Lecture 5

Summary of Lecture 4

Chebyshev Polynomials $T_v(x),\ v = 0, 1, 2...$ were described, their properties were illustrated by means of their plots in the $[-1 \leq x \leq 1]$ domain, and a connection to an angle θ was given in terms of $x = \cos\theta$. The last equation shows that the $x's$ are projections on the $x-$axis of the tip of a radius vector of unit length that describes a semi-circle as θ goes from 0 to π. In terms of the θ variable the T_n's are given by

$$T_n = \cos(n\ \theta); \quad 0 \leq \theta \leq \pi, \quad n = 0, 1, 2, ... \tag{1}$$

It is clear from Eq.(1) that $-1 \leq T_n(x) \leq 1$, and that the larger the index n, the more zeros these polynomials have. For $x = 1$ all the $T_n(x) = 1$, and for $x = -1$, $T_n(x) = (-1)^n$. The T_n's are orthogonal to each other with the weight function $\rho(x) = (1 - x^2)^{-1/2}$. The integral $\mathcal{I}$

$$\mathcal{I}_{n,m} = \int_{-1}^{+1} T_n(x)\ T_m(x)\ (1 - x^2)^{-1/2}\ dx = \int_0^{\pi} \cos(n\theta)\ \cos(m\theta)\ d\theta \tag{2}$$

has the value 0 if $n \neq m$, and the values $\pi/2$ if $n = m \neq 0$ and π if $n = m = 0$.

They also obey a discrete orthogonality

$$\frac{\pi}{N}\sum_{k=1}^{N} T_n(\xi_k)\ T_m(\xi_k) = \frac{\pi}{2}\delta_{n.m}(1 + \delta_{0n}) \quad n < N,\ m < N \tag{3}$$

where the support points ξ_k are the zero's of T_N, given by

$$\xi_k = \cos[\frac{\pi}{N}(k - 1/2)], \quad k = 1, 2, ...N. \tag{4}$$

Note that in contrast to the zeros of Legendre or Laguerre polynomials, the support points (4) can be obtained analytically.

The Chebyshev polynomials can be calculated as a function of x by either using Eq. (1), with $\theta = \arccos(x)$, or by means of the recursion relations

$$T_n(x) = 2x\ T_{n-1}(x) - T_{n-2}(x), \quad n \geq 2 \tag{5}$$

Given a function $f(x)$, it is desired to expand it in terms of T_n, $n = 1, 2, ..N$,

$$f^{(N)}(x) = \sum_{j=1}^{N} a_j\ T_{j-1}(x), \quad -1 \leq x \leq 1 \tag{6}$$

However, this quadrature involving Eq. (2) can be avoided by making use Curtis Clenshaw matrix relationThis results in a matrix relation between the column vectors (f) and (a)

$$(f) = (f(\xi_1), f(\xi_2), ...f(\xi_N))^T \text{and } (a) = (a_1, a_2, ...a_N)^T \tag{7}$$

(here T means "transposed"), of the form

$$(f) = C\ (a) \tag{8}$$

and

$$(a) = C^{-1}(f). \tag{9}$$

are column vectors. Because of the discrete orthogonality between Chebyhev polynomials, the $N \times N$ matrices C and C^{-1} are known, and given in terms of Chebyshev Poynomials evaluated at the support points $\xi_k,\ k = 1, 2, ..N$. Hence, all what is needed to obtain the expansion coefficients a_n, $n = 1, 2, ..N$ are the discrete values of the function f at the zeros of T_N, given by Eq. (4).

