something similar for your discrete points in the graphs below.

2. Calculate the corresponding set of $x-$values, and plot x versus θ.

3. Calculate the values of $T_n(x)$ for $n = 0, 1, 2,$ and 3, and plot them as a function of x

4. Calculate the values of $T_n(x)$ for $n = 0, 1, 2,$ and 3, and plot them as a function of θ

5. Check whether the forward recursion relation in Eq.(7) can be trusted, by comparing the result with Eq.(2)

II. EXPANSION OF FUNCTIONS INTO A SERIES OF CHEBYSHEV POLYNOMIALS

Given a function $f(x)$, it is desired to expand it in terms of T_n, $n = 1, 2, ..N$,

$$f^{(N)}(x) = \sum_{j=1}^{N} a_j\ T_{j-1}(x), \quad -1 \leq x \leq 1 \tag{9}$$

and examine the rapidity of the convergence with N. We will find that the rapidity of the convergence depends on the "smoothness" of f, in particular how many singularities this function or its derivatives can have. Theorems exist about these properties, as well as the uniformity of the convergence in the x variable, some of which we will see further on.

There are various ways of obtaining the coefficiens a_j. One way is to multiply Eq. (9) on the left with $T_n(x)$, and by making use of the orthogonality (3), obtain each a_n in terms of the integral $\int_{-1}^{+1} f(x)T_n(x)(1-x^2)^{-1/2}dx$. However, this quadrature can be avoided by making use of the discrete orthogonality (4). All what is needed are the values of the function f at the zeros of T_N, given byEq. (5), and performing the sum

$$\sum_{k=1}^{N} T_n(\xi_k) f(\xi_k) = (1+\delta_{0n})\frac{N}{2}a_{n+1}, \quad n = 0, 1, ..N-1. \tag{10}$$

This results in a matrix relation between the column vectors $[f]$ and $[a]$

$$[f] = (f(\xi_1), f(\xi_2), ...f(\xi_N))^T \text{and } [a] = (a_1, a_2, ...a_N)^T \tag{11}$$

(here T means "transposed"), of the form

$$[f] = C\ [a] \tag{12}$$

and

$$[a] = C^{-1}[f]. \tag{13}$$

Matrix C is a non-sparse $N \times N$ matrix that contains rows of columns of Chebyshevs $T_0, T_1, ...T_{N-1}$evaluated at $\xi_1, \xi_2, ...\xi_N$. None of the $T's$ vanish at these points, because they are the zeros of T_N, and none of the entries depend on the function f, but only on the value of N. Please note that all the values of the $f(\xi$'s) are involved in obtaining each value of a_k. In other words, relationships (12) and (13) are "spectral". Values of C and C^{-1} are written in our earliest paper [3], are availabe in my MATLAB programs, and are made use of routinely in our numerical calculations. Further, the inverse C^{-1} is given in terms of T_n's as a result of Eq. (4, i.e., no matrix inversion is required. The relationship (12) between $[f]$ and $[a]$ can also be obtained [5] by writing Eq. (9) repeatedly for all values of $x = \xi_k,\ k = 1, 2, ..N$, thus obtaining a set of N linear equations between $f(\xi_k)$'s and the a's. The same procedure can also be applied to expansions in terms of other basis functions. The important question is how fast will the error $|f(x) - f^{(N)}(x)|$ decrease with N ?Numerical examples will be given below. Interpolation of a function known only at the discrete ξ_k points to all values of x is obtained by means of Eq. (13), that can be written as

$$f^{(N)}(x) = \sum_{i=1}^{N}\sum_{k=1}^{N} T_{i-1}(x)(C^{-1})_{i,k} f(\xi_k). \tag{14}$$

Another important matrix relation applies to obtaining indefinite integrals.

$$F_L(t) = \int_{-1}^{t} f(x)dx \quad \text{and} \quad F_R(t) \int_{t}^{1} f(x)dx. \tag{15}$$

Here the subscripts R and L stand for "left" and "right", repectively. From the expansion coefficients $[a]$ of f, one can obtain the expansion coefficients $b^{(L),(R)}$ of the functions $F_{L,R}(t)$

$$F_L^{(N)}(t) = \sum_{k=1}^{N} b_k^{(L)} T_{k-1}(t) \text{ and } F_R^{(N)}(t) = \sum_{k=1}^{N} b_k^{(R)} T_{k-1}(t) \tag{16}$$

from the matrix relations

$$[b^{(L)}] = S_L^{(N)}[a] \ \text{ and } [b^{(R)}] = S_R^{(N)}[a]. \tag{17a}$$

Thus, $[F_L^{(N)}] = C(N)S_L(N)C^{-1}(N)[f]$ is the column vector of the values of $F_L^{(N)}$ evaluated at the points ξ_k, and the interpolation to all other points can be obtained from Eq. (16), that can be written in the fom

$$F_L^{(N)}(x) = T_{i-1}(x)(S_L)_{i,j}(C^{-1})_{j,k} f(\xi_k), \tag{18}$$

where sums over all repeated indices i, j, and k are carried out from $1, 2, ..N$. The same results apply to the "R" integrals. The integrals (18), to be called "Gauss-Chebyshev" integrals, have the same accuracy as the expansions (9), as will be shown below.

Assignment 6 b

1. Consider the function

$$f(r) = r^{1/2} \sin(r) \tag{19}$$

for r defined from 0 to π. Define a new variable x that goes from -1 to $+1$ and relate it to the variable r by means of the linear transformation

$$r = a\ x + b \tag{20}$$

and find the values of a and b. Call the new function $\bar{f}(x) = f(r)$

2. In preparation to expanding this function into Chebyshev polynomials

$$\bar{f}(x) = \sum_{k=1}^{N} a_k\ T_{k-1}(x) \tag{21}$$

choose a value of $N = 8$, and find the zeros ξ_i of $T_N(x)$ with $k = 1, 2, ..N$ Use the expressions given in class

$$\xi_k = \cos[\frac{\pi}{N}(k - 1/2)], \quad k = 1, 2, ...N. \tag{22}$$

and check that $T_N(\xi_k) = 0$ for all values of k

3. Use the MATLAB functions $[C, CM1, xz] = C\_CM1(N)$ given in class, and check wether the output vector xz agrees with the vector of the ξ_k values obtained in 2). Note, xz is a column vector

4. For each of the ξ_i values obtain the corresponding $r_k$values. Use the function $r2 = mapxtor(b1, b2, xz)$ with $b1 = 0$ and $b2 = \pi$, where xz was obtained in part 3) to obtain $r2$ and check wether the values of r_k and $r2$ agree. Note thar $r2$ should be a column vector.

5. Obtain the column vector $F = f(r2)$ and calculate the column vector A by means of the matrix*column vector MATLAB operation

$$A = CM1 * F \tag{23}$$

The $A-$vector contains the coefficients a_n of the expansion (21),.and check how fast the coefficients a_n decrease with n.

6. Repeat parts 3, 4, and 5 for $N = 16$

7. Define a new function

$$g(r) = r\ \sin(r) \tag{24}$$

and repeat parts 3, 4, and 5 with $N = 16$. Check that the new expansion coefficients a_n decreas much faster with n than for the expansion of $f(r)$.

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