Lecture 9: Examples for the use of Integral Eqs.

1 Vibration of an inhomogeneous clamped string[1]

Consider a stretched metallic string clamped between two horizontal points The distance between the fixed points is L,and the mass per unit length of the string is not a constant, and hence the speed of propagation of the waves depends on the location along the string. When a disturbance turbance is imparted to the string, the particles in the string vibrate in the vertical direction with a distribution of frequencies that is to be determined.

We denote by $y(x,t)$ the small displacement of a mass-point on the string in the vertical direction away from the equilibrium position $y=0$ at a horizontal distance x of the point from the left end at time t. The wave equation is

$$\frac{\partial^2 y}{\partial x^2} - \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} = 0 \tag{1}$$

where $\rho(x)$is the density of the string material and T is the tension along the string. We define the dimensionless function $\mathcal{R}(x)$ which describes the variation of the density with x as

$$\rho(x) = \rho_0 \mathcal{R}(x) \tag{2}$$

where ρ_0 is some fixed value of the density. If we define the reference speed c by

$$\frac{\rho_0}{T} = \frac{1}{c^2} \tag{3}$$

then Eq. (1) becomes

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\mathcal{R}(x)\frac{\partial^2 y}{\partial t^2} = 0 \tag{4}$$

A separation or variables $y(x,t) = \psi(x)\ A(t)$ yields the two equations

$$\frac{d^2\psi(x)}{dx^2} + \lambda \mathcal{R}(x)\psi(x) = 0 \tag{5}$$

and

$$\frac{d^2 A}{dt^2} = -\lambda c^2 A. \tag{6}$$

If λ is known then the solution of Eq. (6) is

$$A(t) = a\ \cos(wt) + b\sin(wt);\ w = c\sqrt{\lambda}. \tag{7}$$

The values of λ are obtained from the solution of Eq. (5). This is a Sturm Liouville eigenvalue equation with an infinite number solutions determined by

the boundary conditions. The solutions $\psi_n(x)$ with eigenvalues λ_n form a complete set with $n = 1, 2, ..$ in terms of which a general solution of Eq. (4) is given by

$$\psi(x,t) = \sum_{n=1} [a_n \cos(w_n t) + b_n \sin(w_n t)] \ \psi_n(x) \tag{8}$$

The focus of the present example is how to obtain the eigensolutions of Eq. (5) under the boundary conditions

$$\psi_n(0) = \psi_n(L) = 0. \tag{9}$$

and for $t = 0$

$$y(x,0) = f(x), \text{ and } dy(x,0)/dt = g(x). \tag{10}$$

Here $f(x)$ describes the "shape" of the string at $t = 0$, and $g(x)$ describes the initial velocity of the mass-pieces of the string. One can visalize this as a hammer striking the string at a certain point, imparting to the string at this point a "dent", and a velocity to the particles in the region of the dent. Once f and g are given, then the coefficients in Eq. (8) are determined by the integrals

$$a_n = \int_0^L f(x)\psi_n(x)dx \text{ and } b_n = \frac{1}{w_n}\int_0^L g(x)\psi_n(x)dx \tag{11}$$

1.1 Fourier solution of Eq. (9.6)

The fourier functions $\phi_\ell(x)$ that have the same boundary conditions (9) as $\psi_n(x)$ are

$$\phi_\ell(x) = \sqrt{\frac{2}{L}} \sin(k_\ell x), \quad k_\ell = \ell\pi/L, \ \ell = 1, 2, .. \tag{12}$$

These functions form an orthogonal basis in terms of which the functions ψ_n can be represented

$$\psi^{(N)}(x) = \sum_{\ell=1}^{N} d_\ell \phi_\ell(x) \tag{13}$$

The subscript n has been dropped at this stage, but the truncation parameter N has been included. Inserting the expansion (13) into Eq. (5), multiplying on the left with $\phi_{\ell'}(x)$, and integrating over x from 0 to L, one obtains

$$\sum_{\ell=1}^{N} R_{\ell'\ell} d_\ell = \frac{1}{\lambda} k_{\ell'}^2 d_{\ell'} \tag{14}$$

where

$$\mathbf{R}_{\ell'\ell} = R_{\ell'\ell} = \int_0^L \phi_{\ell'}(x)\mathcal{R}(x)\phi_\ell(x)dx. \tag{15}$$

and where $\mathbf{R}$ is a symmetric $N \times N$ matrix.