Values of C and C^{-1} are written in our earliest paper [3], are availabe in my MATLAB programs, and are made use of routinely in our numerical calculations. Interpolation of a function known only at the discrete ξ_k points to all values of x is obtained by means of Eq. (9), that can be written as

$$f^{(N)}(x) = \sum_{i=1}^{N}\sum_{k=1}^{N} T_{i-1}(x)(C^{-1})_{i,k} f(\xi_k). \tag{10}$$

Start of lecture 5: Integrals over functions

Another important matrix relation applies to obtaining *indefinite* integrals.

$$F_L(t) = \int_{-1}^{t} f(x)dx \quad \text{and} \quad F_R(t) \int_{t}^{1} f(x)dx. \tag{11}$$

Here the subscripts R and L stand for "left" and "right", repectively. From the expansion coefficients (a) of f, one can obtain the expansion coefficients $b^{(L),(R)}$ of the functions $F_{L,R}(t)$

$$F_L^{(N)}(t) = \sum_{k=1}^{N} b_k^{(L)} T_{k-1}(t) \text{ and } F_R^{(N)}(t) = \sum_{k=1}^{N} b_k^{(R)} T_{k-1}(t) \tag{12}$$

from the matrix relations

$$(b^{(L)}) = S_L^{(N)}(a) \ \text{ and } (b^{(R)}) = S_R^{(N)}(a). \tag{13a}$$

Again, these matrices S_L and S_R are standard for Chebyshev expansions, are available in our programs, and depend only on the value of the number of Chebychev expansion polynomials, and their associated support points ξ_k, $k = 1, 2, ..N$ in the open domain $(-1, 1)$.

If the definite integral $\mathfrak{I}$ is required

$$\mathfrak{I} = \int_{-1}^{1} f(x)dx \tag{14}$$

then, by making use of the expansion (12) for $F_L^{(N)}(t)$ for $t = 1$, and remembering that all $T_k(1) = 1$, then one finds

$$\mathfrak{I} \simeq \mathfrak{I}^C = \sum_{k=1}^{N} b_k^{(N)}. \tag{15}$$

This result should be identical (same accuracy?) as the Gauss-Chebyshev integral expression

$$\mathfrak{I} \simeq \mathfrak{I}^{GC} = \sum_{k=1}^{N} c_k f(\xi_k), \tag{16}$$

where

$$c_k = \int_{-1}^{1} \mathcal{L}_k(x)dx, \;\; k = 1, 2, ...N \tag{17}$$

Expression (17) comes from the expansion of the "interpolating polynomial" $P_{N-1}(x)$ of the function $f(x)$

$$f^{(N)}(x) = P_{N-1}(x) = \sum_{k=1}^{N} \mathcal{L}_k(x) f(\xi_k), \tag{18}$$

from which results (16) and (17) follow. The $\mathcal{L}_k$ are Lagrange polynomials of order $N-1$, and they vanish at all support points ξ_j with $j \neq k$. The above results are general expressions valid for orthogonal expansion polynomials, for each there is associated a particular set of support points that are the zeros of the orthogonal polynomial of one order higher than the ones used in the expansion of the interpolating polynomial.

For example, for Laguerre polynomials one has

$$\int_0^{\infty} e^{-x} f(x)dx \simeq \sum_{k=1}^{N} c_k f(\xi_k). \tag{19}$$

where

$$c_k = \int_0^{\infty} e^{-x} \mathcal{L}_k(x)dx, \;\; k = 1, 2, ...N. \tag{20}$$

Eq. (20) still has to be confirmed.

In MATLAB there are functions that give the weights and support points for Legendre functions $GLTable(nnode)$ or $GLNode(nnode)$, and $GaussLagQuad$ for Laguerre polynomials. (please check)

Assignment 5 a):

1.Start from an equispaced discrete set of angles θ contained between 0 and π. Use the symbols $'*'$ or something similar for your discrete points in the graphs below.

2. Calculate the corresponding set of $x-$values, and plot x versus θ.

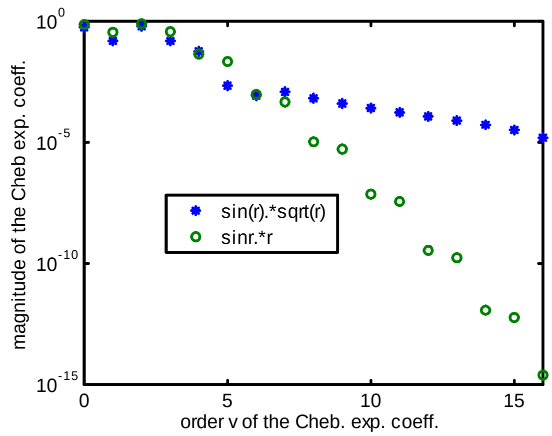


Figure 1: The Chebyshev expansion coefficients as a function of the index v, defined in Eq.(25), for the two functions f_1 and f_2.

3. Calculate the values of $T_n(x)$ for $n = 0, 1, 2,$ and 3, and plot them as a function of x

4. Calculate the values of $T_n(x)$ for $n = 0, 1, 2,$ and 3, and plot them as a function of θ

5. Check whether the forward recursion relation in Eq.(5) can be trusted, by comparing the result with Eq.(1)

Examples of Chebyshev expansions

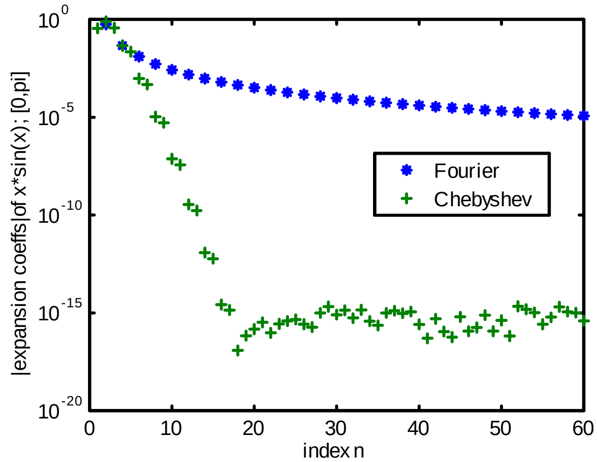
By means of examples we will examine the rapidity of the convergence with N. We will find that the rapidity of the convergence depends on the "smoothness" of f, in particular how many singularities this function or its derivatives can have. Theorems exist about these properties, as well as the uniformity of the convergence in the x variable, some of which we will see further on. The same procedure can also be applied to expansions in terms of other basis functions. The important question is how fast will the error $|f(x) - f^{(N)}(x)|$ decrease with N ? Numerical examples will be given below.

This project requires the Chebyshev expansion coeficients for the functions

$$f_1(r) = \sin(r) * r^{1/2} \tag{21}$$

$$f_2(r) = \sin(r) * r \tag{22}$$

The results are displayed in Fig.(1). Since $f_1(r)$ is not an analytic function of r, (the first derivative has a singularity for $r = 0$), the expansion coefficients decrease with the index v much more slowly than for the analytic function $f_2(r)$. This result was obtained by using the MATLAB functions $[C, CM1, xz] = C\_CM1(N)$ and $r = mapxtor(b1, b2, xz)$;



An expansion into a Fourier series of the function $\sin(r) * r$ is also carried out for comparison with the expansion into Chebyshev polynomials. The fourier expansion functions in the interval $[0, \pi]$ are $\sqrt{2/\pi}\sin(kr)$, $k = 1, 2, ...k_{\max}$. One finds that all a_n vanish for n odd, with the exception for $n = 1$, for which

$$a_1 = \frac{\pi^2}{4}\sqrt{\frac{2}{\pi}} \tag{23}$$

For n even, the corresponding result for a_n is

$$a_n = \sqrt{\frac{2}{\pi}} \left[\frac{1}{(1+n)^2} - \frac{1}{(1-n)^2}\right], \quad n = 2, 4, 6, ... \tag{24}$$

For $n \gg 1$, a_n will approach 0 like $-4\sqrt{\frac{2}{\pi}}(1/n)^3$, i.e., quite slowly. The absolute value of this result is shown in Fig.(2). By comparison with Fig.(1), or directly in Fig one sees that the Fourier expansion coefficients decrease with the index k much more slowly than the Chebyshev expansion coefficients .