Eq. (14) can be transformed into a simple eigenvalue equation by defining the diagonal matrix

$$\mathbf{k^{-1}}= \begin{pmatrix} k_1^{-1} & & & & \\ & k_2^{-1} & & & \\ & & k_3^{-1} & & \\ & & & \ddots & \\ & & & & k_N^{-1} \end{pmatrix} \tag{16}$$

in terms of which the matrix $\mathbf{M}_{fourier}$ can be defined as

$$\mathbf{M}_{fourier} = \mathbf{k^{-1}Rk^{-1}}. \tag{17}$$

The column vectors $(u) = (u_1, u_2, ..u_N)^T$ and $(d) = (d_1, d_2, ..d_N)^T$ related by

$$(u) = \mathbf{k}(d) \tag{18}$$

can also be defined, where $\mathbf{k}$ is the diagonal matrix of the values k_i, $i = 1, 2, ..N$. With these transformations the generalized eigenvalue Eq. (14) can be transformed into an ordinary eigenvalue Eq.

$$\mathbf{M}_{Fourier}(u)_n = \frac{1}{\lambda_n}(u_n),\ n = 1, 2, ..N \tag{19}$$

where $\mathbf{M}_{Fourier}$ is a symmetric matrix and n represents each different eigenvalue and eigenfunction. The eigenvalues are displayed in Fig.2 and 4 of the .pdf page 4 of Ref. [1], corresponding to an inhomogeneity

$$\mathcal{R}(x) = 1 + F_0 x^2 \tag{20}$$

with $F_0 = 2$ or 4 in units of inverse length, and x is the distance along the string. For the present calculation $L = 1$.The inhomogeneity (20) corresponds to a 20% or 40% increase in density along the string. Nevertheless, the frequencies are still harmonic to a surprisingly good approximation, as shown in Fig. 4 of Ref. [1].

The main "computational" observation made in this sub-section is that the eigenvalues of a matrix, even though symmetric, loose accuracy towards the high eigenvalues. An iterative way of calculating eigenvalues will be described in the last section, that serve as a benchmark result because of its high accuracy (10^{-11}), and is not obtained asan eigenvalue of a matrix.

1.2 Eigenvalues in terms of an integral equation.

In the present section we transform Eq. (5) into an integral equation by "inverting" the operator d^2/dx^2 by means of a special energy independent Greens function

$$\frac{1}{\lambda_n}\psi_n = -\int_0^L \mathcal{G}_0(r, r')\mathcal{R}(r')\psi_n(r')dr', \tag{21}$$

with $n = 1, 2, ..N$, where N is the number of Chebyshev polynomials used in the expansion of all the functions involved. Here

$$\begin{aligned} \mathcal{G}_0(r, r') &= -F(r)G(r')/L \text{ for } r \leq r' \\ \mathcal{G}_0(r, r') &= -F(r')G(r)/L \text{ for } r' \leq r \end{aligned} \tag{22}$$

where

$$F(r) = r \text{ and } G(r) = L - r, \tag{23}$$

and the factor $1/L$ is the inverse of the Wronskian between F and G. Both F and G obey $d^2/dr^2(F \text{ and } G) = 0$, they are linearly independent of each other, and provide the appropriate boundary conditions for ψ. The Greens function above is the limiting result for $k \to 0$ of the Greens function $\mathcal{G}_0$ described by Eq. (6) of Lecture 6. An additional difference to Eq. (6) of Lecture 6 is that Eq. (21) lacks a driving term F, i.e., Eq. (21) is now an eigenvalue equation.

Similarly to Eq. (13) of Lecture 6, the spectral matrix that represents Eq. (21) is

$$\mathbf{M}_{IEM} = -C^{-1}(1/L)(L/2)\left[F_D C S_R C^{-1} G_D + G_D C S_L C^{-1} F_D\right] \mathcal{R}_D C \tag{24}$$

and the corresponding eigenvalue equation is

$$\mathbf{M}_{IEM}(a_n) = -\frac{1}{\lambda_n}(a_n). \tag{25}$$

It is remarkable that the two totally different matrices $\mathbf{M}_{Fourier}$ and $\mathbf{M}_{IEM}$ give the same eigenvalues λ_n, albeit with different accuracies. The accuracies are illustrated in Fig. 10 of page 9 of Ref. [1]. The iterative method used to provide an accuracy comparison standard are also described in Ref. [1], but will not be elaborated here.

The basic conclusion of the present lecture is that there are always several different computational methods to solve a problem, and comparison between such methods is desirable in order to gain insight into the accuracy of each of these methods.

References

[1] G. Rawitscher and J. Liss, "*The vibrating inhomogeneous string",* Am. J. Phys. 79, 417-429, (2011)