Assignment 5 b
1. Consider the function

$$f(r) = r^{1/2}\sin(r) \tag{25}$$

for r defined from 0 to π. Define a new variable x that goes from -1 to $+1$ and relate it to the variable r by means of the linear transformation

$$r = a\ x + b$$

and find the values of a and b. Call the new function $\bar{f}(x) = f(r)$

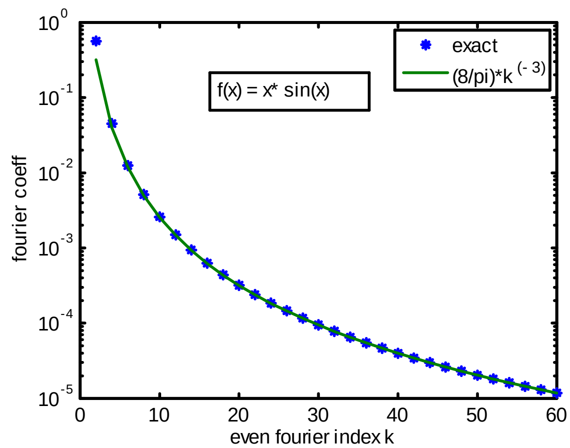


Figure 2: The Fourier expansion coefficients of the function $f(\theta) = \theta * \sin(\theta)$ in the interval $[0, \pi]$ in terms of the basis functions $\sin(k * \theta)$. The analytic result is shown by the symbols $*$. For odd values of k they are zero.

2. In preparation to expanding this function into Chebyshev polynomials choose a value of $N = 8$, and find the zeros ξ_i of $T_N(x)$ with $k = 1, 2, ..N$ Use the expressions given in class

$$\xi_k = \cos[\frac{\pi}{N}(k - 1/2)], \quad k = 1, 2, ...N. \tag{26}$$

and check that $T_N(\xi_k) = 0$ for all values of k

3. Use the MATLAB functions $[C, CM1, xz] = C\_CM1(N)$ given in class, and check wether the output vector xz agrees with the vector of the ξ_k values obtained in 2). Note, xz is a column vector

4. For each of the ξ_i values obtain the corresponding $r_k$values. Use the function $r2 = mapxtor(b1, b2, xz)$ with $b1 = 0$ and $b2 = \pi$, where xz was obtained in part 3) to obtain $r2$ and check wether the values of r_k and $r2$ agree. Note thar $r2$ should be a column vector.

5. Obtain the column vector $F = f(r2)$ and calculate the column vector A by means of the matrix*column vector MATLAB operation

$$A = CM1 * F \tag{27}$$

The $A-$vector contains the coefficients a_n of the expansion (22),.and check how fast the coefficients a_n decrease with n.

6. Repeat parts 3, 4, and 5 for $N = 16$

7. Define a new function

$$g(r) = r \ \sin(r) \tag{28}$$

and repeat parts 3, 4, and 5 with $N = 16$. Check that the new expansion coefficients a_n decreas much faster with n than for the expansion of $f(r)$.

References

[1] L. N. Trefethen, *Spectral Methods in MATLAB*, (SIAM, Philadelphia, PA, 2000);

[2] A. Deloff, Ann. Phys. (NY) **322** (2007) 1373–1419.

[3] R. A. Gonzales, J. Eisert, I Koltracht, M. Neumann and G. Rawitscher, J. of Comput. Phys. 134 (1997) 134.

[4] Y. L. Luke, *Mathematical Functions and their Approximations,* (Academic Press, NY, 1975); John P. Boyd, *Chebyshev and Fourier Spectral Methods,* 2nd revised ed. (Dover Publications, Mineola, NY, 2001);

[5] C.C. Clenshaw, and A.R. Curtis, Numer. Math., 2 (1960) 197;